BOUNDARY LAYER USING DITHERING IN SLIDING MODE CONTROL

Roberto Frasca [*] **Luigi Iannelli** [*] **Francesco Vasca** [*]

[*] *Università del Sannio - Dipartimento di Ingegneria*
P.zza Roma, 21 - 82100 Benevento, Italy
e-mail: frasca, luigi.iannelli, vasca@unisannio.it

Abstract: Practical issues in the implementation of sliding mode control systems, such as finite switching frequency, time delays and unmodelled dynamics, determine the state to belong to a neighborhood of the sliding manifold (the boundary layer) instead of lying on it. Dithering is here proposed as a technique for the regulation of the boundary layer when the actuator has a natural on-off behaviour. It is proved that dithering maintains the state of the system close to the ideal sliding manifold and that state error and practical stability depend on the dither amplitude, period and shape. Simulations on DC/DC power converters show that dither can play an important role for stability and robustness by using sliding mode control.

Keywords: Dither; Sliding Mode; Power Circuits; Averaging Control; Chattering.

1. INTRODUCTION

Sliding mode control is sometimes to be preferred with respect to other control techniques due to the robustness with respect to large disturbances and good dynamic performances, by preserving a simple implementation (Khalil, 2002; Perruquetti and Barbot, 2002). The dynamics in the state space of a sliding mode controlled system are characterized by a *reaching phase*, in which the trajectories starting off the sliding manifold will reach it, and a *sliding phase* in which the motion will be confined on the sliding manifold. In practical applications, due to the presence of delays, unmodelled dynamics typical of sensors and actuators or non ideal realization of the relay characteristic, ideal sliding mode is not possible. In other words, after the reaching phase, the state will belong to a suitable neighborhood of the sliding manifold, instead of lying on it. This approximation of the sliding motion gives rise to the so called chattering phenomenon (Young *et al.*, 1999). Among others, a widely used solution to the chattering problem is based on the boundary layer idea (Khalil, 2002; Perruquetti and Barbot, 2002). In particular, the boundary layer controller consists of a smooth interpolation of the relay characteristic (resulting for instance in a saturation or a sigmoid) and is designed so that, once reached, the state remains inside a boundary layer around the sliding manifold. Despite to its common use, such solution presents some problems which still need to be addressed. Firstly, the boundary layer controller must be designed in order to obtain a good compromise between robustness of the sliding controller, which asks for increasing the controller gain, and chattering mitigation, which needs a reduction of the controller gain (Chen *et al.*, 2002; Young *et al.*, 1999). Second, sliding mode controllers are often implemented by using actuators that by their nature operate in on-off mode. An interesting example of such systems are power converters for which switching represents a natural behaviour (Marino and Vasca, 1995; Habetler and Harley, 2001; Shtessel *et al.*, 2002).

Some applications of the sliding mode control for power converters can be found in (Spiazzi and Mattavelli, 2002; Sira-Ramirez, 1987). In such systems the use of a continuous approximation of the ideal relay sliding mode controller is clearly not natural and introduces further problems for the controller implementation (Young *et al.*, 1999). The dithered sliding mode controller proposed in this paper allows to maintain the state inside a boundary layer around the sliding manifold; it preserves the natural switching mode of the controller; it ensures a finite switching frequency and allows to reduce the chattering phenomenon. A dithered feedback system is a system in which a suitable high frequency signal (the dither) is injected at the input of the discontinuous nonlinearity. It has been recently shown that dithered relay feedback systems can be approximated by a feedback system (the averaged system) without dither, in which the relay is replaced by an averaged nonlinearity whose shape depends on the dither signal waveform (Iannelli *et al.*, 2003; Iannelli *et al.*, 2004). In particular, the error between the state of the dithered system and the state of the averaged system is of order T, where T is the dither period. By exploiting these results, in this paper it is proved that dithering can be used as a boundary layer approach for sliding mode control systems with on-off actuators: the boundary layer can be adapted by simply changing the dither shape and the simple operating mode of the switching controller is preserved. Simulations carried out for DC/DC power converters show the effectiveness of the proposed solution.

