SYSTEMATIC APPROACH TO THE SELECTION OF REDUCED MODELS. APPLICATION TO A PRACTICAL CASE

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Abstract: In the modelling of systems, it is common to obtain high order mathematical models. This can lead to problems in implementation, computing problems or problems in the design of the controller. It is thus preferable to obtain reduced models of the system. The selection of the most suitable reduced model, with real applications in some cases, should not be taken lightly, but rather an exhaustive study should be made of the suitability of the model, examining the main characteristics required in each case and particularly the controller design techniques to be used. This paper presents a methodological proposal for the selection of a reduced model which is applied to the case of a TF-120 high-speed craft model.

Keywords: Model reduction, model selection methodology, QFT robust control, high-speed craft

1. INTRODUCTION

In model reduction (of the plant and/or controller) the control problem (Skogestad and Postlethwaite, 1996) is: Given a high order stable linear model G(s) time-invariant, find a lower order $G_R(s)$ such that the norm-∞ of the difference $\| G(s) - G_R(s) \|_\infty$ is small.

Traditionally, the method used to reduce the order of the control system was the so-called Dominant Pole Approximation. The modern formulation of this idea was devised by Davison (1966) in a paper in which he described a model reduction technique, which consisted in differentiating between the dominant and the non-dominant modes of the original system in order to discard the latter.

Since then, more interest has been taken in finding different techniques for reducing models. Moore (1981) introduced the concept of balanced realizations. This was a great advance allowing the introduction of the so-called balanced truncation method for reducing an asymptotically stable linear system. Enns (1984) introduced the concept of balanced realization with frequency weighting. This method consisted in introducing frequencies with weights in the balanced truncation procedure. Another important advance in the area of model reduction came with the article published by Glover (1984), which described a new characterization of all solutions in state space for the problem of the optimal Hankel norm approach for linear multivariable continuous systems and determined an error bound for the frequency response.

Once a reduced order model is obtained for the system, it is necessary to check whether its behaviour fits the original model and to what extent. This paper presents a methodological approach to the selection of the most appropriate reduced model for the system control under study and its application to the selection of a reduced model for a TF-120 high-speed craft.

2. GENERAL METHOD FOR THE SELECTION OF A REDUCED MODEL

In a control system whose mathematical model is of a high order, there may arise problems of implementation, computing problems in the analysis of the behaviour of the system in response to different signals, or problems in the design of the controller. Thus, for the stages of analysis, design and implementation of the control system, it is a great advantage to be able to work with high order models with the greatest possible reduction.

The most widely used reduction models are: balanced truncation, balanced residualization and optimal Hankel norm approach.

The definition of the control problem posed in the reduction of a model suggests that the selection of the most suitable reduced model should be based on the norm-∞ of the error being sufficiently small. Moreover, for the Hankel norm approach, a reduction is considered good if the Hankel norm of the error is small ($\| G(s) - G_R(s) \|_H$).

The selection of the most suitable reduced model can be made by following the steps described below:

Step 1. In the reduction techniques mentioned above, an error bound is also defined for the reduced model obtained. In the first two cases, the bound consists in limiting the value of the norm-∞ of the error so that it is the same or less than twice the sum of the singular Hankel values (σ_i). In the third case, the Hankel norm of the error must be equal to the singular Hankel k+1 value, k being the order of the reduced model obtained. (See Table 1)

Table 1. Reduced model technique: error bound

	Error bound
Truncation and Residualization	$\|G(s) - G_R(s)\|_\infty \leq 2\sum_{i=k+1}^{n} \sigma_i$
Hankel norm approach	$\|G(s) - G_R(s)\|_H = \sigma_{k+1}$

Step 2. When the order of a system model is reduced, it is necessary to maintain all of the behaviour characteristics of the original system. A dilemma then arises between simplicity and accuracy in the model to be used. If the reduced model presents any behavioural difference with respect to the high order model, the importance of this variation must be assessed for each particular case. The simplest possible model which is usable and which retains the main dynamics of the original system must be found.

Hence, in order to select the most suitable model, its behaviour must be analysed and compared with the original unreduced model and with the real system if possible. There are several characteristics of the systems which can be used for this purpose and which are proposed by this paper to form part of the model selection methodology.

It is necessary first to verify and calculate the parameters shown in Table 2.

