EXACT NONLINEAR MODELLING USING SYMBOLIC LINEAR FRACTIONAL TRANSFORMATIONS

Andrés Marcos, Declan G. Bates, Ian Postlethwaite

Dept. of Engineering, University of Leicester,
University Road, Leicester, LE1 7RH, U.K.,
{*ame12, dgb3, ixp*} *@le.ac.uk*

Abstract: In this paper, a nonlinear modelling framework is presented that combines symbolic modelling and linear fractional transformation (LFT) techniques to obtain a nonlinear symbolic LFT representation. This modelling approach presents three clear advantages: (i) it provides a unifying framework for the different models that stem from the same nonlinear system, (ii) it allows for a highly structured representation of the various nonlinearities present in the system, and (iii) it is closely connected with other analysis and synthesis frameworks such as linear and linear parameter varying robust control and nonlinear systems analysis via describing functions.

Keywords: nonlinear system modelling, linear fractional transformation

1. INTRODUCTION

The development of mathematical models of industrial systems is a vital but increasingly time consuming and expensive task. Often, completely different model representations are generated for the same system depending on the intended use of the model, i.e. for simulation, analysis or design. The problem of ensuring consistency, continuity and connectedness across a range of different models is therefore critical in order to efficiently validate analysis results and controller designs, arising from the use of simpler models, in full nonlinear simulation.

Over the last twenty years, a paradigm shift in the modelling of dynamic systems has occurred with the introduction of modern robust control theory and its associated modelling framework, the linear fractional transformation (Packard, A. and Doyle, J., 1993). Commonly, LFTs are used to represent a nonlinear system as an approximated linear system together with a structured matrix containing the uncertainty present in the system. The structured singular value theory, and its associated software and toolboxes, provides further incentive for the use of LFT models as it allows the designer to analyze (and design for) robust stability and performance of uncertain linear systems in this form (Packard, A. and Doyle, J., 1993; Balas, G.J. *et al.*, 1998). This restriction to linear systems, however, means that although LFT models easily allow for robust design and analysis, it is still necessary to use high-fidelity nonlinear simulation models for the final validation of the closed-loop prior to real testing and implementation.

More recently, with the development of powerful real-time processors and more mature mathematical control theories, many industries are making significant efforts to develop and apply nonlinear synthesis and analysis techniques - see for example, the work reported in (Fielding, C. *et al.*, 2002) for recent progress in the aerospace industry. One approach to this problem which has been very successful in practice is to extend traditional linear design and analysis methods to address nonlinear problems. This is the basis of modern synthesis and analysis techniques such as gain scheduling (Leith, D.J. and Leithead, W.E., 2000), linear parameter varying (LPV) control (Becker, G. and Packard, A, 1994), and integral quadratic constraints (IQC) (Megretski, A. and Rantzer, A., 1995) amongst others. It is noted that many of these techniques work, in one form or another, with LFT models.

The main contribution of this paper is a nonlinear modelling framework that combines symbolic modelling and linear fractional transformation techniques to obtain a nonlinear symbolic LFT representation. The proposed nonlinear symbolic LFT approach has three main advantages. Firstly, it provides a unifying framework for the different models that stem from the same nonlinear system, and thus improves consistency, continuity and connectedness between these various models. Secondly, it results in a highly structured representation of the different nonlinearities present in the system, thus facilitating their study and ameliorating the effect that inappropriate simplifications and approximations have on the overall modelling process. Thirdly, it is easily connected with other analysis and synthesis frameworks. Indeed, it will be shown that the proposed nonlinear symbolic LFT framework includes as special cases the standard linear, LPV and nonlinear (via describing functions) design and analysis frameworks. Finally, it is noted that the nonlinear symbolic LFT itself is directly amenable to nonlinear extensions of μ-analysis and other robustness analysis techniques (Doyle, J. and Packard, A., 1987; Packard, A. and Doyle, J., 1993).

2. SYMBOLIC NONLINEAR LFT MODELLING FRAMEWORK

In this section, the class of nonlinear systems considered and the proposed symbolic LFT modelling methodology are presented.

