MODELING AND CONTROL OF DYNAMICAL SYSTEMS WITH ACTIVE SINGULARITIES AND SENSING IN A SINGULAR MOTION PHASE

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Abstract: Dynamical systems with active, or controlled, singularities are characterized by constraints, either naturally present or created through actuation, capable of radically changing the attainability set of the post-impact system state. The latter is achieved through the admission of the impulsive control actions during the system engagement with the constraint. The goal of the present work is: i) to introduce the first description of mechanical systems with controlled singularities and sensing in the singular phase that admits control over observations, and ii) to demonstrate on an example the computation of a sensor-based optimal control law in the singular phase. It is shown that the optimal control in this class of problems gives rise to new concepts: the interlaced singular phase and the multi-impulse control signal.

Keywords: Mechanical systems, Controlled singularities, Control over observations

1. INTRODUCTION

Rapid progress in fast sensing and actuation compatible with the singular phases of a number of technologically significant systems has a potential of providing qualitative jump in their performance. The resulting systems dynamics, such as that of power systems under faults (Kundur, 1993) and robotic manipulators (J.W. Grizzle and Plestan, 2001), becomes, however, rather nontrivial, and the impact games, such as ping-pong, can help in providing a conceptual clarity to the behavior of these systems. In these games, the player can be viewed as generating a constraint (Pagilla and Tomizuka, 1995), (Spong, 2001) and controlling its properties during the very short duration engagement phase with the ball. This player action gives rise to a concept of *active, or controlled, constraints* (Bentsman and Miller, 2003), either naturally present or created through actuation, capable of radically changing the attainability set of the post-impact system state. The engagement phase of the system with such constraint can then be termed *active singularity*. Based on these concepts, a new class of systems, *dynamical systems with active, or controlled, singularities*, and the corresponding rigorous modeling and optimal control framework have been recently introduced by the authors (Bentsman and Miller, 2003).

Here we consider, however, a new class of the impact systems admitting the impulsive control during singular phase of their motion – systems with uncomplete observations in the singular motion phase. The present work extends the conceptual framework of (Bentsman and Miller, 2003) to include the novel mode of behavior - *the interlaced singular phase*.

2. PROBLEM STATEMENT

Let the controlled dynamical system be described by the state vector $x(t) = (x_p(t), x_v(t))$, $x_p(t) \in R^n, x_v(t) \in R^n$, where vectors x_p and x_v are referred to as the sets of *generalized positions* and *generalized velocities*, respectively.

Suppose that system motion includes interaction with some elastic constraint. Let the elastic deformation of

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the constraint be parametrized by some coefficient μ, so that for finite μ the constraint would admit a system motion, although inhibited, within the area occupied by it. Let the constraint-free domain be given by

$$\{(x_p, t) : G(x_p, t) \geq 0\} \tag{1}$$

where $G(\cdot)$ is a sufficiently smooth function.

2.1 *Motion in the Natural and Singular Phases*

Following (Bentsman and Miller, 2003) the system motion is described by

$$\begin{aligned} \dot{x}_p(t) &= F_p^r(x_p(t), x_v(t), t), \\ \dot{x}_v(t) &= F_v^r(x_p(t), x_v(t), u(t), t)+ \end{aligned} \tag{2}$$

$$+\mu F_v^{rs}(x_p(t), x_v(t), w_2^\mu(\xi, t), t, \mu) I\{G(x_p(t), t) \geq 0\}+$$
$$+\mu F_v^s(x_p(t), x_v(t), w_1^\mu(\xi, t), t, \mu) I\{G(x_p(t), t) < 0\},$$

where $I\{A\}$ is an indicator of the set A, $u(t) \in U \subset R^r$ is a control variable (a measurable function) in the natural phase, U is a compact set, $F_p^r(x_p, x_v, u, t)$ and $F_v^r(x_p, x_v, u, t)$ are the generalized forces in the natural phase, $F_v^s(x_p, x_v, w_1^\mu(\xi, t), t, \mu)$ is a controlled force arising from a contact with the constraint in the inhibited area, $w_1^\mu(\xi, t)$ is a control signal in the singular phase, $F_v^{rs}(x_p, x_v, w_2^\mu(\xi, t), t, \mu)$ is an additional force arising in the case of interlaced singular phase governed by a control signal $w_2^\mu(\xi, t)$ (a measurable function). This force characterizes the impulsive action in singular phase (Miller and Rubinovich, 2003)

Let in the singular phase, when $G(x_p(t), t) < 0$, components of the state vector $(x_p(t),\ x_v(t))$ be unobservable directly, and it be possible to observe only signal $\xi(t) \in R^k$. Then, the control variables in the singular phase can be taken to be measurable functionals of the sensor output signal $\xi(t)$ and time.

