REACHABILITY ANALYSIS UNDER CONTROL-DEPENDENT STOCHASTIC NOISE

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Abstract: The list of important problems of modern control theory includes those of reachability under under unknown disturbances, incomplete information on the system model and possible resets of system dynamics. The present paper deals special type of stochastic disturbances, when a linear system is subjected to perturbations generated by Brownian noise. The latter is assumed to depend on the values of the control, which in its turn may be either unbounded or bounded by hard bounds. The "reach" sets introduced here are deterministic. They consist of all points whose mean-square deviation from some controlled trajectory is small. These are presented in terms of level sets to solutions of appropriate Hamilton-Jacobi-Bellman (HJB) equations. The solutions to the HJB equations then allow explicit representation when the controls are unbounded and are given here in terms of solutions to some dual optimization problems when the controls are bounded. Accordingly, the reach sets are either ellipsoids or Euclidean neighborhoods of the reach sets to the respective averaged system.

Keywords: dynamic programming, system model, state estimation, set-membership approach, stochastic disturbance, principle of optimality.

INTRODUCTION

This paper deals with the problem of reachability — one of the central topics of modern control theory. Its motivation comes from problems in control design, verification of algorithms and other numerous problems in automation, navigation and related areas. Of special interest recently are problems of reachability under disturbances and resets. In the case of unknown but bounded disturbances the problem was treated in papers (Kurzhanski and Varaiya, 2000; Kurzhanski and Varaiya, 2002; Mitchell and Tomlin, 2003), while reachability for hybrid systems was dealt with in (Lygeros *et al.*, 1999; Kurzhanski and Varaiya, 2004).

The present report deals with stochastic disturbances. Considered is a continuous-time linear system subjected to perturbations generated by Brownian noise with characteristics dependent on the values of the control. Such situation are typical for feedback loops organized through communication channels. Here two cases are considered, namely, those, when the controls are either unbounded or bounded by hard bounds.

The reachability sets introduced here are deterministic. They are presented in terms of level sets to solutions of certain types of the Hamilton-Jacobi-

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Bellman (HJB) equation. The respective value functions allow explicit expressions in the domains the controls are unbounded and are presented in terms of solutions to some dual optimization problems when the controls are bounded.

Other settings of the problem of stochastic reachability are considered in the paper (Lygeros and Watkins, 2003).

1. THE SYSTEM AND THE REACHABILITY PROBLEM

Given is a continuous-time stochastic control system (Astrom, 1970; Fleming and Rishel, 1975; Fleming and Soner, 1993; Liptser and Shiryayev, 1977; Kurzhanski, 1965)

$$dx = A(t)xdt + b(t)udt + r(t)ud\omega, \qquad (1)$$

with state vector $x \in \mathbb{R}^n$, control $u \in \mathbb{R}$, continuous matrix $A(t)$, and vector $b(t)$ and control-dependent noise $r(t)ud\omega$.

Here $d\omega$ stands for a normalized scalar Brownian motion (Doob, 1953), so that

$$E(\omega(t'') - \omega(t')) = 0,$$
$$E(\omega(t'') - \omega(t'))^2 = |t'' - t'|.$$

Suppose that $r'(t)r(t) \neq 0$, $\forall t \in [\alpha, \beta]$, $\alpha < \tau$, $\beta > \vartheta$.

An important problem is to describe the *reachability set* (*reach set*) for system (1) (see (Lygeros and Watkins, 2003) for other approaches).

Definiton 1.1. The γ — reach set $\mathcal{Z}_\gamma[\vartheta] = \mathcal{Z}_\gamma(\vartheta, \tau, x_\tau)$ of system (1) at time ϑ, from position $\{\tau, x_\tau\}$, is the set of all points $z \in \mathbb{R}^n$ for which the inequality

$$V(\tau, x_\tau, z) =$$
$$\min_u \{E\{(\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\}|$$
$$|u(\cdot) \in \mathcal{U}[\tau, \vartheta]\} \leq \gamma^2$$

is true.