2.1 Nonlinear System Class

The class of nonlinear systems considered is defined by the following ordinary differential equations (ODE) where the states x, outputs y and inputs u depend on time but the dependency is removed for ease of presentation:

$$\dot{x} = f(x,u) = f_1(x)\,x + f_2(x)\,u + f_3(x) \qquad (1)$$

$$y = g(x,u) = g_1(x)\,x + g_2(x)\,u + g_3(x) \qquad (2)$$

The first-order derivative condition for the states is without loss of generality as higher-order derivatives can be substituted by new state variable definitions that transform the higher-order system into a first-order ODE. It is assumed that the nonlinear functions $f_i(x), g_i(x)$ are a polynomial mix of analytic expressions and tabular data that arise from first-principles modelling (e.g. Newton laws of motion, mass-moment conservation, etc).

The main structural restriction for this class of systems is the linear dependency of the nonlinear functions on the input vector u, e.g. $f(x,u)$ is a function of $f_1(x)x,\ f_2(x)u$ and $f_3(x)$. This assumption is indeed quite general and standard for mechanical systems, nevertheless an extension to systems with nonlinear dependency on the inputs is also given in Section 4.3. The inclusion of the functions $f_3(x), g_3(x)$ which represent those terms (nonlinear, time-varying or constant) that cannot be represented as linear in the states, significantly expands the set of nonlinear systems that can be considered. For example, these extra functions often arise in aerospace systems, where their consideration is critical (Stevens, B. and Lewis, F., 1992).

The standard representation of a system for modern control design and analysis is based on the state-space 2×2 block format:

$$\begin{bmatrix}\dot{x}\\ y\end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]\begin{bmatrix}x\\ u\end{bmatrix} \qquad (3)$$

The outputs are the first-order derivatives of the states $\dot{x}$ and the sensed signals y; the inputs are the state x and the control input u vectors. The system (A,B,C,D) is only restricted to be affine on the input vectors $[x\ \ u]^\top$.

In order to write equations (1) and (2), in the standard representation given by (3), a fictitious constant signal u_f is introduced:

$$\dot{x} = f(x,u) = f_1(x)\,x + f_2(x)\,u + f_3(x)\,u_f \qquad (4)$$

$$y = g(x,u) = g_1(x)\,x + g_2(x)\,u + g_3(x)\,u_f \qquad (5)$$

$$u_f = 1\ \ \forall\ t \qquad (6)$$

The nonlinear state-space system in 2×2 block format is then:

$$\begin{bmatrix}\dot{x}\\ y\end{bmatrix} = \left[\begin{array}{c|cc} f_1(x) & f_2(x) & f_3(x) \\ \hline g_1(x) & g_2(x) & g_3(x)\end{array}\right]\begin{bmatrix}x\\ u\\ u_f\end{bmatrix} \qquad (7)$$

At this stage, the modelling engineer will try to simplify and/or linearize the nonlinear model for ease of manipulation while maintaining a minimum required level of fidelity with respect to the true nonlinear system. Managing the trade-off between these conflicting objectives is usually a very costly and lengthy task based on the perceived requirements on the model arising from its final use, i.e. analysis and/or design based on a specific technique.

2.2 Nonlinear Symbolic LFT

In order to better manage the trade-off between modelling simplicity and system fidelity, the proposed methodology uses a linear fractional transformation (LFT) framework.

The proposed method declares as symbolic parameters ρ_k all those terms that are nonlinear or time-varying in nature (e.g. trigonometric relationships, tabular data, exponential terms) as well as those physical parameters that can vary with time and/or operational condition (e.g. for an aircraft these might be mass, center of gravity, etc). Indeed, the guiding principle proposed at this stage of model development is *to select everything that is not a known constant c_j as a symbolic parameter ρ_k (including the number of repetitions $n1, n2, \ldots, nk$)*:

$$f_i(x) = f_i(\rho_1^{n1}, \rho_2^{n2}, \ldots, \rho_k^{nk}, c_1, c_2, \ldots, c_j) \qquad (8)$$

After the symbolic parameter declaration, the resulting symbolic nonlinear equations can be arranged in the nonlinear state-space system format of equation (7). Due to the symbolic declaration, the state-space system becomes a multivariate symbolic polynomial matrix.

