HOMOGENEOUS EIGENVALUE ANALYSIS OF HOMOGENEOUS SYSTEMS

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Abstract: This paper focuses on the problem of stability in homogeneous systems with dilation. First, we propose an 'homogeneous eigenvalue' for homogeneous systems. Next, we analyze the stability of homogeneous systems using homgeneous eigenvalues, and we show that the use of positive real homogeneous eigenvalues implies instability. Finally, we show the effectiveness of the proposed method through an example.

Keywords: nonlinear control systems, eigenvalue problems, instability, stability analysis

1. INTRODUCTION

Weighted homogeneity, as elucidated by Rothschild et al. (Rothschild and Stein, 1976), plays an important role in nonlinear control theory. Homogeneous systems include linear systems. However, among these homogeneous systems include the finite-time controlled system (Bhat and Bernstein, 1997), (Bhat and Bernstein, 1998), (Hong *et al.*, 1999), (Hong, 2001), the exact differentiator system and the higher-order sliding-mode controlled system (Levant, 2001),(Nakamura *et al.*, 2004), and the nonholonomic controlled system (Nakamura *et al.*, 2002),(Nakamura *et al.*, 2003); such systems cannot be analyzed by linear control theory.

Rosier (Rosier, 1992) proved the converse of Lyapunov's second theorem for homogeneous systems: An asymptotically stable homogeneous system permits use of s some homogeneous Lyapunov functions. Sepulchre and Aeyels (Sepulchre and Aeyels, 1996) analyzed the properties of homogeneous Lyapunov functions and homogeneous systems. Grüne (Grüne, 2000) proposed construction of a Lyapunov function for a homogeneous system. However, the construction of homogeneous Lyapunov functions is usually very difficult.

Accordingly, this paper provides another method for analyzing homogeneous systems. In the next section we clarify the basic properties of homogeneity. Next, we propose the 'homogeneous eigenvalue' for homogeneous systems. In the fourth section, we analyze the stability of homogeneous systems using homgeneous eigenvalues, and we show that use of the positive real homogeneous eigenvalues implies instability while the negative real homogeneous eigenvalues permit stable solutions. Finally, we show the effectiveness of the proposed method through an example.

2. HOMOGENEOUS SYSTEMS

In this section, we show the basic definitions and properties of homogeneous systems with dilations.

Definition 1. (dilation). Dilation Δ_ϵ^r is a mapping, depending on positive dilation coefficients $r = (r_1, r_2, \ldots, r_n)$ $(r_i > 0,\ \ 1 \le i \le n)$, which assigns to every $\varepsilon > 0$ a global diffeomorphism

$$\Delta_\varepsilon^r(x) = (\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n), \qquad (1)$$

where $(x_1, \ldots, x_n)$ are suitable coordinates on $\mathbb{R}^n$.

Definition 2. (homogeneous function). A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of degree $q \in \mathbb{R}$ with respect to the dilation Δ_ε^r, if

$$V(\Delta_\varepsilon^r(x)) = \varepsilon^q V(x). \qquad (2)$$

Definition 3. (homogeneous vector field). A vector field $f(x) = (f_1(x), \ldots, f_n(x)) : \mathbb{R}^n \to \mathbb{R}^n$ is called homogeneous of degree $k \in \mathbb{R}$ with respect to the dilation Δ_ε^r, if there exists $k \in \mathbb{R}$ such that

$$f_i(\Delta_\varepsilon^r(x)) = \varepsilon^{k+r_i} f_i(x), \quad i = 1, \ldots, n. \qquad (3)$$

A system $\dot{x} = f(x)$ is called homogeneous if its vector field $f(x)$ is homogeneous.

