STATE DERIVATIVE FEEDBACK BY LQR FOR LINEAR TIME-INVARIANT SYSTEMS

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Abstract: In this paper, state-derivative and especially output-derivative feedbacks for linear time-invariant systems are derived using control approach similar to linear quadratic regulator (LQR). The optimal feedback gain matrices are derived for the desired performance. This problem is always solvable for any controllable system if the open-loop system matrix is nonsingular. Explicit expression of the state-derivative gain matrix is derived. Finally, simulation results are included to show the effectiveness of the proposed approach.

Keywords: state derivative feedback, output derivative feedback, LQR.

1. INTRODUCTION

The state feedback control problem for linear time-invariant systems has been investigated in control community during the last four decades using pole placement approaches or using optimal control approaches. However, this paper focuses on a special feedback using only state derivatives instead of state feedback. Therefore this feedback is called state derivative feedback. The problem of system stabilization and/or arbitrary pole placement using state-derivative feedback naturally arises. To the best knowledge of the authors there have been yet no general study solving this feedback by pole placement or by optimal control. The problem of state derivative feedback has been investigated within the treatment of generalized class of singular linear dynamic systems using geometric approach in (Lewis and Syrmos, 1991)) and (Kucera and Loiseau, 1994). Only recently, the authors have derived (Abdelaziz and Valášek, 2004) a pole placement technique by state-derivative feedback for SISO time-invariant and time-varying linear systems and then have generalized them for MIMO systems.

The motivation for the state derivative feedback in this paper comes from controlled vibration suppression of mechanical systems. The main sensors of vibration are accelerometers. From accelerations it is possible to reconstruct velocities with reasonable accuracy but not any longer the displacements. Therefore the available signals for feedback are accelerations and velocities only and these are exactly the derivatives of states of the mechanical systems that are the velocities and displacements. There have been published many papers (e.g. (Preumont et al., 1993), (Bayon de Noyer et al., 1997), (Olgac et al., 1997), (Dyke, 1996), (Kejval et al., 2000)) describing the acceleration feedback for controlled vibration suppression. However, the pole placement approach for feedback gain determination has not been used at all or has not been solved generally.

The other problem with state derivative feedback for controlled vibration suppression of mechanical systems is that only several states are measured and are available for control. Thus output derivative feedback naturally arises. This paper deals with the application of control similar to linear quadratic regulator (LQR) for this purpose. It utilizes the optimal output feedback control of linear systems that has been solved by in papers by (Levine and Athans, 1970), (Moerder and Calise, 1985) and others with survey in (Syrmos et al., 1997).

2. STATE-DERIVATIVE FEEDBACK BY LQR

In this section, state-derivative feedback for linear time-invariant systems using LQR similar approach is derived.

2.1 LQR problem formulation

Consider a continuous, time-invariant, linear system

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t),\ \ \boldsymbol{x}(t_0)=\boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state, and $\boldsymbol{u}(t)\in\mathbb{R}^m$ is the control vector, ($m \le n$), while $\boldsymbol{A}\in\mathbb{R}^{n\times n}$ and $\boldsymbol{B}\in\mathbb{R}^{n\times m}$ are the system and control gain matrices, respectively. The fundamental assumption imposed on the system is that the system is completely controllable. Further it is assumed that system matrix $\boldsymbol{A}$ is of full rank.

The objective is to stabilize the system by means of a linear state-derivative feedback

$$\boldsymbol{u}(t)=-\boldsymbol{K}\dot{\boldsymbol{x}}(t) \tag{2}$$

that stabilizes the system and achieves the desired performance. The closed-loop system dynamics is

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{A}_c\boldsymbol{x}(t)\ ,\ \ \boldsymbol{A}_c=(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{A} \tag{3}$$

where $\boldsymbol{I}_n$ is the $n\times n$ identity matrix. It is further assumed that the matrix $(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K})$ is of full rank in order that the closed-loop system is well defined.