Recently, algorithmic implementations have appeared (Magni, J.F., 2004; Marcos, A. *et al.*, 2005), of the structured-tree decomposition (Cockburn, J.C. and Morton, B.G., 1997) and the Horner-tree decomposition (Marcos, A. *et al.*, 2005) respectively. These order-reduction techniques transform a symbolic multivariate polynomial matrix into a close-to-minimal equivalent system by performing matrix factorizations and sum decompositions (minimal representations are in general very difficult to obtain except for some simple cases). This lower-order system can be represented as an LFT which separates the constant terms from the symbolic parameters in two different matrices: a varying matrix $\Delta(\rho)$, containing the possibly nonlinear, time-varying and uncertain symbolic parameters, and a constant matrix M, containing the symbolic constants and the linear feedback interconnection information, see Figure 1.

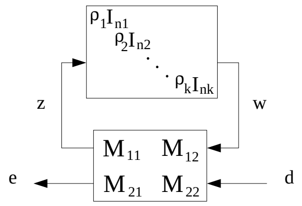

Fig. 1. Nonlinear symbolic LFT - $\mathcal{F}_u(M,\Delta(\rho))$.

In the course of this symbolic LFT process, the matrix $\Delta(\rho)$ acquires a diagonal structure, following the well-known fact that uncertainty at component level becomes structured uncertainty at system level (Packard, A. and Doyle, J., 1993), 'uncertainty' signifying in this case symbolic parameterization. The parameters are repeated $n1,n2,\ldots,nk$ times depending on their "presence" degree, position (numerator vs denominator) and operation performed (sum or product) in the original nonlinear functions. Note, that the approach results in an LFT representation which is identical to the original nonlinear system given by equations (1-2).

3. FURTHER MANIPULATIONS OF THE NONLINEAR SYMBOLIC LFT

An advantage of the proposed modelling approach is that it results in a structured representation of the nonlinearities, which facilitates their analysis and ameliorates the effect that inappropriate simplifications and approximations have on the overall modelling process. This advantage arises due to the LFT nature of the framework together with the diagonal structure of the symbolic nonlinearities in $\Delta(\rho)$.

The LFT nature of the framework presents a direct way to substitute symbolic nonlinear parameters by their approximations, since the latter can also generally be represented in LFT form, and the interconnection of LFTs yield another LFT. On the other hand, the diagonal representation of the nonlinearities provides a clearer understanding of the effects each nonlinearity has on the performance and robustness of the nonlinear system. Furthermore, the combination of the diagonal representation of the nonlinearities and the LFT nature of the approach provides modelling modularity which reduces the effect modelling mistakes and misjudgements can have on the modelling process. These three characteristics (diagonality, LFT nature and modularity) can be exploited to further manipulate the symbolic nonlinear LFT in order to simplify, reduce and/or approximate the nonlinear, time-varying and uncertain terms represented by the symbolic parameters.

In order to exploit the above advantages, four additional modelling stages in the nonlinear symbolic LFT approach are proposed: simplification, reduction, approximation, and uncertainty characterization. At each of these stages, new constant M_i and varying $\Delta(\rho)_i$ matrices (with $i=1,2,\ldots p$) are obtained, and the number of independent symbolic parameters is reduced. Note, however, that the resulting LFT model is still a symbolic *nonlinear* model (specifically, note that 'approximation' in this context does not correspond to 'linearization').

1) Simplifying assumptions can be based on knowledge about the physical system (e.g. mass in an aircraft can be considered constant for a relatively short period of time) and established approximations (e.g. small angle assumption). The diagonal structure of $\Delta(\rho)$ greatly facilitates this task by allowing direct and independent (from the other parameters) operations on the LFT. A great advantage of LFT manipulation is that algebraic operations such as series and parallel connections preserve the structure, see references (Packard, A. and Doyle, J., 1993). Furthermore, implementation of these operations has recently been simplified by the appearance of an LFT MATLAB toolbox (Magni, J.F., 2004).

2) Model reduction techniques such as singular value decomposition, balanced truncation and sensitivity analysis can also be used to identify and remove negligible symbolic parameters. As the physical nature of the parameters is kept intact in the previous stages, it is more meaningful to reduce the model at this stage. The removal of a symbolic parameter involves only a simple row and column cancelation on the constant matrix M and a corresponding reduction in the order of the varying matrix $\Delta(\rho)$ (equivalent to the number of repetitions for that parameter).