Definition 4. (homogeneous norm). A homogeneous norm is a map $x \to \|x\|_{\{r,p\}}$ of degree 1 with respect to the dilation Δ_ϵ^r and satisfies the following equation:

$$||x||_{\{r,p\}} = (|x_1|^{\frac{p}{r_1}} + \cdots + |x_n|^{\frac{p}{r_n}})^{\frac{1}{p}}.w \qquad (4)$$

Definition 5. (Euler vector field). The Euler vector field associated to a dilation coefficient $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for each $i \in \{1, \ldots, n\}$ is defined by the following equation:

$$\nu(x) = \begin{bmatrix} r_1 x_1 \\ \vdots \\ r_n x_n \end{bmatrix}. \qquad (5)$$

Proposition 1. (Hong: 2001) Let a function $V : \mathbb{R}^n \to \mathbb{R}$ be homogeneous of degree $k \in \mathbb{R}$ with dilation coefficients $r = (r_1, \ldots, r_n)$. Then, V is homogeneous of degree pk with dilation coefficient $r = (pr_1, \ldots, pr_n)$, $p > 0$.

Proposition 2. Let a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ be homogeneous of degree $k \in \mathbb{R}$ with dilation coefficients $r = (r_1, \ldots, r_n)$. Then, f is homogeneous of degree pk with dilation coefficients $r = (pr_1, \ldots, pr_n)$, $p > 0$.

Lemma 1. (Rosier: 1992) Let a function $V : \mathbb{R}^n \to \mathbb{R}$ be homogeneous of degree $k \in \mathbb{R}^+$ with dilation coefficients $r = (r_1, \ldots, r_n)$. Then,

$$\forall x = (x_i)_{i=1,\ldots,n} \in \mathbb{R}^n \backslash \{0\}, \forall t > 0 \qquad (6)$$

$$\frac{\partial V}{\partial x_i}(t^{r_1} x_1, \ldots, t^{r_n} x_n) = t^{k-r_i} \frac{\partial V}{\partial x_i}(x_1, \ldots, x_n). \qquad (7)$$

For classical homogeneous functions ($r_i = 1$), the Euler property is important. For homogeneous functions with dilations, almost the same property holds. Sepulchre and Ayels proved an extended version of euler property (Sepulchre and Aeyels, 1996). However, the property is restrictive. Here, we show a more general version of the extended Euler property.

Theorem 1. (Extended Euler property). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 homogeneous function of degree k with respect to a dilation coefficient $r = (r_1, r_2, \ldots, r_n)$, where $r_i > 0 (i = 1, 2, \ldots, n)$. Then,

$$\frac{\partial V}{\partial x}(x)\nu(x) = kV(x). \qquad (8)$$

Proof . Since V is homogeneous of degree k,

$$V(\Delta_\varepsilon^r(x)) = \varepsilon^k V(x). \qquad (9)$$

By differentiating both sides of eq. (9) with respect to ε, we can obtain the following equation:

$$\sum_{i=1}^{n} \frac{\partial V(\tilde{x}_i)}{\tilde{x}_i} \cdot r_i \varepsilon^{r_i - 1} x_i = k\varepsilon^{k-1} V(x), \qquad (10)$$

where $\tilde{x} = \Delta_\varepsilon^r(x)$. Equation (10) evaluated with $\varepsilon = 1$ gives eq. eq. (8).

3. HOMOGENEOUS EIGENVALUES

Eigenvalue analysis plays an important role in linear control theory. For nonlinear systems, there are various approaches to eigenvalue analysis (Grüne, 2000), (Jingxian and Bendong, 1997), (Fujimoto, 2004). However, the results of these papers do not cover the relationship between stability and nonlinear eigenvalues. Accordingly, we propose 'homogeneous eigenvalues' in this paper and analyze the stability of homogeneous systems.

In this paper, we consider the following continuous homogeneous system:

$$\dot{x} = f(x). \qquad (11)$$

To analyze the stability of homogeneous systems, the following theorem proven by Rosier (Rosier, 1992) is very useful.

Theorem 2. (Rosier: 1992) Assume a system $\dot{x} = f(x)$ such that f is continuous and homogeneous with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$. Then, there exist C^p-smooth homogeneous Lyapunov functions of degree k such that $k \ge p \cdot \max_{1 \le i \le n} r_i$.