2. PRELIMINARIES

2.1 Sliding mode system

Consider the following nonlinear system represented in the so-called regular form

$$\dot{\eta} = f(\eta, \xi) \tag{1a}$$
$$\dot{\xi} = f_a(\eta, \xi) + G_a(\eta, \xi)u \tag{1b}$$

where $\eta \in \mathbb{R}^{n-p}$, $\xi \in \mathbb{R}^p$, $u \in \mathbb{R}^p$ is the input of the system, $G_a(\eta, \xi)$ is non singular and $f(\eta, \xi)$ and $f_a(\eta, \xi)$ are globally Lipschitz functions. From (1) the sliding mode controlled system can be represented through the following model (Khalil, 2002):

$$\dot{\eta} = f(\eta, \xi) \tag{2a}$$
$$\dot{\xi} = f_0(\eta, \xi) - \alpha(\eta, \xi)w \tag{2b}$$

where $\alpha(\eta, \xi)$ is a scalar function satisfying the condition $\alpha(\eta, \xi) \geq \alpha_0 > 0$, f_0 is a globally Lipschitz function obtained from the control law

$$u = G_a^{-1}[-f_a(\eta, \xi) + \frac{\partial \phi(\eta)}{\partial \eta} f(\eta, \xi)] + G_a^{-1} v \tag{3a}$$
$$v = -\alpha(\eta, \xi)w \tag{3b}$$

with $\phi(\eta)$ a Lypschitz function that stabilizes the system $\dot{\eta} = f(\eta, \phi(\eta))$ and the control variable $w \in \mathbb{R}^p$ is given by the relay characteristic:

$$w = n(\sigma) = \text{sgn}(\sigma) \tag{4}$$

with

$$\sigma(\eta, \xi) = \xi - \phi(\eta) \tag{5}$$

where $\sigma = 0$ represents the sliding manifold. Note that the equation (4) means that for every component of w holds:

$$w_i = n(\sigma_i) = \text{sgn}(\sigma_i) \triangleq \begin{cases} +1, & \sigma_i > 0 \\ -1, & \sigma_i < 0. \end{cases} \tag{6}$$

A block diagram of the sliding mode controlled system is reported in Fig. 1.

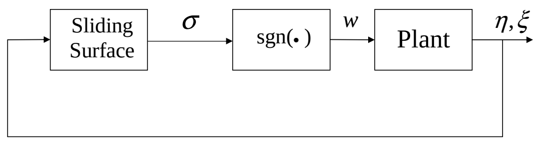

Fig. 1. Block diagram of the sliding mode controlled system.

2.2 Boundary layer system

In order to mitigate the chattering, it is possible to use a continuous approximation of the discontinuous sliding mode controller. A typical choice consists of replacing the relay characteristic (4) by a saturation function, i.e. by using in (2)

$$w_i = \text{sat}\left(\frac{\sigma_i}{\epsilon}\right) \tag{7}$$

for $i = 1, \ldots, p$, where $1/\epsilon$ is the slope of the linear part of the saturation. The resulting system is the so-called boundary layer system. In order to have a high gain feedback system, the use of a small ϵ is needed, but a too small ϵ will increase the chattering problem (Young *et al.*, 1999). Thus a compromise between these two conflicting needs, which are present also if a different smooth nonlinearity is used to replace the relay in (2), must be obtained. To this aim, the proposed dithered system can represent a valuable and simply implementable solution.

2.3 Dithered system

The dithered system corresponding to (2), shown in Fig. 2, is defined as

$$\dot{\eta}_d = f(\eta_d, \xi_d) \quad (8a)$$

$$\dot{\xi}_d = f_0(\eta_d, \xi_d) - \alpha(\eta_d, \xi_d) w_d \quad (8b)$$

$$w_d = n(\sigma(\eta_d, \xi_d) + \delta) \quad (8c)$$

$$\sigma(\eta_d, \xi_d) = \xi_d - \phi(\eta_d) \quad (8d)$$

where the components of the dither signal δ are assumed to be periodic of period T and the relay characteristic n is defined in (4). Note that σ is Lipschitz.