The stabilizing control with good dynamic behaviour is achieved control design that minimize a quadratic cost or performance index of the type

$$J(\dot{\boldsymbol{x}}(t),\boldsymbol{u}(t))=\min_u\int_0^\infty(\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{x}}(t)+\boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{u}(t))dt \tag{4}$$

where $\boldsymbol{Q}$ is an $n\times n$ positive-definite (or positive-semidefinite) symmetric state-derivative weighting matrix and $\boldsymbol{R}$ is an $m\times m$ positive-definite symmetric control weighting matrix. This formulation is only similar to the original LQR one as the performance index is based on state derivatives instead of states. Nevertheless, similar properties as original LQR will be derived.

Substituting (2) into J, the performance index is

$$\begin{aligned}J&=\int_0^\infty(\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{Q}\dot{\boldsymbol{x}}+(\boldsymbol{K}\dot{\boldsymbol{x}})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{K}\dot{\boldsymbol{x}}))dt=\\&=\int_0^\infty(\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K})\dot{\boldsymbol{x}})dt\end{aligned} \tag{5}$$

The design problem is to select the feedback gain $\boldsymbol{K}$ so that J is minimized subject to the dynamical constraint (3). Then the LQR problem with state-derivative feedback for linear systems is formulated as follows:

<u>Problem 1:</u> Given the linear dynamical system (1) and the symmetric matrices $\boldsymbol{Q}\ge 0$ and $\boldsymbol{R}>0$. Find the real feedback gain matrix $\boldsymbol{K}\in\mathbb{R}^{m\times n}$ in the control input (2) that minimizes the value of quadratic performance index (5) and stabilizes the closed-loop system (3) for any initial state x_0.

2.2 Linear quadratic regulator analysis

Our main objective is to minimize the performance index function in (5) with respect to the feedback gain $\boldsymbol{K}$. Suppose that we can find a constant positive-semidefinite symmetric matrix $\boldsymbol{P}$ that satisfy (5), then

$$\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K})\dot{\boldsymbol{x}}=-\frac{d}{dt}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x})=-\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{x}} \tag{6}$$

Therefore, the performance index can be evaluated as

$$\begin{aligned}J&=\int_0^\infty(\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K})\dot{\boldsymbol{x}})dt=-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}\Big|_0^\infty=\\&=-\boldsymbol{x}^{\mathrm{T}}(\infty)\boldsymbol{P}\boldsymbol{x}(\infty)+\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)\end{aligned} \tag{7}$$

Assuming that the closed-loop system is asymptotically stable, i.e. all eigenvalues of $\boldsymbol{A}_c$ have negative real parts, so that $\boldsymbol{x}(t)$ vanishes with time and $\boldsymbol{x}(\infty)\rightarrow 0$. Therefore, the performance index converges to the positive optimal value

$$J=\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0) \tag{8}$$

Thus the performance index J can be obtained in terms of the initial conditions $\boldsymbol{x}(0)$ and matrix $\boldsymbol{P}$. From (3) one can obtain the following relation

$$\boldsymbol{x}=\boldsymbol{A}_c^{-1}\dot{\boldsymbol{x}},\ \ \boldsymbol{A}_c^{-1}=\boldsymbol{A}^{-1}(\boldsymbol{I}_n+\boldsymbol{B}\boldsymbol{K}) \tag{9}$$

Then, equation (6) can be rewritten as

$$\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K})\dot{\boldsymbol{x}}=-\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{A}_c^{-1}+\boldsymbol{A}_c^{-\mathrm{T}}\boldsymbol{P})\dot{\boldsymbol{x}} \tag{10}$$

Comparing both sides of the above equation

$$\boldsymbol{P}\boldsymbol{A}_c^{-1}+\boldsymbol{A}_c^{-\mathrm{T}}\boldsymbol{P}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{K}+\boldsymbol{Q}=0 \tag{11}$$