For eq. (11), when there exist $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ that satisfy the following equation:

$$f(x) = \lambda x, \tag{12}$$

λ is usually said to be a nonlinear eigenvalue. However, we cannot analyze even homogeneous systems by using this definition. Thus, we consider the following equation:

$$f(x) = \lambda \nu(x). \tag{13}$$

Assume λ satisfies the above equation. Then, by Rosier's theorem and the extended euler property, the following proposition is obtained.

Proposition 3. Assume a system $\dot{x} = f(x)$ such that f is continuous and homogeneous with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$. If there exist x_0 and $\lambda \geq 0$ such that

$$f(x_0) = \lambda \nu(x_0), \tag{14}$$

the system is not asymptotically stable.

Proof . If the system $\dot{x} = f(x)$ is asymptotically stable, it allows a smooth homogeneous Lyapunov function V of degree k. The derivative of V can be described by the following equation:

$$\dot{V} = \frac{\partial V}{\partial x} f(x). \tag{15}$$

For x_0,

$$\frac{\partial V}{\partial x}(x_0) f(x_0) = \frac{\partial V}{\partial x}(x_0) \lambda \nu(x_0). \tag{16}$$

However, by the extended euler property,

$$\frac{\partial V}{\partial x}(x_0) f(x_0) = \lambda k V(x_0) > 0. \tag{17}$$

This contradicts the assumption that V is a Lyapunov function.

The above proposition implies that there exists a relationship between λ and the stability of homogeneous systems. However, when we look for λ, we must solve the following equation:

$$f(x) = \lambda \nu(x). \tag{18}$$

Equation (18) is not always a homogeneous equation. Therefore, the equation is difficult to solve. Then, we define an 'homogeneous' eigenvalue for the homogeneous vector field.

Definition 6. (homogeneous eigenvalue). Assume that $f(x)$ is a vector field that is continuous and homogeneous of degree l with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$. $\lambda \in \mathbb{R}$ is called an homogeneous eigenvalue if there exist $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that

$$f(x_0) = \lambda \|x_0\|^l_{\{r,2\}} \nu(x_0). \tag{19}$$

Moreover, x_0 is called an homogeneous eigenvector.

Then, the following proposition is obtained.

Proposition 4. Assume a vector field $f(x)$ such that f is continuous and homogeneous of degree l with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$. The following eigenequation is an homogeneous equation:

$$f(x) = \lambda \|x\|^l_{\{r,2\}} \nu(x). \tag{20}$$

The proof is easy and thus omitted. This definition of homogeneous eigenvalues is very useful for the analysis of homogeneous systems. We show an important property for an homogeneous eigenvalue in the next section.

4. MAIN RESULTS

In this section, we show the main results of this paper.

Theorem 3. Assume a system $\dot{x} = f(x)$ such that f is continuous and homogeneous of degree l with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$, and then assume that there exist a $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that

$$f(x_0) = \lambda \|x\|^l_{\{r,2\}} \nu(x_0). \tag{21}$$

(1) If $\lambda > 0$, the system is unstable.
(2) If $\lambda = 0$, the system is not asymptotically stable.
(3) If $\lambda < 0$, there exists a solution that converges to the origin.

To prove the above theorem, we prepare some lemmas.

Lemma 2. Assume a system $\dot{x} = f(x)$ such that f is continuous and homogeneous of degree 0 with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$, and further assume there exist $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that

$$f(x_0) = \lambda \nu(x_0). \tag{22}$$

Then, the solution for an initial state $x(0) = x_0 = (x_{01}, \ldots, x_{0n})^T$ is shown in the following equations:

$$\begin{aligned} x_1(t) &= x_{01} e^{\lambda r_1 t} \\ x_2(t) &= x_{02} e^{\lambda r_2 t} \\ &\vdots \\ x_n(t) &= x_{0n} e^{\lambda r_n t}. \end{aligned} \tag{23}$$