Table 2. Prior data

			Original System	*Reduced System*
Transfer function			G(s)	$G_R(s)$
Time response			c(t)	$c_R(t)$
Frequency response	*Module*		m(ω)	$m_R(\omega)$
	Phase		f(ω)	$f_R(\omega)$
Energy spectrum	*Spectral momentum*		M_A	M_{AR}
System norm	*Norm H_2*		$\|G(s)\|_2$	$\|G_R(s)\|_2$
	Norm H_∞		$\|G(s)\|_\infty$	$\|G_R(s)\|_\infty$
	Norm Hankel		$\|G(s)\|_H$	$\|G_R(s)\|_H$
Singular values (if MIMO)	*Function S*	*Max*	$\bar{\sigma}_S(\omega)$	$\bar{\sigma}_{SR}(\omega)$
		Min	$\underline{\sigma}_S(\omega)$	$\underline{\sigma}_{SR}(\omega)$
	Function T	*Max*	$\bar{\sigma}_T(\omega)$	$\bar{\sigma}_{TR}(\omega)$
		Min	$\underline{\sigma}_T(\omega)$	$\underline{\sigma}_{TR}(\omega)$
	Function S + T	*Max*	$\bar{\sigma}_{ST}(\omega)$	$\bar{\sigma}_{STR}(\omega)$
		Min	$\underline{\sigma}_{ST}(\omega)$	$\underline{\sigma}_{STR}(\omega)$

Step 3. One way of analyzing the behaviour of a control system is to study the time response characteristics (impulse, step), or the frequency response characteristics (Bode, Nichols). Since the reduced system must contain the main dynamics of the original system, the time and frequency responses of both systems should be similar as possible. Thus, a good indicator of the appropriacy of a model could be that the difference between the responses of the two systems is as small as possible. This can be obtained either by simple observation of the graphic representation of the response of both systems or by calculating analytically the absolute and/or relative error. Since the difference depends on the instant of time (time response) or the frequency (frequency response), the error can be calculated as a geometrical average of the difference (norm-2).

If the system is a multivariable control system, it is of great interest to know its singular values (main gains), and particularly its maximum and minimum values (for each frequency) as these limit the area in which the system gain will always be found. The reduced model must have maximum and minimum values very similar to those of the original system. It is possible then to calculate, as a measure of the appropriacy of the reduced model, the error in the singular maximum and minimum values of both models in graphic or numerical form.

A system norm also provides important information: norm-2 gives an idea as to whether the system response for a bound input is also bound, and indicates the degree of amplification of this input signal as it passes through the system, norm-∞ indicates the maximum of the input-output relation of the system and the Hankel norm indicates the maximum singular Hankel value. Depending on the type of behaviour to be prioritised in the system, the selection of the reduced model must be conditioned by whether the values of the norms H_2, H_∞ and/or Hankel of both systems are as similar as possible. These conditions are more restrictive than the definition of the reduction problem itself because:

$$\| G(s) \|_p - \| G_R(s) \|_p \leq \| G(s) - G_R(s) \|_p \qquad (1)$$

Another widely used measure in signal analysis is the energy spectrum of the signal and the spectral momentum, which provide statistical information on the behaviour of the signal (Ochi, 1998). The signal considered for this study could be, for example, the frequency response of both systems (original and reduced). The energy spectrum and the spectral momentum should be similar.

Table 3 (at the end of this paper) shows all of the methods proposed for the selection of the most suitable reduced model. The selection must be conditioned to the highest priority characteristics in each case. For example, if it is important that a specific signal should not exceed a certain maximum, it is appropriate to prioritise that the error in the norm-∞ is as small as possible, rather than prioritising the error in norm-2.

Step 4. Another validation method for testing whether a signal fits certain criteria is to use methods which minimise the norm-2 of the error: Akaike information criteria (AIC) and final prediction error (FPE). Both values are a function of the number of parameters of the model p, the data register length N and a loss function V. The V function will be the measurement to be considered for the selection of the model, such as the norm-2 of the error in the frequency response of the reduced system with respect to the original system response. The most suitable model is determined as that for which the AIC and FPE values are the smallest. Table 4 includes the formulas for calculating both indices.

Table 4. Model selection: error indices

Error indices
$AIC = ln\left[V\left(1+\frac{2p}{N}\right)\right]$
$FPE = V\frac{1+\frac{p}{N}}{1-\frac{p}{N}}$

3. SELECTION METHOD USING CONTROL SYSTEM DESIGN TECHNIQUE

As well as the model selection methods described up to now, an analysis can also be made of the main characteristic points of the control design technique to be used.