Here $x[\cdot] = x(\cdot, \tau, x_\tau)$, is the trajectory of system (1), emanated from position $\{\tau, x_\tau\}$ and $\mathcal{U}[\tau, \vartheta]$ is the class of admissible controls for the reachability problem (we shall consider two types of such classes.)

In order that $\mathcal{Z}_\gamma[\vartheta] \neq \emptyset$ it is necessary that $\gamma^2 \geq \gamma_0^2$ where γ_0^2 will be indicated below.

The problem to be studied is to calculate the set $\mathcal{Z}_\gamma[\vartheta]$ for the following types of controlled systems:

(a) $u \in \mathbb{R}$,
(b) $|u(t)| \leq \mu$.

So the main problem is find value function

$$V(\tau, x_\tau, z) = \min_u E\{\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\}$$

under control constrains (a) or (b).

We shall reduce this problem to an optimization procedure, starting with case (a).

2. REACHABILITY UNDER UNBOUNDED CONTROL

Consider case (a) (vector $u \in \mathbb{R}$).

Problem (a). *Find value function*

$$V^{(a)}(\tau, x_\tau, z) = \min_u \left\{ E\{\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\} \right|$$
$$\left| u(\cdot) \in \mathcal{U}^{(a)}[\tau, \vartheta] \right\}.$$

Class $\mathcal{U}^{(a)}[t', t'']$ in its feedback representation consists of all continuous functions of $\{t, x\}$, $t \in [t', t'']$, $x \in \mathbb{R}^n$, Lipschitz in x and such that allow extension of solutions $x[t]$ throughout the finite intervals $[\tau, \vartheta]$ under consideration.

Function $V^{(a)}(\tau, x_\tau, z)$ satisfies the equality ("principle of optimality") (Bertsekas, 1995; Krasovskii, 1960; Krasovskii, 1963)

$$V^{(a)}(\tau, x_\tau, z) =$$
$$= \min_u \left\{ E \left\{ \min_u E\{\|x(\vartheta, t, x[t]) - z\|)^2 | t, x[t]\}| \right. \right.$$
$$\left. \left. |u(\cdot) \in \mathcal{U}^{(a)}[t, \vartheta]\} \right| \tau, x_\tau \right\} \left| u(\cdot) \in \mathcal{U}^{(a)}[\tau, t) \right\} =$$
$$= \min_u \left\{ E\{V^{(a)}(t, x[t], z) | \tau, x_\tau\} \middle| u(\cdot) \in \mathcal{U}^{(a)}[\tau, \vartheta] \right\}.$$

Taking $t = \tau + \sigma$, and applying this principle along the standard lines of Dynamic Programming Theory (Fleming and Soner, 1993) we obtain (assuming differentiability of $V^{(a)}$, which further proves to be true), the following sufficient condition for optimality.

Theorem 2.1. Function $V^{(a)}(t, x, z)$ satisfies the relation

$$\min_u \left\{ dV^{(a)}(t, x, z)/dt \middle| u \in \mathbb{R} \right\} = 0, \qquad (2)$$

under boundary condition

$$V^{(a)}(\vartheta, x, z) = (x - z, x - z). \qquad (3)$$

Here

$$dV^{(a)}(t, x, z)/dt =$$
$$= \lim_{\sigma \to +0} \sigma^{-1}(E\{\|x(\vartheta, t + \sigma, x[t + \sigma]) - z\|^2 | t, x\} -$$
$$- V^{(a)}(t, x, z))$$

and equation (2) is actually a backward HJB equation

$$V_t^{(a)} + \min_u \left\{ (V_x^{(a)}, A(t)x + b(t)u) + \right.$$
$$\left. + \frac{1}{2} r'(t) V_{xx}^{(a)} r(t) u^2 \middle| u \in \mathbb{R} \right\} = 0. \qquad (4)$$

Here $V_t^{(a)}$ is the partial derivative of $V^{(a)}(t, x, z)$ in t, $V_x^{(a)}$ — its n-dimensional gradient vector and $V_{xx}^{(a)}$ — its matrix of second partials (the Laplacian).