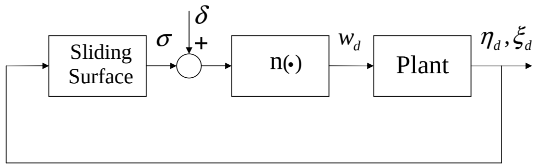

Fig. 2. Block diagram of the dithered system.

2.4 Averaged system

The averaged (smooth) system is defined as

$$\dot{\eta}_s = f(\eta_s, \xi_s) \quad (9a)$$

$$\dot{\xi}_s = f_0(\eta_s, \xi_s) - \alpha(\eta_s, \xi_s) w_s \quad (9b)$$

$$w_s = N(\sigma(\eta_s, \xi_s)) \quad (9c)$$

$$\sigma(\eta_s, \xi_s) = \xi_s - \phi(\eta_s) \quad (9d)$$

where N is the so-called averaged nonlinearity.

A block diagram of the averaged system is reported in Fig. 3. The averaged (or smoothed) nonlinearity $N(z)$ is obtained by evaluating the time average of the output of the relay on a dither period T by assuming z constant:

$$N(z) = \frac{1}{T} \int_0^T n(z + \delta(t)) \mathrm{d}t. \quad (10)$$

The function $N(z)$ depends on the signal δ and its shape (square wave, sawtooth, sinusoid, etc.). If the dither signal is a sawtooth (or a triangular) waveform which varies between $-M_\delta$ and $+M_\delta$ with period T, it is easy to show (by applying (10)) that

$$N(z) = \mathrm{sat}(z/M_\delta) = \begin{cases} -1, & z < -M_\delta \\ z/M_\delta, & |z| \le M_\delta \\ +1, & z > M_\delta. \end{cases} \quad (11)$$

Note that $N(z)$ is Lipschitz (with Lipschitz constant equal to $1/M_\delta$) while the original relay nonlinearity was discontinuous: dither has "smoothed" the relay. Other averaged nonlinearities can be obtained by considering different dither signals (Iannelli *et al.*, 2004).

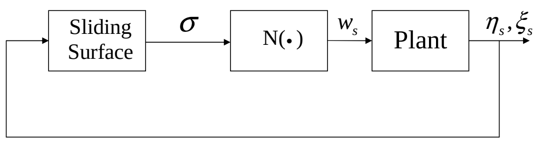

Fig. 3. Block diagram of the averaged system.

3. DITHERED CONTROLLER

The state of the dithered system will reach in finite time a suitable region around the sliding manifold $\sigma(\eta, \xi) = 0$. The size of such set will depend on the dither amplitude. That is proved in the following lemma.

Lemma 1. Consider the dithered system (8) in which δ is any periodic waveform of amplitude M_δ. For any initial condition $(\eta_0, \xi_0) \in \mathbb{R}^n$, the trajectories of the dithered system reach the region

$$\Omega = \{(\eta_d, \xi_d) : |\sigma_i(\eta_d, \xi_d)| \le M_\delta, i = 1, \ldots, p\} \quad (12)$$

after a finite time Δ.

PROOF. Outside the region Ω, the dither signal δ has no effect since when $|\sigma_i| > M_\delta$ the corresponding argument of n in (8) does not change signum. So the dithered system is equal to the system (2). Therefore, outside Ω, from the definition of the sliding manifold one can write:

$$\dot{\sigma}(\eta_d, \xi_d) \equiv \dot{\sigma}(\eta, \xi) = \dot{\xi} - \frac{\partial \phi(\eta)}{\partial \eta} \dot{\eta} =$$

$$= f_0(\eta, \xi) - \alpha(\eta, \xi) w - \frac{\partial \phi(\eta)}{\partial \eta} f(\eta, \xi) =$$

$$= -\alpha(\eta, \xi) w = -\alpha(\eta, \xi) n(\sigma(\eta, \xi)). \quad (13)$$

Now, define $V = \frac{1}{2} \sum_{i=1}^{p} \sigma_i^2$ as a Lyapunov function candidate for the system (13). Considering the i-th component, it follows that:

$$\dot{V}_i = \sigma_i \dot{\sigma}_i = -\sigma_i \alpha(\eta, \xi) n(\sigma). \quad (14)$$

If $|\sigma_i| > M_\delta$ (that is outside Ω), then:

$$\dot{V}_i = -\sigma_i \alpha(\eta, \xi) n(\sigma_i) = -|\sigma_i| \alpha(\eta, \xi) \le -\alpha_0 |\sigma_i|. \quad (15)$$

So, the inequality $\dot{V}_i \le -\alpha_0 |\sigma_i|$ ensures that the trajectory reaches the region (12) in a finite time.