By the second method of Lyapunov, if $\boldsymbol{A}_c$ is stable matrix, there exists a positive-definite matrix $\boldsymbol{P}$ that satisfies the above equation. Hence, our procedure is to determine matrix $\boldsymbol{P}$. From (9) we can write

$$\boldsymbol{A}_c^{-1}=\boldsymbol{A}^{-1}+\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{K},\text{and }\boldsymbol{A}_c^{-\mathrm{T}}=\boldsymbol{A}^{-\mathrm{T}}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-\mathrm{T}} \tag{12}$$

Substituting in (11) one can obtain

$$P(A^{-1}+A^{-1}BK)+(A^{-T}+K^{T}B^{T}A^{-T})P+ \\ +K^{T}RK+Q=0 \tag{13}$$

$$PA^{-1}+A^{-T}P+PA^{-1}BK+K^{T}B^{T}A^{-T}P+ \\ +K^{T}RK+Q=0 \tag{14}$$

Since $\boldsymbol{R}$ is positive-definite symmetric matrix

$$R = T^{T}T, \tag{15}$$

where $\boldsymbol{T}$ is a nonsingular matrix. Substituting in (14)

$$PA^{-1}+A^{-T}P+PA^{-1}BK+K^{T}B^{T}A^{-T}P+ \\ +K^{T}T^{T}TK+Q=0 \tag{16}$$

which can be reformulated as

$$PA^{-1}+A^{-T}P+(TK+T^{-T}B^{T}A^{-T}P)^{T}(TK+ \\ +T^{-T}B^{T}A^{-T}P)-PA^{-1}BR^{-1}B^{T}A^{-T}P+Q=0 \tag{17}$$

The minimization of J requires the minimization of

$$\dot{x}^{T}(TK+T^{-T}B^{T}A^{-T}P)^{T}(TK+T^{-T}B^{T}A^{-T}P)\dot{x} \tag{18}$$

with respect to $\boldsymbol{K}$. Since this last expression is nonnegative, the minimum occurs when it is zero

$$TK=-T^{-T}B^{T}A^{-T}P \tag{19}$$

The optimal gain matrix $\boldsymbol{K}$ is

$$K=-T^{-1}T^{-T}B^{T}A^{-T}P=-R^{-1}B^{T}A^{-T}P \tag{20}$$

Finally, the optimal stabilizing control law is given by

$$u(t)=-K\dot{x}(t)=R^{-1}B^{T}A^{-T}P\dot{x}(t) \tag{21}$$

The matrix P in (21) must satisfy (14) or the following algebraic Riccati equation (ARE)

$$PA^{-1}+A^{-T}P-PA^{-1}BR^{-1}B^{T}A^{-T}P+Q=0 \tag{22}$$

2.3 Linear quadratic regulator solution

The computation of LQR similar state derivative feedback is transformed into the solution of the corresponding matrix Riccati equation (22). There are many efficient algorithms for its solution. Using the well known theorem about unique solvability of Riccati equation (e.g. Lewis, 1992) it follows that the Riccati equation (22) has unique positive semidefinite solution if the pair (A^{-1}, $\boldsymbol{B}$) is stabilizable and ($\sqrt{Q}$, A^{-1}) is observable Based on that the necessary and sufficient conditions for the existence of LQR similar problem with state-derivative feedback can be proven. Then the stabilization problem of state derivative feedback is transformed into the solution of the equation (22).

Theorem 1
The LQR similar problem of state derivative feedback for the real pair ($\boldsymbol{A}$, $\boldsymbol{B}$) is solvable if ($\boldsymbol{A}$, $\boldsymbol{B}$) is stabilizable, ($\sqrt{Q}$, $\boldsymbol{A}$) is observable and $\boldsymbol{A}$ is nonsingular.
Proof: The pair ($\boldsymbol{A}$, $\boldsymbol{B}$) is stabilizable means that the pair ($\boldsymbol{A}$, $\boldsymbol{B}$) is controllable, i.e. the controllability matrix has full rank. The system matrix **A** is nonsingular and using the state transformation by the matrix **A** it follows that the controllability matrix ($\mathbf{A}^{-1}$, **B**) has also full rank. Similarly if the pair ($\sqrt{Q}$, $\boldsymbol{A}$) is observable then also the pair ($\sqrt{Q}$, A^{-1}) is observable. Hence the resulting closed-loop system matrix Ac is stable. ■
Comment: The state-derivative feedback by LQR can be also derived from traditional state feedback by LQR. Let substitute (1) into the performance index (4) obtaining the traditional LQR problem $J =$

