ANALYSIS AND DESIGN OF DISCRETELY CONTROLLED SWITCHED POSITIVE SYSTEMS

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Abstract: Discretely controlled switched positive systems are characterized by interacting continuous and discrete dynamics. Switching must take place not only to move the continuous state from the initial state to a goal state, but also to make the system remain in the surroundings of the goal state. The continuous dynamics are positive. This paper shows that if the continuous positive systems making up the switched system have a certain structure, it is possible to design stabilizing state-feedback controllers which ensure that the trajectories of the switched system cannot diverge to infinity regardless of the way the switching thresholds are selected. The trajectories of the discretely controlled switched positive systems can be restricted to invariant sets (called H-invariant sets) away from the equilibrium points of the continuous system parts. For a planar system, the trajectories within an H-invariant set converge to a stable and unique limit cycle regardless of the initial state. It is shown how this idea can be applied to design controllers which restrict the steady-state values of the continuous states to desired sets. Experimental results concern a manufacturing cell with hybrid dynamics.

Keywords: Hybrid systems, discretely controlled systems, controller design, stability

1. INTRODUCTION

Linear switched systems are dynamical systems where a number of continuous linear systems are switched in a systematic manner in order to achieve an overall control aim. These kind of systems arise in many practical situations where a control objective can only be satisfied if the operation mode of a continuous system is switched in an appropriate manner. In DC-DC converters (cf. e.g. (Krupar and Schwarz 2003)) switching must take place indefinitely in order to maintain the output voltage within a given range. Other examples can be found in the process industry where heated parts are used to transfer heat energy to other parts or fluids in order to maintain them within a given temperature range (cf. Section 6).

In the controller design for switched systems, the interaction between continuous-variable and discrete-variable dynamics has to be considered, and the design procedure results in a hybrid con-

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troller which has a continuous part and a discrete part. A general method for simultaneously designing the discrete and the continuous parts of the controller has been proposed in (Branicki et al. 1998) based on the framework of a controlled general hybrid dynamical model. In this approach, a cost function is defined and the continuous and discrete control signals which minimize the cost function over a given time horizon are computed. A similar optimization approach has been proposed in (Bemporad and Morari 1999) with the hybrid system modelled as a mixed logical dynamical system. Another general method of designing controllers for switched systems has been reported in (Asarin et al. 2000). All these methods result in an open-loop control which has the known disadvantages of missing robustness with respect to model uncertainties and disturbances.

This paper shows how for a specific class of switched systems a closed-loop controller can be found (Fig. 1). It concerns switched systems that have a positive dynamics in all operation modes. The controller has to switch the operation mode in order so satisfy the given control aim. This controller reacts on events that are generated by the continuous-variable system if the state crosses a partition border in the state space. The interesting aspect of these systems is that the partition borders can be freely chosen when designing the controller. Hence, the design problem considered here includes the definition of the state partition that characterises the hybrid closed-loop system. The positivity property of the continuous system makes it possible to analyse the behaviour of the closed-loop system in such a general way that guidelines for this choice of the controller can be obtained.

The paper is organized as follows: After the model of the discretely controlled switched system has been presented in Section 2 Section 3 is devoted to the analysis of the vector fields of the switched system. It is shown that the switched system cannot escape from the non-negative part of the state-space, and the trajectories of the switched system cannot diverge to infinity. The steady-state analysis in Section 4 shows that the trajectories of the discretely controlled switched positive system can be restricted to invariant sets away from the equilibrium points of the constituent systems making up the switched system. These invariant sets are called $\mathcal{H}$-invariant sets. For planar systems, the trajectories within a $\mathcal{H}$-invariant set converge to a stable and unique limit cycle regardless of the initial state. In Section 5 this idea is applied to design controllers which restrict the steady-state values of the continuous states to desired sets regardless of the initial state. A practical example is given in Section 6.

Due to space limitations, the propositions in this paper are stated without proofs which can be found in (Kamau 2004). The paper deals with the two-dimensional case, whereas generalisations can be found in (Kamau 2004).