Proof . The solution eq. (23) can be described by the following equation:

$$x_i(t) = x_{0i} (e^{\lambda t})^{r_i}. \tag{24}$$

By using dilation Δ^r_ε, we can transform the above equation to the following equation:

$$x(t) = \Delta^r_{e^{\lambda t}}(x_0). \tag{25}$$

On the other hand, since $\dot{x} = f(x)$ is homogeneous of degree 0,

$$\begin{aligned} f_i(\Delta^r_{e^{\lambda t}}(x_0)) &= (e^{\lambda t})^{r_i} f_i(x_0) &(26)\\ &= (e^{\lambda t})^{r_i} \lambda r_i x_{0i}. &(27)\end{aligned}$$

Differentiating both sides of eq. (23),

$$\begin{aligned} \dot{x}_i(t) &= \lambda r_i x_{0i} e^{\lambda r_1 t} &(28)\\ &= (e^{\lambda t})^{r_i} \lambda x_{0i} &(29)\\ &= f_i(\Delta^r_{e^{\lambda t}}(x_0)) &(30)\\ &= f(x(t)). &(31)\end{aligned}$$

Therefore, eq. (23) is a solution for $\dot{x} = f(x)$.

Lemma 3. Consider the following system:

$$\dot{x} = f(x), \tag{32}$$

where f is continuous and homogeneous of degree l with dilation coefficient $r = (r_1, \dots, r_n)$ and $f(0) = 0$, and assume that there exist $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that

$$f(x_0) = \lambda \|x_0\|^l_{\{r,2\}} \nu(x_0). \tag{33}$$

Then, the solution for an initial state $x(0) = x_0 = (x_{01}, \dots, x_{0n})^T$ is shown in the following equations:

$$\begin{aligned} x_1(t) &= \frac{x_{01}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_1}{l}}}\\ x_2(t) &= \frac{x_{02}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_2}{l}}}\\ &\vdots\\ x_n(t) &= \frac{x_{0n}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_n}{l}}}. \end{aligned} \tag{34}$$

Proof . Consider the new time scale τ that satisfies the follwing equation:

$$\frac{d\tau}{dt} = \|x(t)\|^l_{\{r,2\}}. \tag{35}$$

Then, eq. (33) becomes the following equation:

$$\frac{dx}{d\tau} = \|x\|^{-l}_{\{r,2\}} f(x). \tag{36}$$

Note that the above system is homogeneous of degree 0. For x_0, we can obtain the following equation:

$$\begin{aligned} \left.\frac{dx}{d\tau}\right|_{x=x(0)} &= \|x_0\|^{-l}_{\{r,2\}} \|x_0\|^l_{\{r,2\}} \lambda \nu(x_0) &(37)\\ &= \lambda \nu(x_0). &(38)\end{aligned}$$

Then, assume $\tau = 0$ at $t = 0$, a solution for system (36) can be obtained by the following equation:

$$\begin{aligned} x_1(\tau) &= x_{01} e^{\lambda r_1 \tau}\\ x_2(\tau) &= x_{02} e^{\lambda r_2 \tau}\\ &\vdots\\ x_n(\tau) &= x_{0n} e^{\lambda r_n \tau}. \end{aligned} \tag{39}$$

Therefore,

$$\|x(\tau)\|^l_{\{r,2\}} = \|x_0\|^l_{\{r,2\}} e^{\lambda l \tau}. \tag{40}$$

Now, eq. (35) becomes the following equation:

$$\frac{d\tau}{dt} = \|x_0\|^l_{\{r,2\}} e^{\lambda l \tau}. \tag{41}$$

Consequently, we can obtain

$$\tau = \ln(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{1}{\lambda l}}. \tag{42}$$

Therefore,

$$\begin{aligned} x_1(t) &= \frac{x_{01}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_1}{l}}}\\ x_2(t) &= \frac{x_{02}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_2}{l}}}\\ &\vdots\\ x_n(t) &= \frac{x_{0n}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_n}{l}}}. \end{aligned} \tag{43}$$

Remark . Grüne used a similar time-scale transformation in (Grüne, 2000).