For example, in the robust design control technique, QFT (Quantitative Feedback Theory), developed by Horowitz (1992) for systems with uncertainty, the controller tuning is performed by adjusting the open loop function:

$$L(j\omega) = G(j\omega)P_0(j\omega) \qquad (2)$$

where *G(jω)* is the transfer function of the controller and *P_0(jω)* is the transfer function of the nominal plant.

This function must be such that it does no violate the limits of certain forbidden regions (bounds), which are determined by the uncertainty of the model and the design specifications. It may be of interest then to verify the form and value of the restriction curves of the reduced model and to compare them with those of the original model.

4. APPLICATION OF SELECTION METHODOLOGY TO THE MODEL OF THE TURBO FERRY TF-120

In the project CICYT TAP97-0607-C03, the mathematical model for Turbo Ferry TF-120 performance for ssn 4 (sea state number) and speeds of 20, 30 and 40 knots was obtained. The objective of the control design system is to reduce the effect of the waves on the vertical movement of the craft so that the motion sickness incidence (MSI) decreases.

The vertical dynamics of the craft is composed of various continuous linear SISO models, which were identified from the program PRECAL simulated data (De la Cruz, et al., 2004), at a speed of 40 knots. A linear model was also developed to describe the behaviour of the active actuators (one T-foil and two flaps) intended to reduce the vertical accelerations and movement, because the actuators counteract the effect of the waves. This also reduces the MSI, which allows a higher speed to be maintained.

Using the rules of block diagram algebra, it has been verified that the process model is as shown in Figure 1.

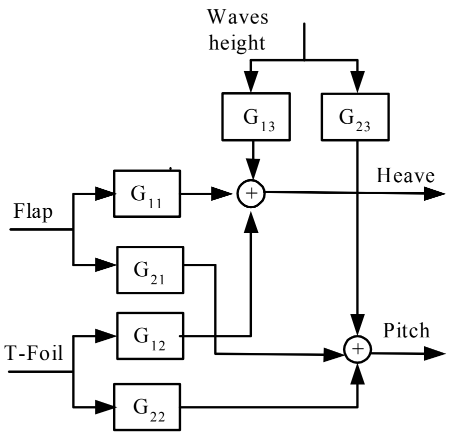

Fig. 1. Linearised process model

The transfer functions G_{11}, G_{12}, G_{21}, G_{22} of the multivariable 2x2 system of the craft are in the order of 17. These values make it desirable to look for some lower order model with a dynamic performance close to that of the original model to facilitate controller design.

Applying balanced truncation techniques, balanced residualization and optimal Hankel norm approach, several reduced models have been obtained (Rueda, 2004, Rueda et al., 2004). Among these, two options are presented here (model 1 with transfer functions of order 4^{th}, and model with transfer functions of order 5^{th}, both obtained by residualization, for ssn 4 and 40 knots).

Model 1 (order 4^{th}):

$$G11=\frac{28.95*10^{-4}\left(s^2+0.4323s+1.36\right)\left(s^2-2.143s+4.976\right)}{\left(s^2+0.4939s+1.5\right)\left(s^2+0.7928s+2.895\right)}$$

$$G12=\frac{-10.406*10^{-4}\left(s^2+0.4981s+1.78\right)\left(s^2-3.139s+10.18\right)}{\left(s^2+0.4939s+1.5\right)\left(s^2+0.7928s+2.895\right)}$$

$$G21=\frac{-37.66*10^{-4}\left(s-1.164\right)\left(s+0.267\right)\left(s^2+1.155s+3.201\right)}{\left(s^2+0.4939s+1.5\right)\left(s^2+0.7928s+2.895\right)}$$

$$G22=\frac{-68.787*10^{-4}\left(s-1.113\right)\left(s+0.2294\right)\left(s^2+0.6905s+2.877\right)}{\left(s^2+0.4939s+1.5\right)\left(s^2+0.7928s+2.895\right)}$$