$$=\int_{0}^{\infty}(x^{T}A^{T}QAx+u^{T}(R+B^{T}QB)u+2x^{T}A^{T}QBu)dt$$

It results into the optimal feedback gain K_{opt}. Then

$$u_{opt}=-K_{opt}x=-K\dot{x}=-K(Ax+Bu_{opt})= \\ =-K(Ax+B(-K_{opt}x))=-K(Ax-BK_{opt})x.$$

It gives $K=K_{opt}(A-BK_{opt})^{-1}$. However, to derive the output-derivative feedback by LQR from traditional output feedback by LQR is difficult. Therefore the derivation is provided in this way.

3. OUTPUT-DERIVATIVE FEEDBACK BY LQR

In many practical applications, a complete set of state-derivatives is not directly available for feedback purposes. Therefore, the LQR with output-derivative is proposed that utilize only a few measurements of the system. Consider a time-invariant linear system

$$\dot{x}(t)=Ax(t)+Bu(t),\ \dot{x}(t_0)=\dot{x}_0 \\ \dot{y}(t)=C\dot{x}(t) \tag{23}$$

where $x(t)\in\mathbb{R}^{n}$ is the state, $\dot{y}(t)\in\mathbb{R}^{r}$ is the measured output and $u(t)\in\mathbb{R}^{m}$ is the control input, ($m \le n$), while $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{r\times m}$ are the system, control and output gain matrices, respectively. Again the system is supposed to be completely controllable and observable and the system matrix $\boldsymbol{A}$ to have full rank. The objective is to stabilize the system by means of a linear output-derivative feedback control

$$u(t)=-F\dot{y}(t) \tag{24}$$

that stabilize the system and achieve the desired performance of the closed-loop system

$$\dot{x}(t)=A_{co}x(t),\ A_{co}=(I_n+BFC)^{-1}A \tag{25}$$

In what follows, we assume that ($I_n + BFC$) has a full rank in order that the closed-loop system is well defined. Additionally, A_{co} is to be asymptotically stable. This may achieved by selecting the control input $\boldsymbol{u}(t)$ to minimize a quadratic performance index

$$J(\dot{\boldsymbol{x}}(t), \boldsymbol{u}(t)) = \min_{u} \int_0^\infty (\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{Q}\dot{\boldsymbol{x}}(t) + \boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{u}(t))dt \tag{26}$$

where $\boldsymbol{Q} \geq 0$ is symmetric state-derivative weighting matrix and $\boldsymbol{R} > 0$ is symmetric control weighting matrix. Substituting (24) into J, the PI is

$$\begin{aligned} J &= \int_0^\infty (\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{Q}\dot{\boldsymbol{x}} + (\boldsymbol{F}\boldsymbol{C}\dot{\boldsymbol{x}})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{F}\boldsymbol{C}\dot{\boldsymbol{x}}))dt = \\ &= \int_0^\infty (\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\dot{\boldsymbol{x}})dt \end{aligned} \tag{27}$$

Then the LQR problem with output-derivative feedback for linear systems is formulated as follows:
<u>Problem 2:</u> Given the linear dynamical system (23) and the symmetric matrices $\boldsymbol{Q} \geq 0$ and $\boldsymbol{R} > 0$. Find the feedback gain matrix $\boldsymbol{F} \in \mathbb{R}^{m \times r}$ in the control (24) that minimizes the value of PI (27) and stabilizes the closed-loop system (25) for any initial state x_0.