Lemma 4. Consider the following system:

$$\dot{x} = f(x), \tag{44}$$

where f is continuous and homogeneous of degree l with dilation coefficients $r = (r_1, \dots, r_n)$ and $f(0) = 0$, and assume that there exist $\lambda < 0$ and $x_0 \in \mathbb{R}^n$ such that

$$f(x_0) = \lambda \|x_0\|^l_{\{r,2\}} \nu(x_0). \tag{45}$$

Then, there exists a solution for an initial state $x(0) = x_0 = (x_{01}, \dots, x_{0n})^T$ such that $x(t) \to 0$ when $t \to +\infty$.

Proof . If $l = 0$, it is clear that there exists a solution such that $x(t) \to 0$ when $t \to +\infty$ by Lemma 2.

If $l > 0$, by Lemma 3, there exists the following solution:

$$\begin{aligned} x_1(t) &= \frac{x_{01}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_1}{l}}}\\ x_2(t) &= \frac{x_{02}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_2}{l}}}\\ &\vdots\\ x_n(t) &= \frac{x_{0n}}{(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_n}{l}}}. \end{aligned} \tag{46}$$

Since $l > 0, \lambda < 0$, it is clear that $x_i(t) \to 0$ when $t \to +\infty$.

If $l < 0$, by Lemma 3, there exists the following solution:

$$\begin{aligned} x_1(t) &= x_{01}(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_1}{l}}\\ x_2(t) &= x_{02}(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_2}{l}}\\ &\vdots\\ x_n(t) &= x_{0n}(1-\lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_n}{l}}. \end{aligned} \tag{47}$$

When $1 - \lambda l \|x_0\|^l_{\{r,2\}} t = 0$, the above $x_i(t) = 0$ applies for $(i = 1, \ldots, n)$. Therefore, the following equations also give the solution of eq. (50).

(1) When $1 - \lambda l \|x_0\|^l_{\{r,2\}} > 0$,

$$\begin{aligned} x_1(t) &= x_{01}(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_1}{l}} \\ x_2(t) &= x_{02}(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_2}{l}} \\ &\vdots \\ x_n(t) &= x_{0n}(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_n}{l}}. \end{aligned} \tag{48}$$

(2) When $1 - \lambda l \|x_0\|^l_{\{r,2\}} \leq 0$,

$$\begin{aligned} x_1(t) &= 0 \\ x_2(t) &= 0 \\ &\vdots \\ x_n(t) &= 0. \end{aligned} \tag{49}$$

Therefore, there exists a solution $x(t)$ such that $x(t) \to 0$ when $t \to +\infty$

Remark . For asymptotically stable homogeneous systems, the system is finite-time stable if $l < 0$ and exponentially stable if $l = 0$ (Bacciotti and Rosier, 2001). By the proofs of Lemma 2 and Lemma 4, if there exists a negative real homogeneous eigenvalue $\lambda < 0$, the corresponding solutions have the finite-time convergence property and the exponential convergence property, respectively.

Lemma 5. Consider the following system:

$$\dot{x} = f(x), \tag{50}$$

where f is continuous and homogeneous of degree l with dilation coefficients $r = (r_1, \ldots, r_n)$ and $f(0) = 0$, and there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ such that

$$f(x_0) = \lambda \|x_0\|^l_{\{r,2\}} \nu(x_0). \tag{51}$$

Then, the system is unstable.

Proof . If $l = 0$, it is clear that there exists a solution such that $\|x(t)\| \to \infty$ when $t \to +\infty$ by Lemma 2.

If $l < 0$, by Lemma 3, there exists the following solution:

$$\begin{aligned} x_1(t) &= x_{01}(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_1}{l}} \\ x_2(t) &= x_{02}(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_2}{l}} \\ &\vdots \\ x_n(t) &= x_{0n}(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{-\frac{r_n}{l}}. \end{aligned} \tag{52}$$

Since $l < 0, \lambda > 0$, it is clear that $\|x_i(t)\| \to \infty$ when $t \to +\infty$.