Minimizing over u, we have

$$u_*(t,x,z) =$$
$$= -(r'(t)V_{xx}r(t))^{-1}b'(t)V_x^{(a)},$$

where u_* is the minimizer in (4).

Function $V^{(a)}(t,x,z)$ may be found in explicit form, namely, as

$$V^{(a)}(t,x,z) = (x-z, P(t)(x-z)) + 2q'(t)(x-z) +$$
$$+\kappa(t),$$

where matrix $P(t) = P'(t) > 0$ and vector $q(t)$ with scalar function $\kappa(t)$ are continuous. Then

$$u_*(t,x,z) =$$
$$= -(r'(t)P(t)r(t))^{-1}b'(t)(P(t)(x-z)+q(t)). \quad (5)$$

Substituting $V^{(a)}(t,x,z)$, $u_*(t,x,z)$ in (4) and equalizing the terms with multipliers x_ix_j, x_i and also the free terms, we come to the next system of equations.

These are:

$$\dot{P}(t) + A'(t)P(t) + P(t)A(t)-$$
$$-P(t)M(t)P(t) = 0 \quad (6)$$
$$\dot{q}(t) + (A'(t) - P(t)M(t))q(t) + P(t)A(t)z = 0, \quad (7)$$
$$\dot{\kappa}(t) + 2q'(t)A(t)z - q'(t)M(t)q(t) = 0 \quad (8)$$

with boundary condition

$$P(\vartheta) = I, \quad q(\vartheta) = 0, \quad \kappa(\vartheta) = 0, \quad (9)$$

which follows from (3). Here

$$M(t) = b(t)(r'(t)P(t)r(t))^{-1}b'(t).$$

System (6)–(9) is solvable, consisting of a well-posed matrix Riccati equation in $P(t)$, a linear equation in vector $q(t)$ and an integral equality for $\kappa(t)$. This fact indicates that $V^{(a)}(t,x,z)$ is differentiable.

As it is following from the Definition 1.1, the requested reach set $\mathcal{Z}_\gamma[\vartheta] = \mathcal{Z}_\gamma^{(a)}[\vartheta]$ may now be described within the next statement.

Theorem 2.2. The following reprezentation is true

$$\mathcal{Z}_\gamma^{(a)}[\vartheta] = \{z : V^{(a)}(\tau, x_\tau, z) \le \gamma\}.$$

Note that $q(t)$ and $\kappa(t)$ may be represented as $q(t) = F(t,\vartheta)z$, $\kappa(t) = (z, H(t,\vartheta)z)$, where

$$F(t,\vartheta) = \int_t^\vartheta X_K(t,s)P(s)A(s)ds,$$
$$K(t) = A'(t) - P(t)b(t)b'(t)(r'(t)P(t)r(t))^{-1},$$
$$\dot{X}_K(t,s) = -K(t)X_K(t,s), \quad X_K(s,s) = I,$$
$$H(t,\vartheta) = \int_t^\vartheta (2A'(s)F(s,\vartheta)-$$
$$-F'(s,\vartheta)M(t)F(s,\vartheta))ds.$$

Therefore

$$V^{(a)}(t,x,z) = (x-z, P(t)(x-z))+$$
$$+2(z, F'(t,\vartheta)(x-z) + (z, H(t,\vartheta)z) =$$
$$= (z-x, \mathcal{P}(t,\vartheta)(z-x)) - 2(x, \mathcal{N}(t,\vartheta)(z-x)-$$
$$-(x, H(t,\vartheta)x).$$

Here

$$\mathcal{P}(t,\vartheta) = P(t) - 2F'(t,\vartheta) + H(t,\vartheta),$$
$$\mathcal{N}(t,\vartheta) = F'(t,\vartheta) - H(t,\vartheta).$$