3.1 Output linear quadratic regulator analysis

Suppose that we can find a constant, positive-semidefinite, symmetric matrix P that satisfy (27), then

$$\begin{aligned} \dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\dot{\boldsymbol{x}} &= -\frac{d}{dt}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}) = \\ &= -\dot{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{x}} \end{aligned} \tag{28}$$

The performance index can be evaluated as

$$\begin{aligned} J &= \int_0^\infty (\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\dot{\boldsymbol{x}})dt = -\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}\Big|_0^\infty = \\ &= -\boldsymbol{x}^{\mathrm{T}}(\infty)\boldsymbol{P}\boldsymbol{x}(\infty) + \boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0) \end{aligned} \tag{29}$$

Since we assume that the closed-loop system is asymptotically stable, then $x(\infty) \to 0$, it holds

$$J = \boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0) \tag{30}$$

Thus, the performance index can be obtained in terms of the initial conditions $x(0)$ and $\boldsymbol{P}$. Using

$$\boldsymbol{x} = \boldsymbol{A}_{co}^{-1}\dot{\boldsymbol{x}},\ \boldsymbol{A}_{co}^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{I}_n + \boldsymbol{B}\boldsymbol{F}\boldsymbol{C}) \tag{31}$$

the equation (28) can be rewritten as

$$\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{Q} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}\boldsymbol{C})\dot{\boldsymbol{x}} = -\dot{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{A}_{co}^{-1} + \boldsymbol{A}_{co}^{-\mathrm{T}}\boldsymbol{P})\dot{\boldsymbol{x}} \tag{32}$$

Comparing both sides of (32) for all state-derivative trajectories, Lyapunov equation is obtained

$$g \equiv \boldsymbol{P}\boldsymbol{A}_{co}^{-1} + \boldsymbol{A}_{co}^{-\mathrm{T}}\boldsymbol{P} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}\boldsymbol{C} + \boldsymbol{Q} = 0 \tag{33}$$

If $\boldsymbol{F}$ and $\boldsymbol{Q}$ are given a constant, symmetric, positive-semidefinite matrix $\boldsymbol{P}$ may be computed from this equation. Now, we may write the PI as

$$J = \mathrm{tr}(\boldsymbol{P}\boldsymbol{X}) \tag{34}$$

where the $n \times n$ symmetric matrix $\boldsymbol{X}$ is defined by

$$\boldsymbol{X} \equiv \boldsymbol{x}(0)\boldsymbol{x}^{\mathrm{T}}(0) \tag{35}$$

Therefore, the problem of selecting $\boldsymbol{F}$ to minimize J subject to the dynamical constraint (25) on the state-derivative is equivalent to the algebraic problem of selecting $\boldsymbol{F}$ to minimize J subject to constraint g on the auxiliary matrix $\boldsymbol{P}$. To solve this modified problem, we use the Lagrange multiplier approach to modify the problem yet again according to (Lewis, 1992). Thus, adjoin the constraint to the PI by defining the Hamiltonian function

$$H = \mathrm{tr}(\boldsymbol{P}\boldsymbol{X}) + \mathrm{tr}(\boldsymbol{g}\boldsymbol{S}) \tag{36}$$

with $\boldsymbol{S}$ a symmetric $n \times n$ matrix of Lagrange multipliers which still needs to be determined. Then our constrained optimization problem is equivalent to the simpler problem of minimizing (36) without constraints. Taking the partial derivatives of H with respect to all the independent variables $\boldsymbol{P}$, $\boldsymbol{S}$ and $\boldsymbol{F}$ equal to zero and utilizing that

$$\boldsymbol{A}_{co}^{-1} = \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{F}\boldsymbol{C},\ \boldsymbol{A}_{co}^{-\mathrm{T}} = \boldsymbol{A}^{-\mathrm{T}} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-\mathrm{T}} \tag{37}$$