Model 2 (order 5^{th}):

$$G11=\frac{23.75*10^{-4}\left(s+0.4213\right)\left(s^2+0.9915s+2.363\right)\left(s^2-2.315s+5.691\right)}{\left(s+0.4182\right)\left(s^2+0.9545s+2.465\right)\left(s^2+0.7915s+2.895\right)}$$

$$G12=\frac{-19.073*10^{-4}\left(s+0.414\right)\left(s^2+0.841s+2.57\right)\left(s^2-2.503s+6.382\right)}{\left(s+0.4182\right)\left(s^2+0.9545s+2.465\right)\left(s^2+0.7915s+2.895\right)}$$

$$G21=\frac{17.36*10^{-4}\left(s+0.0413\right)\left(s^2+0.8921s+2.959\right)\left(s^2-4.908s+12.14\right)}{\left(s+0.4182\right)\left(s^2+0.9545s+2.465\right)\left(s^2+0.7915s+2.895\right)}$$

$$G22=\frac{23.162*10^{-4}\left(s+0.04102\right)\left(s^2+0.7589s+2.892\right)\left(s^2-5.171s+12.63\right)}{\left(s+0.4182\right)\left(s^2+0.9545s+2.465\right)\left(s^2+0.7915s+2.895\right)}$$

In order to analyse their appropriacy, all of the values have been calculated using the methodology summarised in Tables 3 and 4. Table 5 (bottom in the next page) shows some of these. The aim of the controller design is to reduce the effect of the waves on the movement of the craft, so it is especially interesting to verify whether the frequency response of the original model and the reduced model are similar, since the system input signal depends on the frequency. Moreover, if the vertical movement of the craft is to be reduced in a measured way, it is also important to observe the differences in the H_2 norm. The error is verified in the singular maximum and minimum values because it is a multivariable system. Finally, the Akaike information criterion is included for norm-2 of the frequency response of the system.

In view of the data obtained, either of the two models could be suitable, since the error is small in both cases. Model 1 could be chosen as it is of a lower order, which facilitates its implementation and simulation, and reduces the complexity of the control system design.

However, the appropriacy of both models in the feedback system must be verified if this varies according to the control design technique to be used, in this case QFT. The aim of the design is to reduce the system sensitivity to waves, to minimize the control effort of ship actuators and to obtain robust stability. In both model 1 and model 2 there is uncertainty in the system parameters, whose values depend on whether the ship's speed is 20, 30 or 40 knots. The nominal plant chosen for the design was the one corresponding to 40 knots. Bearing in mind that MSI has a maximum for frequencies close to 1 rad/sec (Lloyd, 1989) the following set of frequencies have been used for the design: $\Omega=\{0.7,0.85,1,1.2,1.5,2,2.5,3,7\}\,rad/\sec$.

Figures 2, 3 and 4 show the bounds, which are not to be violated by the unreduced system, and those for both reduced models. It can be observed that although model 1 looks adequate, the regions limited by the bounds differ more than those of the unreduced model than those of model 2. Thus, if the QFT design technique is to be used, it is better to select model 2, because the controller adjusted using this model is a priori a good design also for the original system, as has been shown in Velasco, et al. (2004).

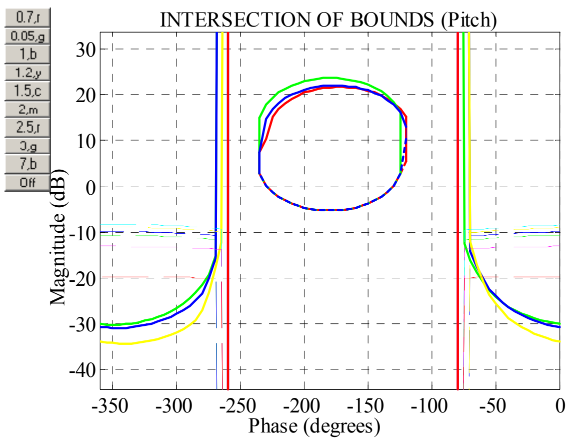

Fig 2. Bounds of unreduced model

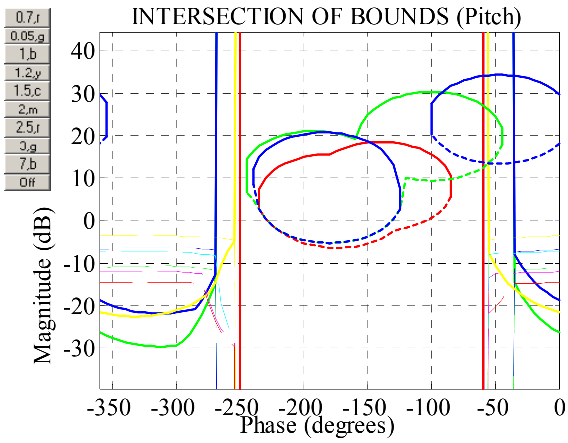

Fig 3. Bounds of model 1 (order 4^{th})