One may further observe, after some calculations that function $V^{(a)}(\tau, x_\tau, z)$ may be represented in following form

$$V^{(a)}(\tau, x_\tau, z) =$$
$$= \|z - x_\tau - \mathcal{P}^{-1}(\tau,\vartheta)\mathcal{N}'(\tau,\vartheta)x_\tau\|^2_{\mathcal{P}(\tau,\vartheta)}-$$
$$-k^2(\vartheta,\tau,x_\tau),$$

where

$$k^2(\vartheta,\tau,x_\tau) =$$
$$= (x_\tau, \mathcal{N}(\tau,\vartheta)\mathcal{P}^{-1}(\tau,\vartheta)\mathcal{N}'(\tau,\vartheta)x_\tau)+$$
$$+(x_\tau, H(t,\vartheta)x_\tau).$$

Therefore, the following theorem is true.

Theorem 2.3. The γ — reach set of system (1) $\mathcal{Z}_\gamma^{(a)}[\vartheta]$ may be represented as an ellipsoid

$$\mathcal{Z}_\gamma^{(a)}[\vartheta] = \mathcal{E}(x^0[\vartheta], X^0[\vartheta]) =$$
$$= \{z : (z - x^0[\vartheta], X^0[\vartheta](z - x^0[\vartheta])) \le \gamma^2\}$$

with center

$$x^0[\vartheta] = \left(I + \mathcal{P}^{-1}(\tau,\vartheta)\mathcal{N}'(\tau,\vartheta)\right)x_\tau$$

and shape matrix

$$X^0[\vartheta] = \mathcal{P}(\tau,\vartheta),$$

which is nonempty iff

$$\gamma^2 \ge \gamma_0^2 = k^2(\vartheta,\tau,x_\tau).$$

3. REACHABILITY UNDER HARD BOUNDS ON THE CONTROLS

Consider system (1) where control u restricted by additional *hard bound*:

$$|u| \le \mu. \quad (10)$$

Here the solution follows Section 2, and is solved within class of functions $\mathcal{U}^{(a)}[\tau,\vartheta]$, provided the control $u_*(t,x,z)$ satisfies the hard bound. This property holds within the domain

$$\mathcal{D}^{(a)}) = \{t, x : |u_*(t,x,z)| \le \mu\}.$$

where $u_*(t,x,z)$ is defined above, in (5). Beyond domain $\mathcal{D}^{(a)}$ lies the domain $\mathcal{D}^{(b)} = \mathbb{R}^n \setminus \mathcal{D}^{(a)}$. Here we have to deal with problem

Problem (b). *Find value function*

$$V^{(b)}(\tau, x_\tau, z) =$$
$$= \min_u \Big\{ E\{\|x[\vartheta] - z\|^2 | \tau, x(\tau) = x_\tau\} \Big|$$
$$\Big| u(\cdot) \in \mathcal{U}^{(b)}[\tau, \vartheta] \Big\}.$$

where $\mathcal{U}^{(b)}[\tau, \vartheta]$ consists of controls $U(t, x)$ taken as set-valued maps with compact interval values, upper semicontinuous in $\{t, x\}$.

Let $G(t, \vartheta)x$ stand for the fundamental transition matrix of the homogeneous part of system (1). Applying transformation $z = G^{-1}(t, \vartheta)x$ to equation (1) and returning to the original notations we come to equation

$$dx = b(t)udt + r(t)ud\omega, \tag{11}$$

Without loss of generality we will now solve Problem (b) for equation (11) (see also (Kats and Kurzhanski, 1970; Kurzhanski and Vályi, 1997)).