Then the necessary conditions for the solution of LQR problem with output-derivative feedback are

$$0 = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{S}} = g = \boldsymbol{P}\boldsymbol{A}_{co}^{-1} + \boldsymbol{A}_{co}^{-\mathrm{T}}\boldsymbol{P} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{F}\boldsymbol{C} + \boldsymbol{Q} \tag{38}$$

$$0 = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{P}} = \boldsymbol{A}_{co}^{-1}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{A}_{co}^{-\mathrm{T}} + \boldsymbol{X} \tag{39}$$

$$0 = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{F}} = 2\boldsymbol{R}\boldsymbol{F}\boldsymbol{C}\boldsymbol{S}\boldsymbol{C}^{\mathrm{T}} + 2\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{P}\boldsymbol{S}\boldsymbol{C}^{\mathrm{T}} \tag{40}$$

The first two of these are Lyapunov equations and the third is an equation for the feedback gain $\boldsymbol{F}$. If $\boldsymbol{S} > 0$ in (41) then $\boldsymbol{CSC}^{\mathrm{T}}$ is nonsingular, then (40) can be solved to obtain the optimal output-derivative feedback gain as

$$\boldsymbol{F} = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{P}\boldsymbol{S}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{C}\boldsymbol{S}\boldsymbol{C}^{\mathrm{T}})^{-1} \tag{41}$$

Unfortunately, in many applications the initial state-derivatives of the system $\boldsymbol{x}(0)$ are usually unknown, so the optimal performance index can be obtained but with its expected value, that is $\mathrm{E}\{J\}$. We assume that $\boldsymbol{x}(0)$ is uniformly distributed on the unit sphere and $\boldsymbol{X} = \boldsymbol{I}_n$. Then

$$J = \mathrm{tr}(\boldsymbol{P}) \tag{42}$$

To obtain the output-derivative feedback gain minimizing the performance index (26), we need to solve the three coupled equations (38)-(39). The equations for $\boldsymbol{P}$, $\boldsymbol{S}$ and $\boldsymbol{F}$ are coupled nonlinear matrix equations in three unknown. Numerical techniques can be used for solving these matrix equations (Moerder and Calise, 1985, Lewis, 1992). The iterative numerical technique varies $\boldsymbol{F}$ based on changes in J. There are more than one local minimum and global optimality is not guaranteed. The found optimal gain may depend on the initial guess that must guarantee an initial stabilizing controller, which is also a nontrivial problem. However, the determination of the globally optimal solution is still a difficult task. The computational algorithm for solving the LQR problem with output-derivative feedback is following:

Algorithm: *Input:* Real matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, where $\boldsymbol{A}$ is nonsingular, and symmetric weighting matrices $\boldsymbol{Q} \geq 0$ and $\boldsymbol{R} > 0$.

Step 1: Initialize: Set $k = 0$, and determine a gain $\boldsymbol{F}_0$ so that $(\boldsymbol{I}_n + \boldsymbol{BF}_0\boldsymbol{C})^{-1}\boldsymbol{A}$ is asymptotically stable.

Step 2: k-th iteration: Set $\boldsymbol{A}_k = (\boldsymbol{I}_n + \boldsymbol{BF}_k\boldsymbol{C})^{-1}\boldsymbol{A}$, and solve for $\boldsymbol{P}_k$ and $\boldsymbol{S}_k$ in

$$\begin{aligned} 0 &= \boldsymbol{PA}_k^{-1} + \boldsymbol{A}_k^{-\mathrm{T}}\boldsymbol{P} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{RFC} + \boldsymbol{Q} \\ 0 &= \boldsymbol{A}_k^{-1}\boldsymbol{S} + \boldsymbol{SA}_k^{-\mathrm{T}} + \boldsymbol{X} \end{aligned} \tag{43}$$

Set $J_k = \mathrm{tr}(\boldsymbol{P}_k\boldsymbol{X})$ and evaluate the gain update direction

$$\Delta\boldsymbol{F} = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{PSC}^{\mathrm{T}}(\boldsymbol{CSC}^{\mathrm{T}})^{-1} - \boldsymbol{F}_k \tag{44}$$

Update the gain by $\boldsymbol{F}_{k+1} = \boldsymbol{F}_k + \alpha\Delta\boldsymbol{F}$ where α is chosen so that $(\boldsymbol{I}_n + \boldsymbol{BF}_{k+1}\boldsymbol{C})^{-1}\boldsymbol{A}$ is asymptotically stable and $J_{k+1} = \mathrm{tr}(\boldsymbol{P}_{k+1}\boldsymbol{X}) \leq J_k$. If J_{k+1} and J_k are close enough to each other, go to Step 3, otherwise, set $k = k+1$ and go to Step 2.