2.2 *Sensor Equations and Admissible Control in the Singular Phase*

To admit control of the sensing environment, let the sensor output signal $\xi(t)$ satisfy the equation

$$\begin{aligned} \dot{\xi}(t) &= \mu H(x_p(t), x_v(t), \alpha^\mu(\xi, t), t, \mu)\times \\ &\times I\{G(x_p(t), t) < 0\} \end{aligned} \tag{3}$$

where $\alpha^\mu(\xi, t)$ is a control signal.

Let the motion in the singular phase begin at an instant τ, where τ is the first instant when

$$G(x_p(\tau), \tau) = 0, \quad \left.\frac{d}{dt}\right|_{F_p^r} G(x_p(\tau), \tau) < 0. \tag{4}$$

Denoting by γ any of the controls $w_1, w_2,\ \alpha$, define its dependence on t and μ in the singular phase as

$$\gamma^\mu(\xi, t) = \begin{cases} \gamma(\xi, \sqrt{\mu}(t-\tau)), & t \geq \tau, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Let the following Lipschitz condition takes place

$$|\gamma(\xi', t) - \gamma(\xi'', t)| \leq L\|\xi' - \xi''\|_t, \quad L = \text{const} \tag{6}$$

where

$$\|\xi\|_t = \operatorname*{ess\,sup}_{\tau \leq s \leq t} |g(s)|, \quad \text{or} \quad \|\xi\|_t = \left(\int_\tau^t |\xi(s)|\, ds\right)^{1/2}.$$

Definition 1. Admissible control $w_1^\mu(\xi, t)$ in a singular phase is a restricted measurable functional, where dependence on τ, t, μ, ξ is given by (5) (6) and a restriction has the form $w_1^\mu(\xi, t) \in W_1 \subset R^{r_1}$. Here W_1 is a compact set. Admissible controls $w_2^\mu(\xi, t)$, $\alpha^\mu(\xi, t)$ and $\beta^\mu(\xi, t)$ are defined analogously.

It is assumed that the right hand sides of (2)-(3) are sufficiently smooth to guarantee unique solution of (2)-(3) for any admissible controls.

As in (Bentsman and Miller, 2003), the objective is to describe the behavior of system (2)-(3) in both natural and singular phases for $\mu \uparrow \infty$ and to find out if there exists the appropriate limit for its solution.

3. SPACE-TIME TRANSFORMATION AND LIMIT BEHAVIOR IN THE SINGULAR PHASE

According to (4), the singular motion phase begins at the first time τ that the system engages the constraint. Therefore, for a finite value of μ there exists a non-zero time interval of the constraint violation. The representation of motion in singular phase can be obtained with the aid of the following space-time transformation (Bentsman and Miller, 2003)

$$\begin{aligned} s &= \sqrt{\mu}\,(t-\tau), \quad t \geq \tau, \\ y_p^\mu(s) &= x_p(\tau) + \sqrt{\mu}\left[x_p(\tau + \mu^{-1/2}s) - x_p(\tau)\right], \\ y_v^\mu(s) &= x_v(\tau + \mu^{-1/2}s), \\ \eta^\mu(s) &= \xi(\tau + \mu^{-1/2}s), \end{aligned} \tag{7}$$

The next theorem describes the limit behavior of variables (7) as $\mu \to \infty$.

Assumption 1. Suppose that F_v^s (analogously F_v^{rs}) satisfies the Lipschitz condition in the following form: there exists $L > 0$, $\mu_0 > 0$ such that for any $(x_p, x_p^{'}, x_v, x_v^{'})$, $t \in [0, T]$, $w_1 \in W_1$, and $\mu \geq \mu_0$

$$\begin{aligned} &\|F_v^s(x_p, x_v, w_1, t, \mu) - F_v^s(x_p^{'}, x_v^{'}, w_1, t, \mu)\| \leq \\ &\leq L\{\|x_p - x_p^{'}\| + \mu^{-1/2}\|x_v - x_v^{'}\|\}. \end{aligned} \tag{8}$$

Theorem 1. Assume that:

1) for any (x_p, τ) such that $G(x_p, \tau) = 0$ there exists

$$\begin{aligned} &\lim_{\mu\uparrow\infty} F_p^r\Big(\frac{\bar{y}_p - x_p}{\mu^{1/2}} + x_p, \bar{y}_v, \tau + \mu^{-1/2}s\Big) = \\ &\qquad = F_p^r(x_p, \bar{y}_v, \tau), \\ &\lim_{\mu\uparrow\infty} \mu^{1/2} F_v^s\Big(\frac{\bar{y}_p - x_p}{\mu^{1/2}} + x_p, \bar{y}_v, w_1(\eta^\mu, s), \\ &\tau + \mu^{-1/2}s, \mu\Big) = \bar{F}_v^s(\bar{y}_p, \bar{y}_v, w_1(\bar{\eta}, s), x_p, \tau), \\ &\lim_{\mu\uparrow\infty} \mu^{1/2} F_v^{rs}\Big(\frac{\bar{y}_p - x_p}{\mu^{1/2}} + x_p, \bar{y}_v, w_2(\eta^\mu, s), \\ &\tau + \mu^{-1/2}s, \mu\Big) = \bar{F}_v^{rs}(\bar{y}_p, \bar{y}_v, w_2(\bar{\eta}, s), x_p, \tau), \\ &\lim_{\mu\uparrow\infty} \mu^{1/2} H\Big(\frac{\bar{y}_p - x_p}{\mu^{1/2}} + x_p, \bar{y}_v, \alpha(\eta^\mu, s), \\ &\tau + \mu^{-1/2}s, \mu\Big) = \bar{H}(\bar{y}_v, \alpha(\bar{\eta}, s), x_p, \tau), \end{aligned} \tag{9}$$

where convergence is uniform in any bounded vicinity of $(\bar{y}_p, \bar{y}_v, \bar{\eta}, \bar{\zeta}, \tau, s)$;

2) the limit system of differential equations, i.e.

$$
\begin{aligned}
&\dot{\bar{y}}_p(s) = F_p^r(x_p(\tau), \bar{y}_v(s), \tau),\\
&\dot{\bar{y}}_v(s) = \bar{F}_v^s(\bar{y}_p(s), \bar{y}_v(s), w_1(\bar{\eta}, s), x_p(\tau), \tau)+\\
&+\bar{F}_v^{rs}(\bar{y}_p(s), \bar{y}_v(s), w_2(\bar{\eta}, s), x_p(\tau), \tau),\\
&\dot{\bar{\eta}}(s) = \bar{H}(\bar{y}_v, \alpha(\bar{\eta}, s), x_p(\tau), \tau),\\
&\bar{\eta}(0) = \xi(\tau)
\end{aligned}
\tag{10}
$$

has the unique solution on some interval $[0, s^* + \varepsilon]$, where $\varepsilon > 0$ and

$$
s^* = \inf_{s>0} \left\{ \begin{array}{l} \bar{G}_t'\Big|_{(x_p(\bar{\tau}),\bar{\tau})} s \; + \bar{G}_x'\Big|_{(x_p(\bar{\tau}),\bar{\tau})} \times \\ \times\,(\bar{y}_p(s) - x_p(\bar{\tau})) = 0, \\ \bar{G}_t'\Big|_{(x_p(\bar{\tau}),\bar{\tau})} \; + \; \bar{G}_x'\Big|_{(x_p(\bar{\tau}),\bar{\tau})} \times \\ \times\,\bar{F}_p(\bar{y}_v(s), x_p(\bar{\tau}), \bar{\tau}) > 0 \end{array} \right\}. \tag{11}
$$

Then, if $\mu \to \infty$,

$$
(y_p^{\mu}(s), y_v^{\mu}(s), \eta^{\mu}(s),) \to (\bar{y}_p(s), \bar{y}_v(s), \bar{\eta}(s))
$$

uniformly on $[0, s^* + \varepsilon]$, and for all sufficiently large μ there exists

$$
s_\mu^* = \inf_{s>0} \left\{ \begin{array}{l} G(x_p(\tau + \mu^{-1/2}s), \tau + \mu^{-1/2}s, \mu) = 0, \\ G_t'\Big|_{(x_p(\tau+\mu^{-1/2}s),\tau+\mu^{-1/2}s,\mu)} + \\ + \; G_x'\Big|_{(x_p(\tau+\mu^{-1/2}s),\tau+\mu^{-1/2}s,\mu)} \times \\ \times F_p^r(x_p(\tau + \mu^{-1/2}s), x_v(\tau + \mu^{-1/2}s), \\ \tau + \mu^{-1/2}s) > 0, \end{array} \right\} \tag{12}
$$

such that $s_\mu^* \to s^*$.

The single jump representation is given by Corollary.

Corollary 1. For sufficiently small $\varepsilon > 0$ on the interval $[0, \tau + \varepsilon)$, solution of the original system (2) converges to some discontinuous functions $(\bar{x}_p(t), \bar{x}_v(t))$, such that $\bar{x}_p(t) = x_p(t)$, $\bar{x}_v(t) = x_v(t)$, $t < \tau$, and $\bar{x}_p(\tau+) = \lim_{\mu\uparrow\infty} x_p(\tau + \mu^{-1/2}s_\mu^*) = x_p(\tau)$, $\bar{x}_v(\tau+) = \lim_{\mu\uparrow\infty} x_v(\tau + \mu^{-1/2}s_\mu^*) = \bar{y}_v(s^*)$.