Similarly to the previous section we come to the HJB equation

$$V_t^{(b)} + \min_u \Big\{ (V_x^{(b)}, b(t)u) +$$
$$+ \frac{1}{2} r'(t) V_{xx}^{(b)} r(t) u^2 \Big| |u| \leq \mu \Big\} = 0. \tag{12}$$

under boundary condition

$$V^{(b)}(\vartheta, x, z) = (x - z, x - z). \tag{13}$$

Our further calculations will justify this assumption. After a minimization we then come to the equation

$$V_t + \mathcal{H}(t, x, V_x, V_{xx}) = 0$$

where in the domain

$$\mathcal{D}^{(a)} = \{t, x : (r'(t)V_{xx}r(t))^{-1}|V_x' b(x)| \leq \mu\}$$

we have

$$\mathcal{H}(t, x, V_x, V_{xx}) = -\frac{1}{2}(V_x' b(t) b'(t) V_x)(r'(t) V_{xx} r(t))^{-1},$$

and in the domain $\mathcal{D}^{(b)}$ we have

$$\mathcal{H}(t, x, V_x, V_{xx}) = -\mu |b'(t) V_x| + \frac{1}{2}\mu^2 r'(t) V_{xx} r(t)$$

Thus, function $V(t, x, z)$ must coincide with $V^{(a)}(t, x, z)$ in $\mathcal{D}^{(a)}$ and with $V^{(b)}(t, x, z)$ in $\mathcal{D}^{(b)}$.

As indicated in Section 2, in the domain $\mathcal{D}^{(a)}$ function $V^{(a)}(t, x, z) = V(t, x, z)$ is quadratic and control $u_*(t, x, z)$ is affine.

The relation (5) implies that $\mathcal{D}^{(a)}$ can be represented as

$$\mathcal{D}^{(a)} = \{t, x :$$
$$(r'(t)P(t)r(t))^{-1} |b'(t)(P(t)(x - z) + q(t))| \leq \mu\}.$$

Therefore we now have to find function $V^{(b)}(t, x, z) = V(t, x, z)$ in $\mathcal{D}^{(b)}$.

We further assume $V_{xx}^{(b)} = \alpha I = const,\ \alpha > 0$. Equation

$$V_t - \mu |b'(t) V_x| + \frac{1}{2}\alpha\mu^2 r'(t) r(t) = 0, \tag{14}$$

with boundary condition (13) is also generated by the deterministic control problem of finding

$$\bar{V}(t, x, z) = \min \Big\{ \frac{1}{2} \int_t^\vartheta \alpha r'(s) r(s) u^2(s) ds +$$
$$+ (x(\vartheta, t, x) - z, x(\vartheta, t, x) - z) \ |x(t) = x, |u| \leq \mu \Big\},$$

for the averaged equation

$$\dot{x} = b(t)u, \tag{15}$$

so that $\bar{V}(t, x, z) = V(t, x, z)$.

Applying the techniques of convex analysis along the lines of (Kurzhanski and Vályi, 1997; Kurzhanski and Varaiya, 2002), we find

$$\bar{V}(t, x, z) =$$
$$\max_l \Big\{ (x - z, l) - \mu \int_t^\vartheta |b'(s) l| ds - 1/4(l, l)$$
$$+ \frac{1}{2} \int_t^\vartheta \alpha\mu^2 r'(s) r(s) ds \Big\} =$$
$$= d^2(z, x + \mathcal{X}(\vartheta, t)) + \beta^2(t), \tag{16}$$

where

$$\mathcal{X}(\vartheta, t) = \int_t^\vartheta b(s) \mathcal{S}_\mu(s) ds,\ \mathcal{S}_\mu(s) = [-\mu, \mu] \tag{17}$$

and

$$\beta^2(t) = \frac{1}{2} \int_t^\vartheta \alpha\mu^2 r'(s) r(s) ds. \tag{18}$$

With $t = \vartheta$ we have $\mathcal{X}(\vartheta) = \{0\}$, $d^2(x, \mathcal{X}(\vartheta)) = (x - z, x - z)$, and $\beta(\vartheta) = 0$. The boundary condition (13) is therefore fulfilled.

Let $l_* = l_*(t, x, z)$ be the unique maximizer in (16). Then

$$\bar{V}_t = \mu |b'(t) l_*| - \frac{1}{2}\alpha\mu^2 r'(t) r(t)$$

and $\bar{V}_x = l_*$.