2. THE MODEL

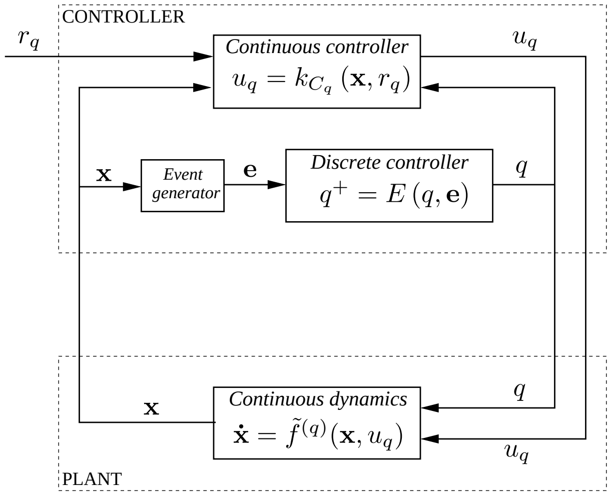

Fig. 1. The model of the closed-loop system

Figure 1 shows the closed-loop system consisting of the plant and the controller. The symbols used in the figure are defined as follows:

- $\mathbf{x} = [x_1 \ x_2 \ \ldots x_n]^T \in X \subseteq \mathbb{R}^n$ is the continuous state vector.
- $q \in Q \subseteq \mathbb{N}^M$ is the discrete state prescribed by the discrete part of the controller.
- $u_q \in U \subseteq \mathbb{R}$ is the continuous control signal.
- $\mathbf{e} = \{e_1, e_2, \ldots, e_M\}$, with $e_q \in \{0, 1\}$ is a vector of discrete events. The event e_q is generated when the continuous state $\mathbf{x}$ crosses the threshold $\Phi_q(\mathbf{x})$ in the continuous state space, i.e.

$$e_q = \begin{cases} 0 & \text{if} \quad \Phi_q(\mathbf{x}) \leq 0 \\ 1 & \text{otherwise} \end{cases} . \qquad (1)$$

- $r_q \in \mathbb{R}$ is a reference input for the continuous controller.

There is a unique continuous map $\tilde{f}^{(q)}(\cdot)$ associated with each discrete state q, and the continuous dynamics which are active at any given time are determined by the discrete controller. There is a separate continuous controller k_{C_q} for each discrete state, i.e. the continuous controller is switched together with the continuous plant.

In this paper, the following **modelling assumptions** are made:

- The continuous state $\mathbf{x}$ does not jump at the switching instants.

- The continuous systems $\tilde{f}^{(q)}(\cdot),\ q = 1, \ldots, M$ are positive linear systems of the form
$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}^{(q)}\mathbf{x} + \tilde{\mathbf{b}}^{(q)}u_q. \quad (2)$$
- The continuous controllers preserve the positivity of the associated continuous dynamics.
- The continuous systems are observable.
- None of the continuous systems $\tilde{f}^{(q)}(\cdot),\ q = 1, \ldots, M$ is completely state controllable by virtue of some inputs not being directly or indirectly connected to the states. This assumption is made to exclude the trivial solution where the switched system can be transferred from the initial state to the goal state without switching. It is assumed that the poles corresponding to the uncontrollable states lie strictly on the left-half plane.
- The switching surfaces $\Phi_q(\mathbf{x})$ are linear functions of the form $\mathbf{c}_q^T\mathbf{x} - d_q = 0,\ q = 1, \ldots, M$.

Specific continuous models. In the analysis described in the next section, it is assumed that two positive continuous-time systems making up the switched system. Each has only one controllable state, but the combination of the controllable states for both the continuous systems covers the whole state space. In a 2-dimensional setting, the continuous systems are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11}^{(1)} & \tilde{a}_{12}^{(1)} \\ 0 & \tilde{a}_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{b}_1^{(1)} \\ 0 \end{bmatrix} u_1 \quad (3)$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11}^{(2)} & 0 \\ \tilde{a}_{21}^{(2)} & \tilde{a}_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{b}_2^{(2)} \end{bmatrix} u_2, \quad (4)$$

where the matrices or vectors occurring in both equations are denoted by $\tilde{\mathbf{A}}^{(1)}$ and $\tilde{\mathbf{A}}^{(2)}$ or $\tilde{\mathbf{b}}^{(1)}$ and $\tilde{\mathbf{b}}^{(2)}$, respectively.