Definition 2. Admissible control $w_1^{\mu}(\xi, t)$ $\left(w_2^{\mu}(\xi, t)\right)$ for (2) is referred to as the *multi-impulse control* if $w_1(\bar{\eta}, s)$ $\left(w_2(\bar{\eta}, s)\right)$ in (10) exists only on the disjoint subset of the time subintervals within the interlaced singular phase time interval corresponding to the system motion within the inhibited domain.

This type of control is shown in the example of collision damping by the multi-impulse control signal.

4. MULTI-IMPULSE COLLISION DAMPING

4.1 *System Representation*

As an example, consider a ball of the unit mass colliding in a free fall with a racket of mass M moving along the vertical axis with a constant speed, as shown in Fig. 1. This system has the phase state vector $Z = (x_p, x_v, X_p, X_v)$, where x_p, x_v and X_p, X_v are the positions and the velocities of the ball and the racket, respectively. The area free of constraint is described

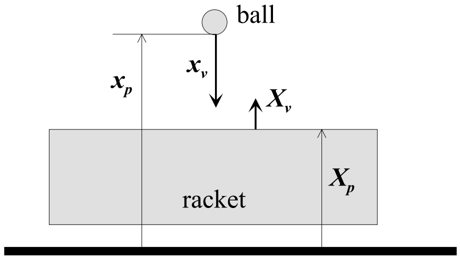

Fig. 1. The constraint-free motion phase

by the inequality

$$
G(Z) = x_p - X_p \geq 0. \tag{13}
$$

In this area the equations of motion have the form

$$
\begin{aligned}
&\dot{x}_p(t) = x_v(t), \;\; \dot{X}_p(t) = X_v(t),\\
&\dot{x}_v(t) = -g, \qquad \dot{X}_v(t) = 0,
\end{aligned}
\tag{14}
$$

where g is the acceleration due to gravity. In the inhibited area, depicted in Fig. 2, the motion is described by the equations

$$
\begin{aligned}
&\dot{x}_p(t) = x_v(t), \qquad \dot{X}_p(t) = X_v(t),\\
&\dot{x}_v(t) = -g - \mu F_v^s(Z(t), \mu),\\
&\dot{X}_v(t) = \mu M^{-1}\big[F(t, \mu) + F_v^s(Z(t), \mu)\big],
\end{aligned}
\tag{15}
$$

where $F_v^s(Z(t), \mu)$ is a visco-elastic force during the contact of the ball and the racket described by

$$
F_v^s(Z(t), \mu) = x_p(t) - X_p(t) + 2\kappa\mu^{-1/2}\left(x_v(t) - X_v(t)\right), \tag{16}
$$

and $F(t, \mu)$ could be interpreted as an external impulsive control force acting on the racket during the contact phase. This force admits the representation

$$
F(t, \mu) = \sqrt{\mu}\, w(\sqrt{\mu}(t - \tau)). \tag{17}
$$

Here $\mu > 0$ is the elasticity coefficient, $0 \leq \kappa \leq 1$ is the damping, τ is the impact time, and $w(\cdot)$ is a control variable satisfying the constraint

$$
|w(\cdot)| \leq w_0 < \infty. \tag{18}
$$

Suppose a player (or robot) has a possibility to measure the pressure on the racket during the contact phase. It means that the sensor output signal $\xi(t)$ is equal to the visco-elastic force acting on the racket:

$$
\xi(t) = x_p(t) - X_p(t) + 2\mu^{-1/2}\kappa\left(x_v(t) - X_v(t)\right). \tag{19}
$$

Equations (13)-(19) describe the continuous motion in the case of $\mu < \infty$.

4.2 *Modeling and Control Objectives*

The modeling objective is to obtain the velocity jump representation corresponding to the limit motion as $\mu \to \infty$. The control obejective is to find an impulsive control law which minimizes the velocity of the ball bounce after the impact.

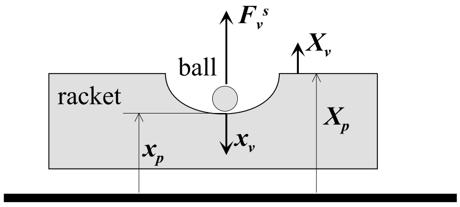

Fig. 2. The singular motion phase: motion in the inhibited domain

Applying Theorem 1 yields the following system for new variables $(y_p, y_v, Y_p, Y_v, \eta)$, describing the motion in the enlarged space-time scale:

$$\begin{aligned} &\dot{y}_p(s) = y_v(s), \qquad \dot{Y}_p(s) = Y_v(s), \\ &\dot{y}_v(s) = -y_p(s) + Y_p(s) - 2\kappa y_v(s) + 2\kappa Y_v(s), \\ &\dot{Y}_v(s) = M^{-1}\big(w(s) + y_p(s) - Y_p(s) + \\ &\qquad\qquad +2\kappa(y_v(s) - Y_v(s))\big), \qquad (20) \\ &\eta(s) = y_p(s) - Y_p(s) + 2\kappa\big(y_v(s) - Y_v(s)\big), \end{aligned}$$

with the initial conditions:

$$\begin{aligned} &y_p(0) = Y_p(0) = 0, \quad y_v(0) = x_v(\tau-), \qquad (21) \\ &Y_v(0) = X_v(\tau-),\ \eta(0) = 2\kappa\big(x_v(\tau-) - X_v(\tau-)\big). \end{aligned}$$

To address the objectives, introduce relative coordinates $q(s) = (q_p(s), q_v(s))$

$$q_p(s) = y_p(s) - Y_p(s), \quad q_v(s) = y_v(s) - Y_v(s),$$

and let $u(s) \triangleq M^{-1}w(s)$. Then, (20) takes the form

$$\begin{aligned} &\dot{q}_p(s) = q_v(s), \\ &\dot{q}_v(s) = -u(s) - aq_p(s) - 2a\kappa q_v(s), \qquad (22) \\ &\dot{y}_v(s) = -q_p(s) - 2\kappa q_v(s) = -\eta(s), \end{aligned}$$

where $a = 1 + M^{-1}$, $q_p(0) = 0$, and $q_v(0) = x_v(\tau-) - X_v(\tau-) < 0$.

4.3 *Optimal Control Problem*

Suppose the visco-elastic force is characterized by the so-called restitution (repulsive) property (Bentsman and Miller, 2003), i.e. guarantees the repulsion of the ball from the inhibited domain in a finite time without an external force. This means that the following rebound conditions take place at the instant s^* given by (11):

$$q_p(s^*) = 0, \quad \dot{q}_p(s^*) = q_v(s^*) > 0. \qquad (23)$$

In this case, system (22) admits an explicit solution

$$\begin{aligned} q_p(s) = e^{-\lambda s}\, \Bigg[&q_v(0)\,\frac{\sin(\omega s)}{\omega} - \\ &- \int_0^s e^{\lambda s'}\,\frac{\sin\omega(s - s')}{\omega}\,u(s')\,ds' \Bigg], \end{aligned} \qquad (24)$$

where $\lambda = \kappa a$, $\omega^2 = a - \lambda^2 > 0$, and, hence, the restitution condition takes the form $a^{-1/2} > \kappa$.

Setting, without loss of generality, $Y_v(0) = 0$, suppose, first, that $u(s) \equiv 0$. Then, from (22),(24)

$$y_{v0}(s_0^*) = a^{-1} y_v(0)\Big(M^{-1} - \exp\big(-\lambda\pi\omega^{-1}\big)\Big),$$

where a subscript 0 in y_{v0} and s_0^* corresponds to zero control value. In view of $y_v(0) < 0$, this implies that

(a) $y_{v0}(s_0^*) > 0$, if $M > \exp\big(\lambda\pi\omega^{-1}\big)$,

(b) $y_{v0}(s_0^*) < 0$, if $M < \exp\big(\lambda\pi\omega^{-1}\big)$,

(c) $y_{v0}(s_0^*) = 0$, if $M = \exp\big(\lambda\pi\omega^{-1}\big)$.

In the case (c), the ball stops without any external force. However, in the cases (a) and (b), the racket needs to be forced to reduce the ball velocity at instant s^*. In the latter two cases, it is natural to take

$$y_v^2(s^*) \to \min \qquad (25)$$

as a performance criterion. Applying the Pontrjagin's maximum principle to the optimal control problem (22)-(25), Hamiltonian $\mathcal{H} = \mathcal{H}(q_p, q_v, y_v, \psi_p, \psi_v, \psi_y, u)$ takes the form

$$\mathcal{H} = \psi_p q_v - \psi_v u - \big(a\psi_v + \psi_y\big)\big(q_p + 2\kappa q_v\big) \to \max, \qquad (26)$$

and the resulting optimal control is given by

$$\tilde{u}(s) = -u_0\,\mathrm{sign}\big(\psi_v(s)\big) \qquad (27)$$

where $u_0 \triangleq M^{-1}w_0$. Further on, the adjoint system is given by

$$\begin{aligned} &\dot{\psi}_p(s) = a\psi_v(s) + \psi_y(s), \\ &\dot{\psi}_v(s) = -\psi_p(s) + 2a\kappa\psi_v(s) + 2\kappa\psi_y(s), \qquad (28) \\ &\dot{\psi}_y(s) = 0, \quad \text{i.e. } \psi_y = C_y = \text{const}, \end{aligned}$$