Substituting the last values in (14) with $\alpha = 1$ we observe that $\bar{V}(t, x, z) = V(t, x, z) = V^{(b)}(t, x, z)$ is a solution of (12) which is also (14).

The synthesized control in domain $\mathcal{D}^{(b)}$ is a "bang-bang" solution $u_*(t, x, z) = -\mu\, \mathrm{sign}(b'(t) l_*)$ which is correctly represented in set-valued form as

$$U_*(t, x, z) = \begin{cases} -\mu\, \mathrm{sign}\Big(b'(t) l_*\Big), & b'(t) l_* \neq 0, \\ \mathcal{S}_\mu(t), & b'(t) l_* = 0, \end{cases}$$

The overall system may be presented as a stochastic differential inclusion (Aubin and Da Prato, 1998)

$$dx = b(t)udt + r(t)ud\omega(t), \quad u \in U_*(t, x, z).$$

Note that domains $\mathcal{D}^{(a)}, \mathcal{D}^{(b)}$ intersect along surfaces

$$(r'(t)P(t)r(t)|)^{-1} |b'(t)(P(t)(x - z) + q(t))| = \mu$$

Direct calculation shows that functions $V^{(a)}(t,x,z)$, $V^{(b)}(t,x,z)$ intersect with the last surfaces along the same curves so that the overall function $V(t,x,z)$ is continuous.

Summarizing the obtained results we come to the following theorem.

Theorem 3.1. (i) The γ — reach set $\mathcal{Z}_\gamma[\vartheta]$ under control-dependent noise with hard bound on the control is of two forms. Namely,in the domain $\mathcal{D}^{(a)}$ it is an ellipsoid (see Section 2)

$$\mathcal{Z}_\gamma[\vartheta] = \mathcal{Z}_\gamma^{(a)}[\vartheta] =$$

$$= \{z : (z - x^0[\vartheta], X^0[\vartheta](z - x^0[\vartheta])) \leq \gamma^2\}.$$

In the complementary domain $\mathcal{D}^{(b)}$ $\mathcal{Z}_\gamma[\vartheta]$ is the Euclid neighborhood of the convex set

$$\mathcal{X}(\vartheta, \tau, x) = x + \mathcal{X}(\vartheta, \tau)$$

— the reachability set at time ϑ from starting position $\{\tau, x\}$ of deterministic system (15) under the concstrain $|u| \leq \mu$. So that

$$\mathcal{Z}_\gamma[\vartheta] =$$

$$= \{z : d^2(z, \mathcal{X}(\vartheta, \tau, x)) \leq \gamma^2 - \beta^2(\tau)\}.$$

(ii) The respective controls which define the trajectories from the starting point $\{t, x\}$ to a boundary point z of set $\mathcal{Z}[\vartheta]$, turn out to be affine in the domain $\mathcal{D}^{(a)}$:

$$u_*(t,x,z) =$$

$$= -(r'(t)V_{xx}r(t))^{-1}b'(t)V_x$$

and of the bang-bang type in the complementary domain $\mathcal{D}^{(b)}$:

$$u_*(t,x,z)) = -\mu\,\mathrm{sign}(b'(t)V_x),$$

when $b'(t)V_x \neq 0$.

4. CONCLUSION

This paper introduces the notion of reachability for controlled systems subjected to stochastic Brownian noise which depends on the control parameters that may be either unbounded or bounded by hard bounds. It indicates the routes to calculate reach sets in both cases. The reach set of each type is described through level sets of solutions to appropriate stochastic HJB equations. The set-valued dynamics of the reach sets introduced here are described either explicitly (for Problem (a) and Problem (b) in domain $\mathcal{D}^{(a)}$) or (for the domain $\mathcal{D}^{(b)}$) through techniques of ellipsoidal calculus (Kurzhanski and Vályi, 1997; Kurzhanski and Varaiya , 2002). The result presented here thus lead to efficient computational algorithms crucial to control design and verification under bounded controls and stochastic noise.

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