It is now possible to show that over an infinite time horizon the state of the dithered system will remain close to the state of the averaged system when it asymptotically tends to the origin. Note that this result is valid everywhere and not only outside Ω where it is obvious since the sliding mode system (2), the dithered system (8) and the averaged system (9) are the same. As a consequence of this result one can conclude that the state of the dithered system will remain around the sliding manifold over an infinite time horizon.

***Lemma** 2.* Consider the averaged system (9). Suppose that the nonlinearity $N(z)$ can be written as $N(z) = \overline{N}(z/M_\delta)$ that satisfies the condition

$$z\overline{N}(z) \geq \gamma z^2 \quad \forall |z| \leq 1 \tag{16}$$

with $\gamma > 0$. So, if the origin of the system $\dot{\eta}_s = f(\eta_s, \phi(\eta_s))$ is globally exponentially stable, than there exists $M_\delta^* > 0$ such that for all $0 < M_\delta < M_\delta^*$ the origin of the averaged system is globally uniformly asymptotically stable.

PROOF. The result can be obtained following the same steps of the proof of Theorem 14.2 in (Khalil, 2002).

This result is well known in the literature (see also (Canudas de Wit and Perruquetti, 2002)). In that work, if an additive disturbance is considered in equation (1b), the memoryless nonlinearity must have a high gain around the origin. Since here the disturbance is not considered, it is possible to have a weaker constraint on N around the origin by just requiring that the nonlinearity belongs to the sector $[\gamma, \infty)$ (condition (16)) and that N must have a non zero slope in the origin (condition $\gamma > 0$).

The following lemma shows that if the dithered system (8) has no sliding solutions, the error between the state of the dithered system and the state of the averaged system is of order of the dither period.

***Lemma** 3.* Consider the dithered system (8) and the averaged system (9) with the following assumptions:

(1) the dithered system has an absolutely continuous solution;
(2) f_0 and f are globally Lipschitz;
(3) the dither δ is T-periodic and such that the averaged nonlinearity N is Lipschitz continuous.

Then the averaged system (9) has a unique absolutely continuous solution on $[0, \infty)$. Moreover, for any given $\Delta > 0$ and any initial condition x_0, it holds that

$$|x_d(t, x_0) - x_s(t, x_0)| = O(T), \quad \forall t \in [0, \Delta], \tag{17}$$

where

$$x_d = \begin{pmatrix} \eta_d \\ \xi_d \end{pmatrix}, \quad x_s = \begin{pmatrix} \eta_s \\ \xi_s \end{pmatrix}.$$

PROOF. The result can be obtained as a particular case of the averaging theorem proved in (Iannelli *et al.*, 2004).

***Lemma** 4.* Consider the averaged system (9) and the dithered system (8). Suppose that the hypotheses of Lemma 3 hold and the origin of the state space of the averaged system (9) is globally uniformly asymptotically stable. Then the origin of the state space of the dithered system (8) is practically globally uniformly asymptotically stable.

PROOF. The result can be obtained as a particular case of the theorem proved in (Moreau and Aeyels, 2000).

It is now possible to prove the main result of the paper.

***Theorem** 1.* Consider the dithered system (8) and the averaged system (9) in which the nonlinearity N satisfies the condition (16). Suppose that the origin of the system $\dot{\eta}_s = f(\eta_s, \phi(\eta_s))$ is globally exponentially stable and the hypotheses of Lemma 2 hold. Then, for any initial condition x_0, it holds that

$$|x_d(t, x_0) - x_s(t, x_0)| = O(T), \quad \forall t \in [0, \infty). \tag{18}$$

PROOF. Applying Lemma 2 it follows that the origin of the the averaged system is globally uniformly asymptotically stable. So, in force of Lemma 3 and Lemma 4 the condition (17) holds over an infinite time horizon.