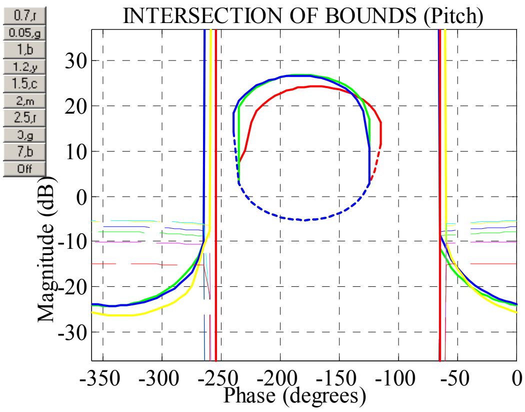

Fig 4. Bounds of model 2 (order 5th)

5. CONCLUSIONS

This paper has described a systematic review of the general methods for the selection of a reduced model with behaviour sufficiently similar to that of the original system.

Similarly, it is suggested that the appropriacy of the reduced model in the closed loop feedback system should be verified according to the design technique to be used.

This methodology is applied systematically for a study case, the model reduction for a TF-120 high-speed craft. A QFT controller is sought to minimise the effects of the waves on the craft in order to reduce the vertical acceleration and the sickness index of passengers and crew.

6. ACKNOWLEDGEMENTS

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Table 5. Sample of reduced model selection

			MODEL 1 (order 4th)				*MODEL 2 (order 5th)*			
			Mean error				*Mean error*			
Frequency response (ω = [0.9, 1.5] rad/sec)	*Distance (Nichols)*		*G11*	2.6476	*G21*	5.0098	*G11*	3.0710	*G21*	1.0301
			G12	5.5812	*G22*	5.0800	*G12*	2.7750	*G22*	0.8777
			Relative Error				*Relative Error*			
Singular values (ω = [0.1, 10] rad/sec)	*Function S*	*Higher*	0.0000				0.0000			
		Lower	0.0000				0.0000			
System Norm	*H_2*		9.2993%				14.5140%			
			Absolute Error				*Absolute Error*			
Error Indices	*AIC*	*Norm-2 of error (Nichols)*	*G11*	1.9545	*G21*	2.5922	*G11*	2.2206	*G21*	1.1282
			G12	2.7002	*G22*	2.6061	*G12*	2.1192	*G22*	0.9682