If $l > 0$, by Lemma 3, there exists the following solution:

$$\begin{aligned} x_1(t) &= \frac{x_{01}}{(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_1}{l}}} \\ x_2(t) &= \frac{x_{02}}{(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_2}{l}}} \\ &\vdots \\ x_n(t) &= \frac{x_{0n}}{(1 - \lambda l \|x_0\|^l_{\{r,2\}} t)^{\frac{r_n}{l}}}. \end{aligned} \tag{53}$$

Since $l > 0, \lambda > 0$, when $1 - \lambda l \|x_0\|^l_{\{r,2\}} t = 0$, $\|x(t)\| = \infty$. Hence, there exists $t_0 > 0$ such that $1 - \lambda l \|x_0\|^l_{\{r,2\}} t_0 = 0$. Therefore, there exist a solution that blows up in finite-time.

In all cases, there exist an unstable solution. Therefore the system is unstable.

Remark . When the homogeneous systems with degree $l > 0$ have positive real homogeneous eigenvalues, the systems may blow up in finite-time; in other words, the system has a finite-escape time. This is dangerous and causes many problems.

Finally, we can prove Theorem 3.

(Proof of Theorem 3). If $\lambda < 0$, there exists a solution that converges to the origin by Lemma 4.

If $\lambda = 0$, there exists the following solution:

$$x(t) = x_0. \tag{54}$$

Therefore, the system is not asymptotically stable.

If $\lambda > 0$, the system is unstable by Lemma 5.

5. EXAMPLE

As an example, we consider the following homogeneous system:

$$\dot{x}_1 = x_2^3 \tag{55}$$

$$\dot{x}_2 = -14(3x_1)^{\frac{1}{3}} - 15x_2. \tag{56}$$

The above system is homogeneous of degree 0 with respect to a dilation coefficient $r = (3, 1)$.

First, we calculate homogeneous eigenvalues. The homogeneous eigenvalues satisfy the following equation for some x_0:

$$\begin{bmatrix} x_2^3 \\ -14(3x_1)^{\frac{1}{3}} - 15x_2 \end{bmatrix} = \lambda \begin{bmatrix} 3x_1 \\ x_2 \end{bmatrix}. \tag{57}$$

Then, one of the solutions for the above equation is

$$\lambda = -1 \tag{58}$$

and

$$x_0 = \begin{bmatrix} 1 \\ -3^{1/3} \end{bmatrix}. \tag{59}$$

By Lemma 2, one of the solutions is obtained by the following equations:

$$x_1 = e^{-3t} \tag{60}$$

$$x_2 = -3^{\frac{1}{3}} e^{-t}. \tag{61}$$

Then,

$$\dot{x}_1 = -3e^{-3t} = (x_2(t))^{\frac{1}{3}} \tag{62}$$

$$\dot{x}_2 = 3^{\frac{1}{3}} e^{-t} = -14(3x_1)^{\frac{1}{3}} - 15x_2. \tag{63}$$

Therefore eq. (60) is a solution for eq. (55). Moreover, the homogeneous eigenvalue -1 is negative. We can confirm that the corresponding solution converges to the origin.

6. CONCLUSION

We have proposed an 'homogeneous eigenvalue' for homogeneous systems in this paper. We have analyzed the stability of homogeneous systems using homgeneous eigenvalues, showing that the positive real homogeneous eigenvalues imply instability. Finally, we have shown the effectiveness of the proposed method through an example.

However, there remain many problems for homogeneous eigenvalues. For example, how many homogeneous eigenvalues actually exist? Solutions to homogeneous eigenequations often becomes complex. How do we manage 'complex eigenvalues'? When all of the eigenvalues are negative, does an homogeneous system become asymptotically stable? These are problems for future research.

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