Step 3: Terminate: Set the optimal output-derivative gain matrix is $\boldsymbol{F} = \boldsymbol{F}_{k+1}$ and $J = J_{k+1}$.

The aforementioned algorithm requires the selection of an initial stabilization gain matrix $\boldsymbol{F}_0$. In this work, first the full state-derivative feedback problem is solved using previous technique, and then it constructs the initial stabilizing output-derivative feedback gain matrix by solving the following equation in the least-square sense

$$\boldsymbol{K} = \boldsymbol{F}_0\boldsymbol{C} \tag{45}$$

where $\boldsymbol{K}$ is the full state-derivative feedback gain matrix.

4 ILLUSTRATIVE EXAMPLE

The mechanical system of vibration isolation is in Fig. 1. The dynamic equation of this system, assuming small angle φ, can be described in the state-space form using the state vector $\boldsymbol{x}(t) = [x_1 \ x_2 \ \dot{x}_1 \ \dot{x}_2]^{\mathrm{T}}$ as:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1c_1 & -k_2c_2 & -b_1c_1 & -b_2c_2 \\ -k_1c_2 & -k_2c_1 & -b_1c_2 & -b_2c_1 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \\ c_2 & c_1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $c_1 = \dfrac{1}{m} + \dfrac{L^2}{I}$, $c_2 = \dfrac{1}{m} - \dfrac{L^2}{I}$, $x_3 = 0.5(x_1 + x_2)$ and $\varphi = 0.5(x_1 - x_2)/L$, m and I represent the mass and inertia of the mass, k_1 and k_2 are the spring constants, b_1 and b_2 are the damper constants, x_1 and x_2 are the mass displacement from both sides, x_3 is the vertical displacement of the center of mass, φ is the inclination angle of the mass with the horizontal, $2L$ is the distance between two supporting points, and u_1 and u_2 are the control inputs.

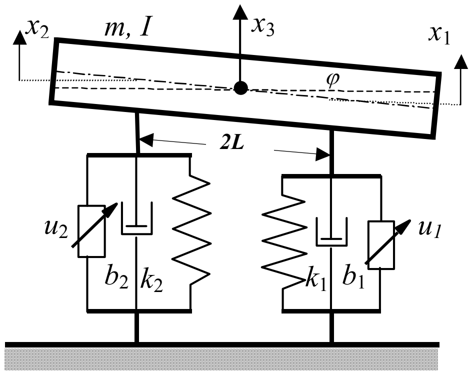

Fig. 1 Vibration isolation example

The model parameters are taken as m = 10 kg, I = 1 kg.m^2, L = 1 m, k_1 = 500 N/m, k_2 = 700 N/m, b_1 = 10 N.s/m and b_2 = 20 N.s/m. The original system poles are {–15.1384±31.1738j and –1.3616±10.7106j}.

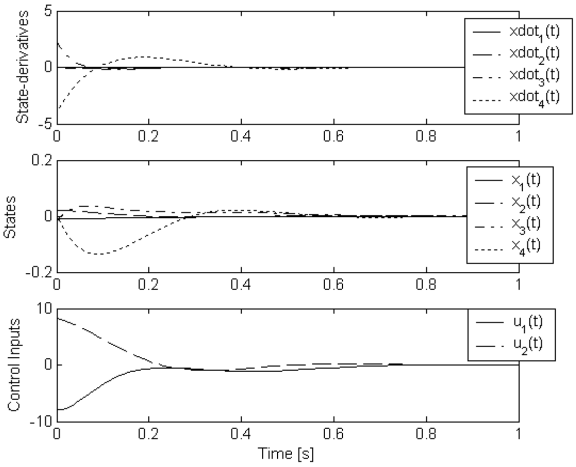

Fig. 2 Response using state-derivative feedback.