and the terminal transversality conditions at $s = s^*$ take the form

$$2y_v\,\delta y_v + \psi_p\,\delta q_p + \psi_v\,\delta q_v + \psi_y\,\delta y_v - \mathcal{H}\,\delta s = 0, \quad (29)$$

where $\delta q_p = 0$ due to condition (23). This yields

$$2y_v(s^*) + C_y = 0,\ \ \psi_v(s^*) = 0,\ \ \mathcal{H}(s^*) = 0. \quad (30)$$

Then, $\mathcal{H}(s^*) = 0$ implies

$$\psi_p(s^*) - 2C_y = 0, \qquad (31)$$

and (28) and (31) imply

$$\dot{\psi}_v(s^*) = \psi_v(s^*) = 0. \qquad (32)$$

Equation (28) for the adjoint variable $\psi_v(s)$ that determines the sign of the optimal control signal has zero terminal conditions and may be easily integrated backwards in time. Indeed, suppose $C_y = -2y(s^*) \neq 0$. Then, defining $\vartheta = s^* - s$ and $\varphi_v(\vartheta) = C_y^{-1}\psi_v(s^* - \vartheta)$, it follows from (28) and (32) that

$$\ddot{\varphi}_v(\vartheta) + 2a\kappa\dot{\varphi}_v(\vartheta) + a\varphi_v(\vartheta) = -1, \quad \varphi_v(0) = 0. \qquad (33)$$

The latter equation admits an explicit solution

$$\varphi_v(\vartheta) = a^{-1}\left[e^{-\lambda\vartheta}\left(\cos\omega\vartheta + \lambda\omega^{-1}\sin\omega\vartheta\right) - 1\right]. \qquad (34)$$

It is easily seen that $\varphi_v(\vartheta) < 0$ for $\vartheta > 0$, so that the optimal control does not change sign. Thus, at the instant s^* two events are possible:

(i) $0 < y_v(s^*) < y_{v0}(s_0^*) < |y_v(0)|$, implying $C_y < 0$ and, further on, $\psi_v(s) = C_y\varphi_v(s^* - s) > 0$, which yields the optimal control $\tilde{u}(s) \equiv -u_0$. The racket in this case is subject to the external impulse $p_{s^*} = -w_0 s^*$ in the negative direction of the coordinate axis. However, the ball rebounds from the racket in the positive direction. The external force impulse p_{s^*} in this case is too small to stop the ball. Since the racket remains behind the ball, the damping problem for the case (i) is completed, with zero velocity of the ball not attainable.

(ii) $y_v(0) < y_{v0}(s_0^*) < y_v(s^*) < 0$, implying $C_y > 0$ and, further on, $\psi_v(s) = C_y\varphi_v(s^* - s) < 0$, which yields the optimal control $\tilde{u}(s) \equiv u_0$. The racket in this case is subject to the external impulse $p_{s^*} = w_0 s^*$ in the positive direction of the coordinate axis, and the ball retains the negative motion direction. In this case, the external force impulse p_{s^*} is too small to stop the ball as well, but the racket is now located in front of the ball, permitting a continuation of the damping problem solution.

In both cases, the terminal time s^* is found by setting the right hand side of (24) to zero, with $u(s') = -u_0$ or $+u_0$, respectively. This gives

$$e^{\lambda s^*} = \cos\omega s^* + \omega^{-1}\big(\lambda \mp u_0^{-1} a\, q_v(0)\big)\sin\omega s^*. \tag{35}$$

The ball velocity increment is calculated as

$$\Delta y_v(s^*) = y_v(s^*) - y_v(0) = -\int_0^{s^*} q_p(s)\, ds.$$

Continuing the case (ii) solution, it is easily seen that the application of the constant control $u(s) = u_0$ after instant s^* yields the next collision after a time interval $T = 2q_v(s^*)u_0^{-1}$ with $q_v(s^* + T) = -q_v(s^*) < 0$ and $q_p(s^* + T) = 0$ under the external force impulse $p_T = w_0 T$. It is seen that the phase vector (q_p, q_v, y_v) components values at $s^* + T$ can be considered as new initial data followed by the case (ii). This process can be repeated several times until the value of the initial relative velocity gets into the interval $\hat{q}_v \le q_v(0) \le 0$, where $\hat{q}_v$ is some threshold velocity. When the latter occurs, the ball gets stuck in the racket. More precisely, it starts exhibiting damped oscillatory motion, but inside an inhibited domain. Physically it means that an inertia force acting on the ball due to the racket acceleration is too large to permit the ball to leave the racket. Fig.3 illustrates this case for the following initial data:

$$a = 2, \quad \kappa = 0.25, \quad u_0 = 0.3, \quad q_p(0) = 0, \tag{36}$$

$$q_v(0) = -0.0822,\ y_v(0) = -0.3329,\ Y_v(0) = -0.2507.$$

Increasing $|q_v(0)|$ leads to the increasing amplitude of the damped oscillation of $q_p(s)$. When $q_v(0) = \hat{q}_v$ is attained, the first local maximum of $q_p(s)$ reaches zero level from below at some instant $\hat{s}$, yielding $q_p(\hat{s}) = 0$ and $q_v(\hat{s}) = 0$. This gives equations for $\hat{s}$ and $\hat{q}_v$ of the form

$$\begin{aligned} e^{\lambda\hat{s}} &= \cos\omega\hat{s} + \omega^{-1}\big(\lambda + u_0^{-1} a\hat{q}_v\big)\sin\omega\hat{s}, \\ 0 &= \hat{q}_v\cos\omega\hat{s} - \omega^{-1}\big(\lambda\hat{q}_v + u_0\big)\sin\omega\hat{s}. \end{aligned} \tag{37}$$

This critical case is illustrated in Fig.4 for the initial data (36), where $Y_v(0) = 0.2507$ has the same value and

$$q_v(0) = \hat{q}_v = -1.2302,\ y_v(0) = -1.4809,\ \hat{s} = 3.41.$$

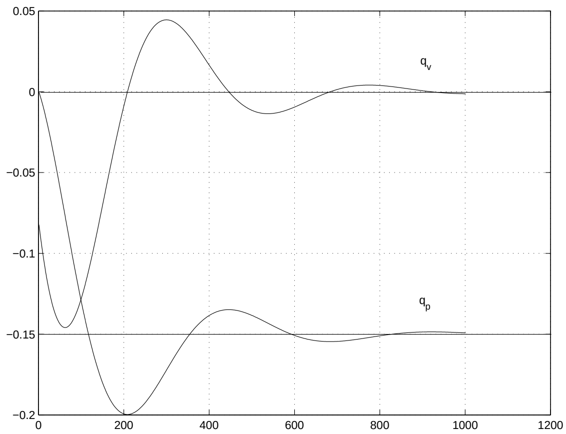

Fig. 3. Damped oscillations of relative coordinates q_p and q_v in the inhibited domain

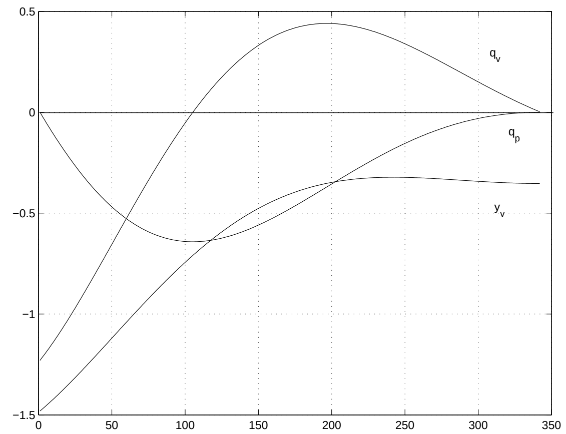

Fig. 4. Critical case: $q_v(0) = \hat{q}$. Variation of coordinates q_p, q_v and y_v in the inhibited domain

The question now arises: what should the subsequent control action be if $\hat{q}_v \le q_v(0) \le 0$? It is plausible that in this case the ball could be stopped. In the latter case, however, the optimal control problem stated above becomes degenerate due to appearance of an additional terminal condition

$$y_v(s^*) = 0. \tag{38}$$

Indeed, as it follows from (30), (31), this condition gives $C_y = \psi_y(s^*) = 0$ and $\psi_p(s^*) = 0$ leading to the trivial solution of the adjoint system (28). It becomes, therefore, necessary to reformulate the original control problem, for example, by considering different performance criteria. As an alternative criterion it is natural to take minimization of the terminal time

$$\int_0^{s^*} C\, ds \to \min, \quad C > 0 \text{ any constant.} \tag{39}$$

Hamiltonian, maximum principle, and an optimal control law have the same form (26), (27), (28) for new time-optimal control problem (22), (23), (38), (39). But due to (38) and (39), transversality conditions have the form

$$\psi_v(s^*) = 0, \qquad \mathcal{H}(s^*) = C.$$

The last equation gives a relation

$$(\psi_p(s^*) - 2\kappa C_y)q_v(s^*) = C.$$

Suppose $C_y \neq 0$. Then, denoting by $\Psi_p(s) = C_y^{-1}\psi_p(s)$, $\Psi_v(s) = C_y^{-1}\psi_v(s)$ and setting $C = C_y$, it is easy to see that these variables satisfy (28) with $\psi_y(s) = 1$ and terminal conditions

$$\Psi_v(s^*) = 0, \quad \big(\Psi_p(s^*) - 2\kappa\big)\, q_v(s^*) = 1. \tag{40}$$