3. ANALYSIS OF THE VECTOR FIELDS

Affine state feedback is used to stabilize the continuous systems while maintaining the positivity of the continuous dynamics, (for details cf. (Kamau 2004)). After the design of the continuous controller, the continuous dynamics of the asymptotically stable closed-loop systems corresponding to systems (3) and (4) are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} \\ 0 & a_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1^{(1)} \\ 0 \end{bmatrix} = \mathbf{A}^{(1)}\mathbf{x} + \mathbf{b}^{(1)}, \quad (5)$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11}^{(2)} & 0 \\ a_{21}^{(2)} & a_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2^{(2)} \end{bmatrix} = \mathbf{A}^{(2)}\mathbf{x} + \mathbf{b}^{(2)}. \quad (6)$$

The continuous controllers are designed in such a way that the inequalities

$$-a_{ii}^{(q)} > \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}^{(q)} \text{ for } i = 1, \ldots, n,\ q = 1, \ldots, n \quad (7)$$

hold. Switching between systems (5) and (6) means superimposing the vector fields of the respective systems. If the continuous systems satisfy condition (7), the vector fields of the switched system have the properties shown in Figure 2. Line l_1 is defined by the equation $a_{11}^{(1)} + a_{12}^{(1)} + b_1^{(1)} = 0$ while line l_2 is given by $a_{21}^{(2)} + a_{22}^{(2)} + b_2^{(2)} = 0$. The point $[\bar{x}_1,\ 0]^T$ is the equilibrium point of system (5) while point $[0,\ \bar{x}_2]^T$ is the equilibrium point of system (6). The arrows show the directions of the vector fields.

The figure shows that the continuous state space $\mathbb{R}^2_{\geq 0}$ can be partitioned into 4 different regions depending on the direction of the vector fields. As a consequence, the following propositions can be proved (Kamau 2004):

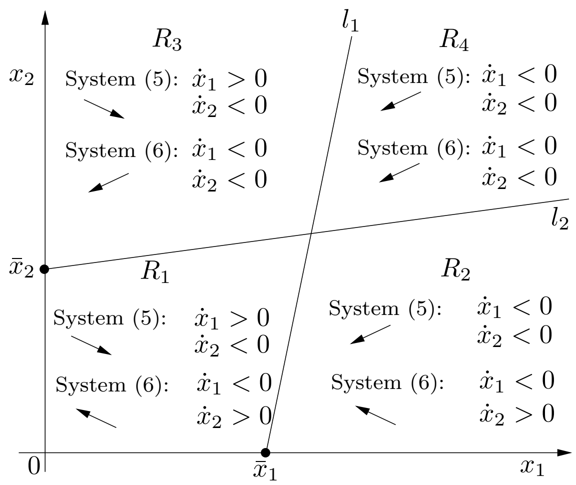

Fig. 2. Vector fields of systems (5) and (6) superimposed

Proposition 3.1. The non-negative part of the state-space (the set $\mathbb{R}^n_{\geq 0}$) is an invariant set for the discretely controlled switched positive systems.

Proposition 3.2. The state of the discretely controlled switched positive system made up of the second-order asymptotically stable positive systems (5) and (6) is bounded, i.e.

$$\|\mathbf{x}(0)\| < \infty \quad \Rightarrow \quad \|\mathbf{x}(t)\| < \infty \quad \forall t \geq 0 \quad (8)$$

This concept of the boundedness of the state can be extended to higher dimensional systems provided the continuous controllers are designed such that each system satisfies condition (7).

4. $\mathcal{H}$-INVARIANT SETS

It is well known for a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{9}$$

that a set $\tilde{X} \subseteq \mathbb{R}^n$ is said to be $\mathbf{A}$-invariant if $\mathbf{A}\mathbf{x} \in \tilde{X}$ for all $\mathbf{x} \in \tilde{X}$. Every trajectory of system (9) starting from the $\mathbf{A}$-invariant set $\tilde{X}$ remains in that set for all future time.

As an analogy, the notion of an $\mathcal{H}$-invariant set ($\mathcal{H}$ for hybrid) is introduced here as follows:

Definition 4.1. A set $H \subseteq \mathbb{R}^n_{\geq 0}$ is called $\mathcal{H}$-invariant if all the trajectories of the discretely controlled switched positive system starting from H remain in that set for all future time.

In the previous section, it was stated that the trajectories of the discretely controlled switched positive system cannot escape from the set $\mathbb{R}^2_{\geq 0}$, which is the entire continuous state space for this type of system.

Corollary 4.1. The set $\mathbb{R}^2_{\geq 0}$ is the largest $\mathcal{H}$-invariant set for the discretely controlled switched positive system.

To illustrate the properties of the $\mathcal{H}$-invariant set, consider Figure 3. The switching thresholds Φ_1 and Φ_2 are straight lines originating along the x_1 axis in region R_1 and extending into region R_3. The switching surfaces do not intersect and do not enclose the equilibrium points $[\bar{x}_1,\ 0]^T$ or $[0,\ \bar{x}_2]^T$ between them. The setting in the figure assumes that the switched system has an initial state along the switching threshold Φ_2 with system (5) active. When the system trajectory crosses the switching threshold Φ_1, system (6) becomes active, and when the trajectory crosses threshold Φ_2, system (5) becomes active, and so on.