3) Approximation techniques can be used next in order to further simplify the model. Typically, most of the nonlinear, varying and uncertain parameters depend on a subset of the system parameters. For example, in the case of an aircraft, almost all of the nonlinear tabular data used to represent the aerodynamic stability derivatives depends only on a handful of the system states (e.g. angle of attack, sideslip, altitude, Mach number). This primogenial dependency can be used to approximate the symbolic parameters and reduce the total number of independent parameters (although it would typically lead to a larger dimension of the resulting varying $\Delta(\rho)$). Some of the approximation techniques that can be used include: polynomial and surface fitting (Spillman, M. *et al.*, 1996), trend and band algorithms (Mannchen, T. *et al.*, 2002), and rational approximations (Hansson, J., 2003). It is noted again, that these approximation techniques will still, if desired, maintain the nonlinear nature of the LFT model (e.g polynomial fits based on parameters with high orders).

Once a parameter in $\Delta(\rho)$ is approximated by a new symbolic expression, this new expression can be directly substituted into the nonlinear symbolic LFT. More importantly, the approximation can also be expressed in LFT form and then substituted in the original LFT model to obtain a new LFT where all the nonlinearities and varying parameters are in $\Delta(\rho)$ and the constants and linear terms in M by means of the following result:

Lemma 1. Consider a lower LFT $y=\mathcal{F}_l(M,\Delta(\rho))u$ as shown in Figure 2 (a). Assume $w_1=\Delta_1(\rho)z_1$ can be represented as another LFT, $\Delta_1(\rho)=\mathcal{F}_l(M^{\Delta_1},\bar{\Delta}_1)$ as shown in Figure 2 (b).

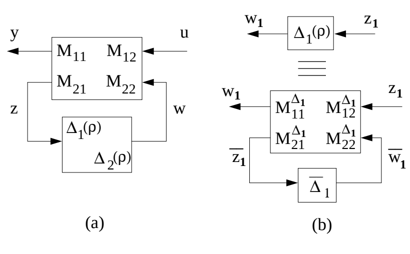

Fig. 2. Nested LFT: initial lower LFTs.

The *nested substitution* of the lower LFT corresponding to $\Delta_1(\rho)$ into $\mathcal{F}_l(M,\Delta(\rho))$ yields another lower

LFT $\mathcal{F}_l(\bar{M},\bar{\Delta})$ with:

$$\bar{\Delta} = \begin{bmatrix} \bar{\Delta}_1 & 0 \\ 0 & \Delta_2(\rho) \end{bmatrix} \tag{9}$$

$$\bar{M} = \begin{bmatrix} \mathcal{F}_l(M,\bar{M}^{\Delta}_{11}) & M_{12}(I-\bar{M}^{\Delta}_{11}M_{22})^{-1}\bar{M}^{\Delta}_{12} \\ \bar{M}^{\Delta}_{21}(I-M_{22}\bar{M}^{\Delta}_{11})^{-1}M_{21} & \mathcal{F}_u(\bar{M}^{\Delta},M_{22}) \end{bmatrix} \tag{10}$$

$$\bar{M}^{\Delta} = \begin{bmatrix} \bar{M}^{\Delta}_{11} & \bar{M}^{\Delta}_{12} \\ \bar{M}^{\Delta}_{21} & \bar{M}^{\Delta}_{22} \end{bmatrix} = \begin{bmatrix} 0_{dim(w)\times dim(z)} & I_{dim(w)\times dim(\bar{w}_1+w_2)} \\ I_{dim(\bar{z}_1+z_2)\times dim(z)} & 0_{dim(\bar{z}_1+z_2)\times dim(\bar{w}_1+w_2)} \end{bmatrix} \tag{11}$$

where $\bar{M}^{\Delta}$ is composed of zero and identity matrices with elements $\bar{M}^{\Delta}_{11}(ii,ii) = M^{\Delta_1}_{11}$, $\bar{M}^{\Delta}_{12}(ii,ii) = M^{\Delta_1}_{12}$, $\bar{M}^{\Delta}_{21}(ii,ii) = M^{\Delta_1}_{21}$, $\bar{M}^{\Delta}_{22}(ii,ii) = M^{\Delta_1}_{22}$. The index (ii,ii) is given by the position of $\Delta_1(\rho)$ in $\Delta(\rho)$. ⋈