As in (33), introduce an inverse time $\vartheta = s^* - s$ and denote $\Phi_v(\vartheta) = \Psi_v(s^* - \vartheta)$. Integrating the system (28) in the inverse time under (40) gives

$$\Phi_v(\vartheta) = \varphi_v(\vartheta) + e^{-\lambda\vartheta}\big(\omega q_v(s^*)\big)^{-1}\sin\omega\vartheta, \tag{41}$$

where $\varphi_v(\vartheta)$ from (34). Due to the additional term in the rhs of (41), the function $\Phi_v(\vartheta)$ changes the sign at some instant $\tilde{s}$ in the segment $[0, s^*]$. This means that the optimal control signal changes the sign as well. Fig.5 illustrates the behavior of variables $q_p(s)$, $q_v(s)$, and $y_v(s)$ for the same initial data. At the instant $\tilde{s}$ the curve q_v has a sharp bend. It is seen that the two-impulse control with impulses $p_1 = w_0\,\tilde{s}$ and $p_2 = -w_0(s^* - \tilde{s})$ stops the ball.

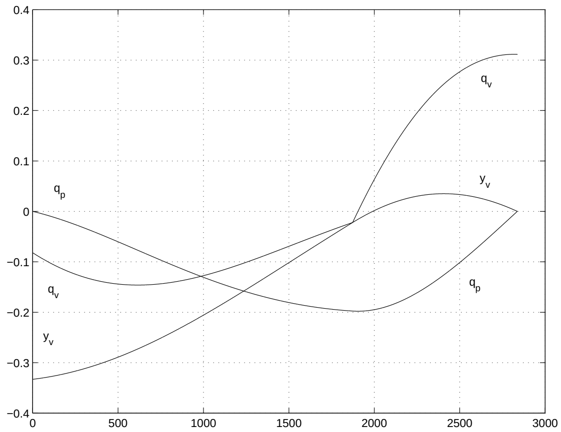

Fig. 5. Variation of q_p, q_v and y_v in the inhibited domain in the case of two impulse control.

Finally, let us consider the case when at the moment s^* the ball and the rocket have zero relative velocity, but the absolute velocity of the ball is still negative. Taking this moment as the initial one, consider the time-optimal damping problem with the initial data

$$q_p(0) = 0, \quad q_v(0) = 0, \quad y_(0) = Y_v(0) < 0.$$

This problem is solved with two-impulse control too.

Now, let us address the original task of using only the sensor output to solve the damping problem. Since the initial relative velocity $q_v(0) < 0$ is unknown, it is also not known which of the cases, (i) or (ii), takes place from the beginning. For this reason, the first control impulse should be negative, i.e. in the direction of the initial velocity of the ball $(u(s) \equiv -u_0)$. Further, the controller begins to integrate the sensor output signal $\eta(s)$ up to the time moment s^* detected by the sensor. On the basis of (22)-(24), this gives

$$J = \int_0^{s^*} \eta(s)\, ds = \int_0^{s^*} q_p(s)\, ds. \tag{42}$$

This integral is easily calculated and permits deducing the initial relative velocity $q_v(0)$ from (42) by an explicit, but rather cumbersome formula. On the bases of $q_v(0)$ and (24) the value of $q_v(s^*)$ is calculated by

$$q_v(s^*) = \frac{1}{e^{\lambda s^*}}\Big[q_v(0)\cos\omega s^* - \big(\lambda q_v(0) - u_0\big)\frac{\sin\omega s^*}{\omega}\Big].$$

By definition, the absolute ball velocity is $y_v(s^*) = q_v(s^*) + Y_v(s^*)$. The velocity of the racket $Y_v(s)$ is known to the controller. If $y_v(s^*) \geq 0$, the control process is completed. If $y_v(s^*) < 0$ the control process is continued by the algorithm describing above until the ball stops. According to Definition 2, the resulting control signal is the multi-impulse one.

From the solution of the limit problem in the singular phase it follows that at an instant τ the controller must form, on the basis of the sensor output signal, a finite number of force impulses applied to the racket, with the first one directed along the initial velocity of the ball.

5. CONCLUSION

This work presents the first input-output model of a dynamical system with active singularity that incorporates both sensing and actuation and admits observations control. The framework of (Bentsman and Miller, 2003) is extended to accommodate solutions of the optimal control problems in the singular phase through the introduction of the interlaced singular phase and the multi-impulse control concepts. An example is given that demonstrates the use of the framework proposed for the design of the observations-based multi-umpulse optimal control law for this class of systems.

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