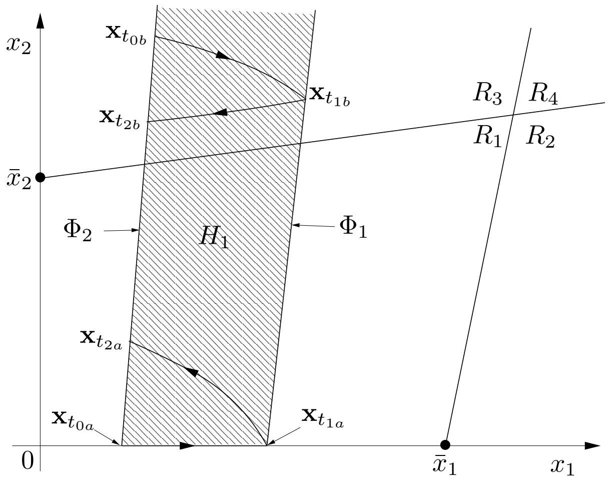

Fig. 3. $\mathcal{H}$-invariant set H_1

The trajectory of the switched system starting at point $\mathbf{x}_{t_{0a}}$ along the x_1-axis in Figure 3 evolves as shown in the figure, and after one cycle, the trajectory of the switched system ends up at a point $\mathbf{x}_{t_{2a}}$ which is higher than $\mathbf{x}_{t_{0a}}$. On the other hand, a trajectory of the switched system starting at point $\mathbf{x}_{t_{0b}}$ in region R_3 evolves as shown in Figure 3, and after one cycle ends up at point $\mathbf{x}_{t_{2b}}$ which is lower than $\mathbf{x}_{t_{0b}}$ (this can be deduced from the directions of the vector fields shown in Figure 2). It follows that trajectories of the switched system starting anywhere between the switching surfaces Φ_1 and Φ_2 (labelled as set H_1 in Figure 3) cannot escape from that set, hence H_1 is a $\mathcal{H}$-invariant set.

If the switching thresholds Φ_1 and Φ_2 are chosen as explained above, it can be shown that there exists an $\mathcal{H}$-invariant set L with the properties shown in Figure 4. The lower boundary of the set is the trajectory of system (5) from $\mathbf{x}_{t_0}$ to $\mathbf{x}_{t_1}$, the upper boundary is the trajectory of system (5) from $\mathbf{x}_{t_0^*}$ to $\mathbf{x}_{t_1^*}$, while the switching thresholds Φ_1 and Φ_2 form the side boundaries of the set.

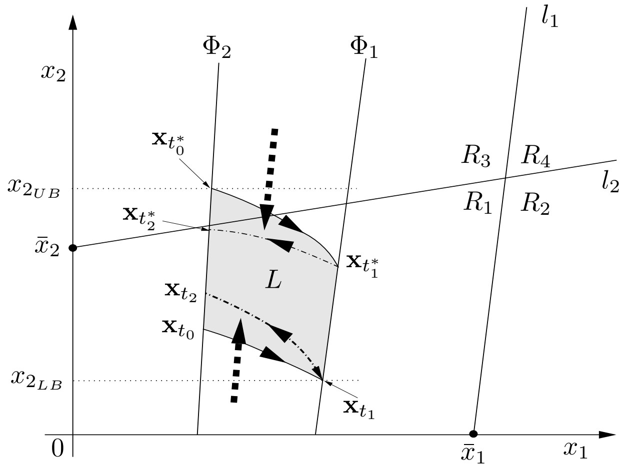

Fig. 4. $\mathcal{H}$-invariant set L

The set L has the important property that with repeated switching, all trajectories starting below the lower boundary eventually enter the set from below and remain within the set for all future time. Similarly, all trajectories starting above the upper boundary eventually enter the set L from above and remain in the set for all future time. Trajectories starting within L remain within the set for all future time. Hence L is a $\mathcal{H}$-invariant set. Since the trajectories cannot escape from set L, the steady-state value of state x_2 is limited to an upper bound of $x_{2_{UB}}$ and a lower bound of $x_{2_{LB}}$ as shown by the dotted lines in Figure 4.