Clearly, a similar result to the above can also be derived for upper LFT representations. It is important to note that the order of the new LFT is equal to the sum of the orders corresponding to $\bar{\Delta}_1$ and Δ_2. The nested substitution, see Figure 3, can be automated, and due to the diagonal structure of the original nonlinear LFT, updated or corrected approximations can be implemented without affecting the other approximations (thus reducing the impact, in terms of time and effort, an inappropriate approximation would have in the overall modelling process).

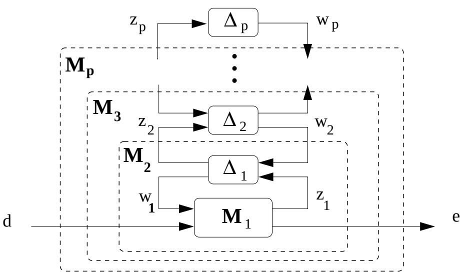

Fig. 3. Nested linear fractional transformations.

4) Uncertainty characterization. As a result of the approximations and/or simplifications described above, it might be desirable to include additional structured or unstructured uncertainty to represent the errors introduced in these stages. The most common structured uncertainty representation for parametric models is the so-called additive uncertainty, i.e. $\rho = \rho + \delta$ where the uncertain parameter δ provides a bound on the maximum expected level of uncertainty. Such δs can also be declared as symbolic parameters under this modelling framework. The inclusion of these additive uncertain terms can be carried out initially from equations (7-8), or more appropriately, after the approximation stage by using the nested LFT substitution. Note that the addition of uncertain terms and the subsequent re-arrangement of the M and $\Delta(\rho,\delta)$ matrices is direct. Other structured uncertainty descriptions (multiplicative, divisive) are also possible as they enter the system in polynomial form. Unstructured uncertainty, which is typically used to characterized neglected high-order dynamics, could also be included, for example *in lieu* of some of the nonlinear terms in equation (7), e.g. $f_3(x)$ could be covered by one unstructured $|\Delta_3| < \alpha$.

4. CONNECTIONS WITH OTHER MODELLING FRAMEWORKS FOR ANALYSIS AND DESIGN

One important advantage of the proposed modelling approach is that it is easily connected with other analysis and synthesis frameworks. These connections are fundamental for the usefulness of the proposed approach, as they allow the use of established design and analysis techniques on models stemming from the original nonlinear symbolic LFT. As these models are particular cases of the nonlinear symbolic LFT model, which exactly represents the true nonlinear system, there is also greater connectedness, continuity and consistency between the different models. Due to space restrictions, connections with standard linear theory are discussed in detail and those with LPV and nonlinear frameworks only briefly.

4.1 Extension to Linear Theory

In order to use linear analysis and design techniques a linear time invariant (LTI) model of the closed-loop system has to be obtained. This LTI system is obtained from the nonlinear system by applying small-perturbation theory and first-order Taylor approximations with respect to an equilibrium point in a process known as linearization. During the linearization process, the terms $f_3(x)$ and $g_3(x)$ from the nonlinear system are embedded in the linear system as terms affine on the deviation states σ_x, i.e. $f_3(x) \approx \left.\frac{\partial f_3(x)}{\partial x}\right|_{eq} \sigma_x$.

The standard linear time invariant LFT modelling approach is based on the technique known as uncertainty "pull-out" whereby the uncertain parameters in the linear plant and/or linear interconnection are placed in the uncertain matrix Δ. Several approaches can be used to obtain the uncertain parameterized LTI LFT model, see (Varga, A. *et al.*, 1998; Bates, D. and Postlethwaite, I., 2002) for a review. One of the first approaches to use symbolic LFT modelling for linear systems was proposed in (Varga, A. *et al.*, 1998; Varga, A. and Looye, G., 1999). These two references propose a two-step modelling approach for the nonlinear system given by equations (4-6). The first step is either to represent numerical LTI state-space matrices as symbolically parameterized polynomials, or alternatively to symbolically linearize the nonlinear system with respect to a generic equilibrium point. Note that by virtue of the linearization process (either symbolic or numerical), the terms $f_3(x)$ and $g_3(x)$ become linear on the deviation states, and the matrices A,B,C,D are parameterized in the selected symbolic parameters, thus a symbolic linear state-space system in the form of equation (3) is obtained. The second step is to use LFT techniques to "pull out" the symbolic parameters into the matrix Δ.