The above theorem allows to conclude that the dithered sliding controller will maintain the state close to the sliding manifold.

4. DITHER DESIGN

After that the sliding mode controller has been designed, the smoothed nonlinearity in the boundary layer system (9) can be chosen in order to satisfy some desired stability or robustness performances. The smoothed nonlinearity will determine the corresponding dither waveform (see (10)). As an example, consider a sawtooth dither signal and the averaged nonlinearity (11). By decreasing M_δ the gain of the saturation increases and approximates with better accuracy the relay characteristic. This will improve the robustness of the closed loop system. If a sinusoidal dither is used instead of a sawtooth, the smoothed nonlinearity will correspond to a sort of variable feedback gain: when the state is close to the sliding manifold the controller gain is “small” and thus the controller effect is weak, whereas the effect of the controller increases when the state tends to go away from the sliding manifold. After choosing the desired smoothed nonlinearity, since (8) approximates (9) with order T, one can simply implement the controller by exploiting the model (8), i.e. by injecting the corresponding dither signal at the input of the relay characteristic of the sliding mode controller, without the need for any complicated adaptation of the nonlinear characteristic of the sliding controller. Thus, the dither will provide a constant switching frequency that can be varied by changing the dither period, the state will remain close to the sliding manifold according to (12), and the local attractiveness of the sliding manifold can be varied by changing the dither shape and its amplitude. Moreover, from a global point of view (i.e. large deviations) the dithered controller provides the same performance of the ideal sliding mode one.

5. APPLICATION TO DC/DC POWER CONVERTERS

As an example, consider the DC/DC buck converter represented in Fig. 4. The main objective of the converter controller is to regulate the capacitor (output) average voltage to a desired value V_{ref}. A sliding mode controller for such converter can be obtained by considering a sliding manifold consisting of a linear combination of the inductor current, through a constant k_1, and the integral of the output voltage error, through a constant k_2 (Ahmed *et al.*, 2003). By choosing as state variables the time integral of the error between the output voltage and the reference voltage, say η_1, the capacitor voltage, say η_2, and the inductor current, say ξ, the converter model can be represented as

$$\dot{\eta}_1 = \eta_2 - V_{ref} \quad (19)$$

$$\dot{\eta}_2 = -\frac{1}{R_2 C}\eta_2 + \frac{1}{C}\xi \quad (20)$$

$$\dot{\xi} = -\frac{R_1}{L}\xi - \frac{1}{L}\eta_2 + \frac{E}{2L} - \frac{E}{2L}n(\sigma) \quad (21)$$

$$\sigma = k_1\xi + k_2\eta_1 \quad (22)$$

which can be simply rewritten in the form (2). Fig. 5 shows the output voltage of the buck converter under different adaptations of the sliding mode controller. As it was expected the compensation through dither (amplitude $M_\delta = 0.2$ and $T = 500\mu s$) still provide good performance also with respect to an hysteresis controller (± 0.2).

Consider the DC/DC boost converter represented in Fig. 6. Now the sliding manifold consists of a linear combination of the inductor current, the capacitor voltage and the integral of the output voltage error. By choosing as state variables the inductor current, say x_1, the capacitor voltage, say x_2 and the time integral of the error between the

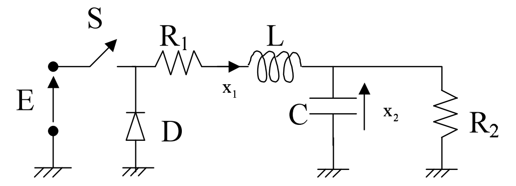

Fig. 4. Buck DC/DC converter.

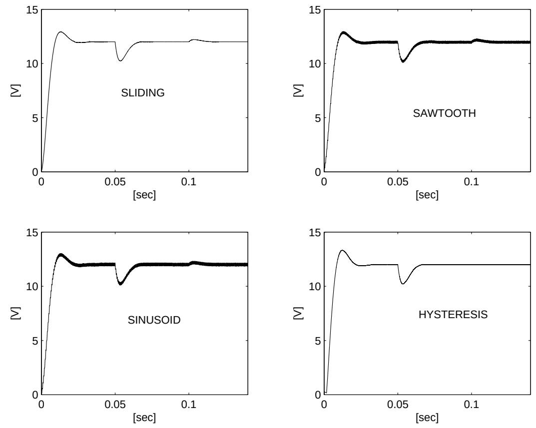

Fig. 5. Output voltage of the buck DC/DC converter: $R_1 = 0.1m\Omega$, $L = 69\mu H$, $C = 220\mu F$, $R_2 = 13\Omega$ (and a step change to 10Ω after $0.05s$), $E = 24V$ (and a step change to $32V$ after $0.1s$), $V_{ref} = 12V$, $k_1 = 0.5$, $k_2 = 10$.