First the <u>*LQR with state-derivative feedback*</u> is computed. The performance index weighting matrices Q and R are chosen as $\boldsymbol{Q}$ = diag{10000, 10, 10, 10} and $\boldsymbol{R}$ = diag{1, 1}. The feedback gain matrix is

$$\boldsymbol{K} = \begin{pmatrix} 98.0081 & -6.5270 & 1.5875 & -1.5119 \\ -4.6622 & 35.6412 & -0.2978 & 1.9490 \end{pmatrix}.$$

Then the eigenvalues of the system are {–23.1893, –5.9365 and –5.6331±10.1242j}. The transient response of the closed-loop system is shown in Fig. 2 from the initial state x_0 = [–0.01, 0.02, –0.02, 0.01]$^{\mathrm{T}}$.

Then the <u>*LQR with output-derivative feedback*</u> is computed. The output vector, that utilize only acceleration measurements of the mass, can be obtained as

$$\dot{\boldsymbol{y}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \dot{\boldsymbol{x}}\,.$$

For simulation, the weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ of the performance index are $\boldsymbol{Q}$ = diag{1e8, 1, 1, 1} and $\boldsymbol{R}$ = diag{1, 1}. The initial gain is taken as

$$\boldsymbol{F}_0 = \begin{pmatrix} -1.1950 & -2.2384 \\ 1.0977 & 0.9412 \end{pmatrix}.$$

The computed feedback gain matrix is

$$\boldsymbol{F} = \begin{pmatrix} -1.0213 & -1.0896 \\ 3.5659 & 1.2723 \end{pmatrix}.$$

The resulting closed-loop eigenvalues are {–313.4212, –49.8538, and –1.3879 ± 10.3777j}. The simulation results are displayed in Fig. 3 from initial state x_0 = [–0.01, 0.02, –0.02, 0.01]^T. In particular, the performance index J decreases during iterations from 2.4615*10^9 to 1.5160*10^9.

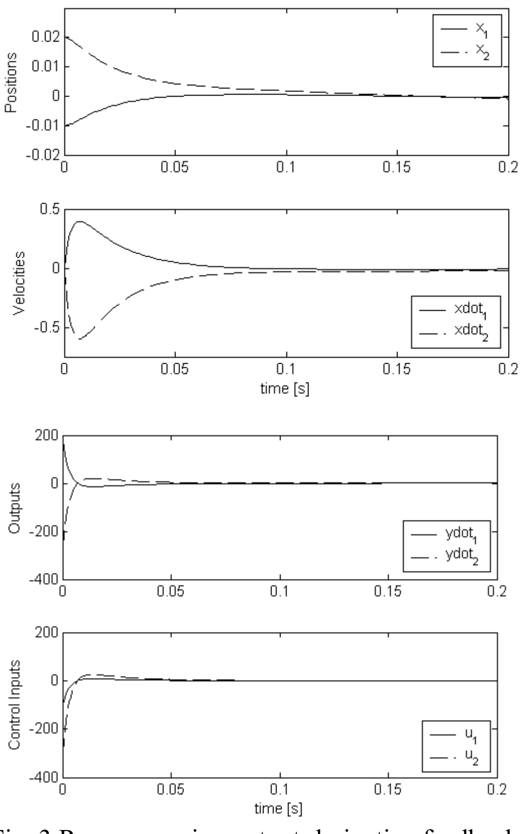

Fig. 3 Response using output-derivative feedback.

5. CONCLUSIONS

This paper has presented a linear quadratic regulator similar control with state-derivative and output-derivative feedbacks for linear time-invariant systems. The optimal gains for the LQR are derived. The necessary conditions to ensure solvability are that the system is controllable and the open-loop system matrix is nonsingular. The main result of this work is an efficient computational algorithm for solving the optimal linear quadratic regulator with state-derivative and output-derivative feedbacks. The simulation results prove the feasibility and effectiveness of the proposed technique.

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