The proposed nonlinear modelling methodology builds on the previous linear symbolic LFT approach of references (Varga, A. *et al.*, 1998; Varga, A. and Looye, G., 1999). The difference between the approach proposed in this paper and the linear symbolic LFT approach is primarily due to the inversion in the order of the steps, which changes the nature of the state-space representation, and the subsequent manipulations on the resulting symbolic LFT. The proposed approach is valid for nonlinear state-space systems, see equation (7), and results in a nonlinear symbolic LFT as opposed to a linear symbolic LFT. As mentioned before, the use of nonlinear symbolic LFTs has several important advantages which are not present in the linear symbolic LFT approach.

A direct connection of the proposed framework to the linear symbolic LFT modelling approach and hence to linear design/analysis theory is obtained by constructing the nonlinear symbolic LFT model as proposed and subsequently performing a symbolic Jacobian linearization on the final matrix $\Delta(\rho)_p$ in order to obtain the linear symbolic LFT model. The following result on symbolic LFT linearization provides a simple and easily automated way to perform this connection (a similar result can be derived for upper LFTs):

Lemma 2. Consider a symbolic well-posed lower LFT $y = \mathcal{F}_l(M,\Delta)u$ where $M = [M_{11}\ M_{12}; M_{21}\ M_{22}]$, $\Delta = diag(\Delta_1, \Delta_2(\rho))$ and $u = [\rho\ \ d]^\top$.
Its *symbolic lower LFT linearization*, see Figure 4, is given by $\sigma_y = \mathcal{F}_l(\bar{M}, \Delta^J)\sigma_u$ where σ_y, σ_u are deviation variables with respect to an equilibrium point (y_{eq}, u_{eq}), e.g. $\sigma_y = y - y_{eq}$; the coefficient matrix $\bar{M}$ is given by:

$$\bar{M} = \begin{bmatrix} M_{11} + M_{12}M_{11}^J & M_{12}M_{12}^J \\ M_{21}^J & M_{22}^J \end{bmatrix} \tag{12}$$

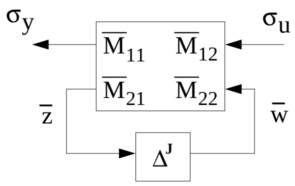

Fig. 4. Symbolic linearized LFT.

and $M^J = [M_{11}^J\ M_{12}^J; M_{21}^J\ M_{22}^J]$ and Δ^J are respectively the coefficient and uncertain matrices from the lower LFT of $\mathcal{L} = \mathcal{F}_l(M^J, \Delta^J)$:

$$\mathcal{L} = \left. \frac{\partial\Big((I - \Delta M_{22})^{-1}\Delta M_{21} u\Big)}{\partial u} \right|_{eq} \tag{13}$$

Δ^J is obtained by selecting as symbolic variables the terms $\Delta_1|_{eq}$, $\Delta_2(\rho)|_{eq}$, u_{eq} and the symbolic derivative $\left.\frac{\partial \Delta_2(\rho)}{\partial u}\right|_{eq}$ for the LFT of the function $\mathcal{L}$. $\bowtie$

In order to use linear analysis and design techniques the analytical form of the linear symbolic LFT must be used. As the symbolic parameters are either parameterized by or independent of the general equilibrium point (y_{eq}, u_{eq}), a simple substitution of the chosen analytic equilibrium point suffices to find the analytic linear system model.