Proposition 4.1. : For a given set of switching surfaces Φ_1 and Φ_2, the $\mathcal{H}$-invariant set L shown in Figure 4 contains one unique and stable limit cycle.

The stability of the limit cycle can also be analyzed by use of a Poincaré map. The trajectory sensitivity matrix after exactly one period T of the limit cycle is known as the monodromy matrix (Hiskens 2001, Seydel 1994), and the eigenvalues of this matrix determine the stability of the limit cycle. The monodromy matrix always has one eigenvalue of 1. If the rest of the eigenvalues (known as characteristic multipliers or Floquet multipliers) are less than 1, the limit cycle is stable.

5. DESIGN RULES FOR THE CONTROLLER

To design the discrete part of the controller means to fix the thresholds Φ_1 and Φ_2, which define the events e_1 and e_2, and to select the control law E (cf. Fig. 1). The uniqueness of the limit cycle as stated in Proposition 4.1 implies that if a limit cycle for surfaces Φ_1' and Φ_2' is found to have vertices $\mathbf{x}_0$ and $\mathbf{x}_1$ (see Figure 5), then the trajectory of the switched system for another set of switching surfaces Φ_1 and Φ_2 passing through points $\mathbf{x}_0$ and $\mathbf{x}_1$ will also converge to the same limit cycle with vertices $\mathbf{x}_0$ and $\mathbf{x}_1$.

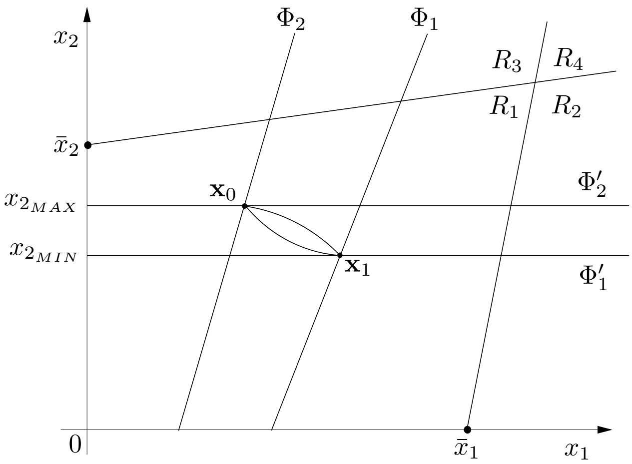

Fig. 5. Restricting state x_2 to $x_{2_{MIN}} \leq x_2 \leq x_{2_{MAX}}$

To design a controller to restrict the steady-state value of state x_2 to the range $x_{2_{MIN}} \leq x_2 \leq x_{2_{MAX}}$, two horizontal switching surfaces

$$\Phi_1'(\mathbf{x}) = -x_2 + x_{2_{MIN}} = 0, \tag{10}$$

and

$$\Phi_2'(\mathbf{x}) = x_2 - x_{2_{MAX}} = 0, \tag{11}$$

extending from region R_1 to region R_2 are selected as shown in Figure 5. The vertices $\mathbf{x}_0$ and $\mathbf{x}_1$ of the limit cycle are then calculated. This limit cycle is restricted to the range $x_{2_{MIN}} \leq x_2 \leq x_{2_{MAX}}$ as shown in the figure.

Another set of non-intersecting switching surfaces Φ_1 and Φ_2 originating along the x_1 axis in region R_1 and extending into region R_3, and passing through points $\mathbf{x}_0$ and $\mathbf{x}_1$ is then selected. These switching surfaces should not enclose any of the equilibrium points between them. For the new set of switching surfaces, the trajectory of the switched system also converges to the same limit cycle with vertices $\mathbf{x}_0$ and $\mathbf{x}_1$. The steady-state value of state x_2 for the switched system is therefore restricted to the desired range of $x_{2_{MIN}} \leq x_2 \leq x_{2_{MAX}}$.

For this design method, the trajectory is guaranteed to converge to the desired limit cycle as long as the initial state is located in between surfaces Φ_1 and Φ_2 (Figure 5). Furthermore, the choice of switching surfaces Φ_1 and Φ_2 is not critical. This method can therefore be applied to systems where the initial state and the plant parameters are not known exactly. Another advantage of this method is that it works for systems which are not completely state controllable.