Table 3. Model Selection Methodology

Selection methodology			*Absolute Error*	*Geometrical average of error*	*Relative error*
Time response			$e(t)=c(t)-c_R(t)$	$e_{abs}(t)=\left\Vert e(t)\right\Vert_2$	$e_{rel}(t)=\frac{\left\Vert e(t)\right\Vert_2}{\left\Vert y(t)\right\Vert_2}$
Frequency response	*Module*		$em(\omega)=m(\omega)-m_R(\omega)$	$em_{abs}(\omega)=\left\Vert em(\omega)\right\Vert_2$	$em_{rel}(\omega)=\frac{\left\Vert em(\omega)\right\Vert_2}{\left\Vert m(\omega)\right\Vert_2}$
	Phase		$ef(\omega)=f(\omega)-f_R(\omega)$	$ef_{abs}(\omega)=\left\Vert ef(\omega)\right\Vert_2$	$ef_{rel}(\omega)=\frac{\left\Vert ef(\omega)\right\Vert_2}{\left\Vert f(\omega)\right\Vert_2}$
	Distance (Nichols)		$ed(\omega)=\sqrt{em^2(\omega)+ef^2(\omega)}$	$ed_{abs}(\omega)=\left\Vert ed(\omega)\right\Vert_2$	
Singular values (if MIMO)	*Function S*	*Max*	$e_{\bar{\sigma}(S)}(\omega)=\bar{\sigma}_S(\omega)-\bar{\sigma}_{S_R}(\omega)$	$e_{\bar{\sigma}(S)\ abs}(\omega)=\left\Vert e_{\bar{\sigma}(S)}(\omega)\right\Vert_2$	$e_{\bar{\sigma}(S)\ rel}(\omega)=\frac{\left\Vert e_{\bar{\sigma}(S)}(\omega)\right\Vert_2}{\left\Vert \bar{\sigma}_S(\omega)\right\Vert_2}$
		Min	$e_{\underline{\sigma}(S)}(\omega)=\underline{\sigma}_S(\omega)-\underline{\sigma}_{S_R}(\omega)$	$e_{\underline{\sigma}(S)\ abs}(\omega)=\left\Vert e_{\underline{\sigma}(S)}(\omega)\right\Vert_2$	$e_{\underline{\sigma}(S)\ rel}(\omega)=\frac{\left\Vert e_{\underline{\sigma}(S)}(\omega)\right\Vert_2}{\left\Vert \underline{\sigma}_S(\omega)\right\Vert_2}$
	Function T	*Max*	$e_{\bar{\sigma}(T)}(\omega)=\bar{\sigma}_T(\omega)-\bar{\sigma}_{T_R}(\omega)$	$e_{\bar{\sigma}(T)\ abs}(\omega)=\left\Vert e_{\bar{\sigma}(T)}(\omega)\right\Vert_2$	$e_{\bar{\sigma}(T)\ rel}(\omega)=\frac{\left\Vert e_{\bar{\sigma}(T)}(\omega)\right\Vert_2}{\left\Vert \bar{\sigma}_T(\omega)\right\Vert_2}$
		Min	$e_{\underline{\sigma}(T)}(\omega)=\underline{\sigma}_T(\omega)-\underline{\sigma}_{T_R}(\omega)$	$e_{\underline{\sigma}(T)\ abs}(\omega)=\left\Vert e_{\underline{\sigma}(T)}(\omega)\right\Vert_2$	$e_{\underline{\sigma}(T)\ rel}(\omega)=\frac{\left\Vert e_{\underline{\sigma}(T)}(\omega)\right\Vert_2}{\left\Vert \underline{\sigma}_T(\omega)\right\Vert_2}$
	Function S + T	*Max*	$e_{\bar{\sigma}(ST)}(\omega)=\bar{\sigma}_{ST}(\omega)-\bar{\sigma}_{ST_R}(\omega)$	$e_{\bar{\sigma}(ST)\ abs}(\omega)=\left\Vert e_{\bar{\sigma}(ST)}(\omega)\right\Vert_2$	$e_{\bar{\sigma}(ST)\ rel}(\omega)=\frac{\left\Vert e_{\bar{\sigma}(ST)}(\omega)\right\Vert_2}{\left\Vert \bar{\sigma}_{ST}(\omega)\right\Vert_2}$
		Min	$e_{\underline{\sigma}(ST)}(\omega)=\underline{\sigma}_{ST}(\omega)-\underline{\sigma}_{ST_R}(\omega)$	$e_{\underline{\sigma}(ST)\ abs}(\omega)=\left\Vert e_{\underline{\sigma}(ST)}(\omega)\right\Vert_2$	$e_{\underline{\sigma}(ST)\ rel}(\omega)=\frac{\left\Vert e_{\underline{\sigma}(ST)}(\omega)\right\Vert_2}{\left\Vert \underline{\sigma}_{ST}(\omega)\right\Vert_2}$
System norm	*Norm H_2*		$e_{n(H_2)}=\left\Vert G(s)\right\Vert_2-\left\Vert G_R(s)\right\Vert_2$		$e_{rel\ n(H_2)}=\frac{e_{n(H_2)}}{\left\Vert G(s)\right\Vert_2}$
	Norm H_∞		$e_{n(H_\infty)}=\left\Vert G(s)\right\Vert_\infty-\left\Vert G_R(s)\right\Vert_\infty$		$e_{rel\ n(H_\infty)}=\frac{e_{n(H_\infty)}}{\left\Vert G(s)\right\Vert_\infty}$
	Norm Hankel		$e_{n(H)}=\left\Vert G(s)\right\Vert_H-\left\Vert G_R(s)\right\Vert_H$		$e_{rel\ n(H)}=\frac{e_{n(H)}}{\left\Vert G(s)\right\Vert_H}$
Energy spectrum	*Spectral momentum*		$e_{M_A}=\left\vert M_A-M_{AR}\right\vert$		$e_{rel\ M_A}=\frac{e_{M_A}}{M_A}$