Finally, note that in order to apply robust linear analysis and synthesis techniques based on $\mathcal{H}_\infty/\mu$ theory, it is required that the matrix Δ is norm-bounded by a positive non-zero value. Typically, the bound is set to one by scaling the constant and uncertain matrices in the LFT form in a process known as normalization. It is recommended that the symbolic parameters are normalized only after all other manipulations have been performed on the symbolic LFT (nonlinear or linear). By performing the normalization last (especially for the nonlinear symbolic LFT), it is ensured that the physical meaning of the symbolic parameters is retained and hence their effects on the system remain easier to understand and study. Furthermore, the diagonal structure of the Δ matrix means that the order of the LFT remains the same after normalization - see (Marcos, A. *et al.*, 2004) for a flight dynamics example of the dramatic effect normalization of the parameters before the LFT process has on the overall LFT order. This problem has been mentioned before, (Cockburn, J.C. and Morton, B.G., 1997; Magni, J.F., 2004), but it is noted that most of the available applications in the literature do not follow the advice contained in these references.

4.2 Extension to LPV Theory

A linear parameter varying system is defined as the class of finite dimensional linear systems whose state-space entries depend continuously on a time-varying parameter vector θ(t):

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & B(\theta(t)) \\ C(\theta(t)) & D(\theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{14}$$

The trajectory of the scheduling variables θ(t) is assumed not to be known in advance, although its value can be accessed (measured) in real time and is constrained *a priori* to lie in a specified bounded set. The scheduling variables can be considered as symbolic parameters, $\rho = \theta(t)$, in the proposed modelling framework without loss of generality.

Note that there is no restriction on the nonlinearity of the scheduling variables within the system matrices. If they enter linearly, a standard linear symbolic LFT can be obtained where the varying matrix Δ contains the symbolic parameters -this is the standard LFT form of a LPV system. If they enter nonlinearly, a symbolic nonlinear state-space in the form of equation (7) is obtained and the proposed modelling framework can be applied. This technique, including the definition of the fictitious input channel u_f, has been used for an aircraft application of the geometric LPV approach for filter detection design in (Szászi, I. *et al.*, 2002).

4.3 Extension to Describing Function Theory

Describing functions (DF) are a powerful mathematical tool for the design and analysis of nonlinear systems (Taylor, J.H., 1999). The basic idea of DFs is to replace each nonlinear element with a quasi-linear approximation based on harmonic linearization, which typically give rise to sinusoidal-input describing functions (SIDF). The main advantage of the SIDF approach is that it allows stability analysis of linear systems with nonlinear inputs or outputs, e.g. saturation of the control signals.

In references (Katebi, M.R. and Zhang, Y., 1995; Ferreres, G. and Fromion, V., 1998), it is shown how the DF approach can be used to transform a linear system with a nonlinear input into an LFT model by using the approximation error of the harmonic linearization and the corresponding SIDF for the nonlinearity. Furthermore, these two references present methods for performing nonlinear analysis, i.e. checking for the presence or absence of limit cycles, based on the use of μ theory -which depends on LFT models.

Hence, the class of nonlinear system considered earlier in equations (1-2) can also be extended to those nonlinear systems where the input enters in a nonlinear manner. The same symbolic nonlinear LFT modelling process is applied except that the nonlinear input is represented by a symbolic nonlinearity affine on a linear input signal, e.g. $nl(u) = \rho_{nl}u$. At the approximation stage in Section 3, the symbolic nonlinearity can be substituted in the varying matrix Δ by its SIDF and associated error, which can be regarded as an extra uncertain parameter. This enables the application of robust nonlinear analysis and design techniques to the resulting LFT model, using a combination of describing function methods and robust control theory.

5. CONCLUSIONS

In this paper a framework for exact nonlinear modelling has been presented. It relies on the use of symbolic parameterization and linear fractional transformations. The resulting nonlinear symbolic LFT is made up of a constant matrix connected in linear feedback with a structured matrix containing all the nonlinear, varying and uncertain terms. This structured presentation of the nonlinearities facilitates their study and ameliorates the effect that inappropriate simplifications and approximations could have on the overall modelling process. The nonlinear modelling framework also improves consistency, continuity and connectedness among the different models that stem from the nonlinear system. Finally, it has been shown that the proposed modelling framework is easily connected with several other analysis and synthesis frameworks, including linear robust control, LPV control and nonlinear describing function theories.

6. ACKNOWLEDGMENTS

This research was supported under EPSRC research grant $GR/S61874/01$.

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