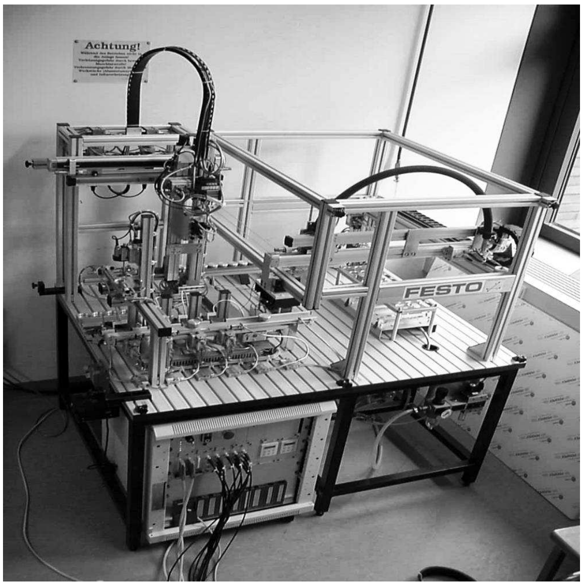

Fig. 6. The Manufacturing Cell with Hybrid Dynamics

6. EXAMPLE

The manufacturing cell shown in Fig. 6 has a hybrid dynamics. The task considered here is to heat a metal block from an initial temperature θ_{B0} to a target temperature $\theta_{B_{SET}}$ by using a workpiece which has been heated by placing it on a heater. This involves the following sequence of tasks:

- placing the workpieces on the heater
- controlling the power output of the heater to achieve the desired workpiece temperature (setpoint).
- placing the workpiece on the metal block

In general, it is not possible for the temperature of the metal block to reach $\theta_{B_{SET}}$ by placing the heated workpiece on it just one time, so the workpiece has to be returned to the heater and re-heated after which it is placed on the metal block and the process is repeated until the target temperature $\theta_{B_{SET}}$ is reached. Figure 7 shows this process.

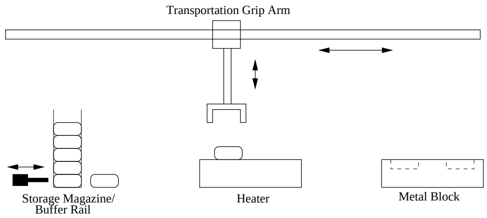

Fig. 7. Heating of a metal block

The time taken by the transportation grip arm to move between the heater and the metal block is is very short in comparison to the time constants of the heating process. Hence, the transportation grip arm is modelled as a discrete variable with two possible values, one value representing the position above the heater and the other value the position above the metal block.

Assuming no disturbances, no measurement noise, and constant room temperature, the open-loop system can be described by the model introduced in Section 2. Details of this model are given in (Kamau 2004).

Denoting by e_1 and e_2 the discrete events associated with the transition guards $\Phi_1 = g_1$ or $\Phi_2 = g_2$, respectively, shown in Fig. 8, the guidelines given in Section 5 yields the discrete control law

$$q^+ = e_1 \bar{q} + e_2 q \,. \tag{12}$$

The results are shown on Figure 8. It can be seen that the switching controller is able to transfer the continuous state of the system from $x_2 = \theta_{B_0} = 4$ to $x_2 = \theta_{B_{SET}} = 15$, so the objective is met.

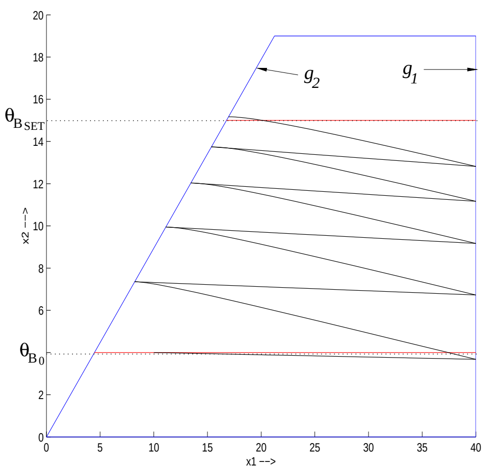

Fig. 8. Simulation Results

7. CONCLUSION

An analysis and control design method for discretely controlled switched positive systems has been described. If the continuous positive systems have the structure described here, it is possible to design stabilizing state-feedback controllers which ensure that the trajectories of the switched system cannot diverge to infinity regardless of the way the switching thresholds are selected.

It was shown that the trajectories of the discretely controlled switched positive system can be restricted to invariant sets called $\mathcal{H}$-invariant sets. For a planar system, the trajectories within an $\mathcal{H}$-invariant set converge to a stable and unique limit cycle as long as the initial state is chosen between the switching surfaces. It was shown how this idea can be applied to design controllers which restrict the steady-state values of the continuous states to desired sets.

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