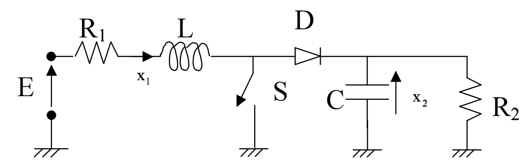

Fig. 6. Boost DC/DC converter.

output voltage and the reference voltage, say x_3, the model of the boost can be represented as

$$\dot{x}_1 = -\frac{R_1}{L}x_1 - \frac{1}{2L}x_2 + \frac{E}{L} - \frac{1}{2L}x_2 n(\widetilde{\sigma}) \quad (23)$$

$$\dot{x}_2 = \frac{1}{2C}x_1 - \frac{1}{R_2 C}x_2 - \frac{1}{2C}x_1 n(\widetilde{\sigma}) \quad (24)$$

$$\dot{x}_3 = x_2 - V_{ref} \quad (25)$$

$$\widetilde{\sigma} = -\frac{1}{L}\sqrt{\frac{L}{C}}x_1 - \frac{K_p}{E}(x_2 - V_{ref}) - \frac{K_i}{E}x_3 \quad (26)$$

Through the state variable transformation:

$$\eta_1 = Lx_1^2 - Cx_2^2 \quad (27)$$

$$\eta_2 = x_3 \quad (28)$$

$$\xi = x_1 \quad (29)$$

it is possible to put the converter model in the form (2). The system stability with the sliding surface (26) is shown in (Spiazzi *et al.*, 1997). Fig. 7 shows the output voltage of the boost converter under different adaptations of the sliding mode controller proposed in (Cortes and Alvarez, 2002). As it was expected the sinusoidal dither (amplitude $M_\delta = 2$ and $T = 100\mu s$) ensures a faster transient only when the state is far from the sliding manifold (the smoothed nonlinearity has a gain larger than that corresponding to the sawtooth dither). In order to obtain similar performances an hysteresis band (± 0.5) much smaller than the dither amplitude must be used.

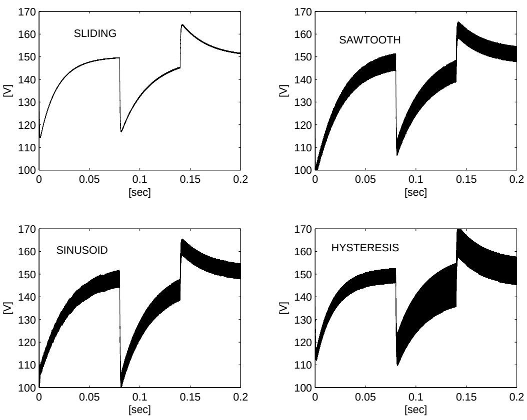

Fig. 7. Output voltage of the boost DC/DC converter: $R_1 = 10m\Omega$, $L = 0.36mH$, $C = 28.2\mu F$, $R_2 = 48\Omega$ (and a step change to 70Ω after $0.15s$), $E = 50V$ (and a step change to $30V$ after $0.06s$), $V_{ref} = 150V$, $k_p = 1$, $k_i = 100$.

6. CONCLUSIONS

In this paper it has been proved that dithering can be used as a boundary layer controller for sliding mode control systems. The boundary layer can be adapted by changing the dither shape and the natural switching operating mode of the actuator is preserved. The proposed approach has been checked through simulations by considering a DC/DC power electronic converter. Verification of the validity of the proposed approach with experiments is in progress.

ACKNOWLEDGMENT

This work was supported by EC within the SICONOS project (contract no. IST2001-37172).

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