ADAPTATION AND NONLINEAR PARAMETRIZATION: NONLINEAR DYNAMICS PROSPECTIVE

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Abstract: We consider adaptive control problems in the presence of nonlinear parametrization of model uncertainties. An approach that foregoes on the need for domination in the control loop during adaptation is proposed. Our approach is based on the notions of attractivity, limit sets, equilibria, and multistability from the theory of nonlinear dynamical systems rather than on the conventional method of Lyapunov functions. As a result of this, our algorithms are applicable to general smooth non-monotonic parametrization and do not require any damping or domination in control inputs.

Keywords: adaptive systems, nonlinear parameterization, finite form algorithms, convergence, nonlinear persistent excitation

1. INTRODUCTION

Nonlinear parametrization of the uncertainty is traditionally addressed in the adaptive control literature using either domination functions (Marino and Tomei, 1993; Lin and Qian, 2002*a*; Lin and Qian, 2002*b*) or damping of the unknown nonlinearity (Loh *et al.*, 1999). Yet, in a wide range of applications this is undesirable desirable as it leads to overshooting, fast wear of the actuators, waste of energy, and undesired chattering. A typical example is traction/braking control. Unknown tyre-road conditions enter into the equations of the slip dynamics as uncertainties that are nonlinear in parameters (Pacejka and Bakker, 1993; Canudas de Wit and Tsiotras, 1999).

In bio-engineering, the motivation to use nonconventional adaptation mechanisms is even stronger. Physiological considerations motivate nondominating adaptation at the level of the single neural units (Webster *et al.*, 2002), of which the mathematical models are nonlinearly parameterized (Dayan and Abbott, 2001).

Problems like these call for gentle adaptation in nonlinear parameterized systems. Present solutions, however, are either local (Karsenti *et al.*, 1996), or assume monotonic parametrization of uncertainties (Tyukin, 2003; Tyukin *et al.*, 2003*b*).

At the same time, conventional design methods in adaptive control theory often favor a Lyapunov-based methodology (Narendra and Annaswamy, 1989; Miroshnik *et al.*, 1999) for its "continuity property": small perturbations in initial conditions result in small deviations from equilibrium. When adaptive control is required, however, deviations in the parameters are likely to be large. Therefore, the requirement of Lyapunov stability does not seem to be necessary unless it provides specific advantages in applications in addition to mere asymptotic reaching of the control goal.

Adaptation processes in many physical and biological phenomena are far from being stable in conventional sense (for example, tremor in the eye (Martinez-Conde *et al.*, 2004), perceptual switching (Ito *et al.*, 2003)). It is often the unperturbed dynamics in these systems that is not asymptotically stable (Moreau and Sontag, 2003; Corral *et al.*, 1995). The relevance unstable regimes for engineering has been discussed, for instance, in (Kaneko and Tsuda, 2000; Kaneko, 1994). These examples motivate us to abandon conventional Lyapunov methodology for problems of adaptation in nonlinear systems.

We start with a review of the design strategies for adaptive systems with nonlinear parametrization of the uncertainties, showing that in general case conventional Lyapunov-based design leads to domination functions in control. As an alternative we suggest a nonlinear dynamics framework in which we replace the stability requirement with mere reaching a neighborhood of the target set. We provide adaptive control algorithms capable of steering the system state to a small neighborhood of the target set. The resulting control algorithms involve neither domination functions nor additional damping of unknown nonlinearity.

Throughout the paper we use the following notations. Symbol $\mathbf{x}(t,\mathbf{x}_0,t_0)$ denotes solution of a system of differential equations starting at the point $\mathbf{x}_0$ at time instant t_0; symbol C^r denotes the space of r times differentiable functions; symbol $\mathbb{R}$ stand for the space of reals; $\mathbb{R}_+$ defines non-negative real numbers, symbol $\mathcal{I}m$ denotes image of the map. We say that $\nu:\mathbb{R}_+\rightarrow\mathbb{R}$ belongs to L_2 iff $L_2(\nu)=\int_0^\infty \nu^2(\tau)d\tau<\infty$. The value $\|\nu\|_2=\sqrt{L_2(\nu)}$ stands for the L_2 norm of $\nu(t)$. Function $\nu:\mathbb{R}_+\rightarrow\mathbb{R}$ belongs to and L_∞ iff $L_\infty(\nu)=\sup_{t\geq 0}\|\nu(t)\|<\infty$, where $\|\cdot\|$ is the Euclidean norm. The value of $\|\nu\|_\infty=L_\infty(\nu)$ stands for the L_∞ norm of $\nu(t)$. Symbol $\mathcal{U}_\epsilon(\mathbf{x})$ denotes the set of all $\mathbf{x}':\|\mathbf{x}-\mathbf{x}'\|\leq\epsilon$. Let $\mathcal{X}\subset\mathbb{R}^n$, distance $\mathrm{dist}(\mathbf{x},\mathcal{X})=\inf_{\mathbf{x}'\in\mathcal{X}}\|\mathbf{x}-\mathbf{x}'\|$. Symbol $\mathcal{U}_\epsilon(\mathcal{X})$ denotes the set $\{\mathbf{x}'\in\mathbb{R}^n|\,\mathbf{x}':\,\mathrm{dist}(\mathbf{x}',\mathcal{X})\leq\epsilon\}$. Symbol $L_\mathbf{f}\psi$ stands for the Lie-derivative of function $\psi(\mathbf{x})$ w.r.t. the vector field $\mathbf{f}(\mathbf{x})$.

2. PROBLEM FORMULATION

Let the uncertain system be given as follows:

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\boldsymbol{\theta})+\mathbf{g}(\mathbf{x})u,\ \mathbf{x}_0\in\Omega_\mathbf{x} \qquad (1)$$

where $\mathbf{x}\in\mathbb{R}^n$ is state vector, $\mathbf{f}:\mathbb{R}^n\times\mathbb{R}^d\rightarrow\mathbb{R}^n$, $\mathbf{g}:\mathbb{R}^n\rightarrow\mathbb{R}^n$ are C^1-smooth vector-fields, $\boldsymbol{\theta}\in\Omega_\theta\subset\mathbb{R}^d$ is a vector of parameters, u is control input, and $\Omega_\mathbf{x}\subset\mathbb{R}^n$ is the set of initial conditions $\mathbf{x}_0$. Functions $\mathbf{f}$, $\mathbf{g}$ are known, the vector $\boldsymbol{\theta}$ is unknown, $\mathbf{g}(\mathbf{x})$ is bounded, and the control goal is to reach asymptotically the following set:

$$\{\mathbf{x}\in\mathbb{R}^n|\mathbf{x}:\psi(\mathbf{x})=0\},\ \psi\in C^2 \qquad (2)$$

In addition to (2) we will require that

$$\psi(\mathbf{x}(t))\in L_\infty\Rightarrow\mathbf{x}(t)\in L_\infty \qquad (3)$$

This ensures that bounded deviations from the target set result in the bounded state $\mathbf{x}(t)$. In addition we assume that $|L_\mathbf{g}(\mathbf{x})\psi(\mathbf{x}))|\geq\delta_\psi>0$. Let us select the class of admissible feedbacks which can compromise between performance and domination issues. The most natural way is to define this class on the ground of the *certainty-equivalence principle*. Consider control function

$$u(\mathbf{x},\hat{\boldsymbol{\theta}})=(L_\mathbf{g}\psi(\mathbf{x}))^{-1}(-L_\mathbf{f}(\mathbf{x},\hat{\boldsymbol{\theta}})\psi(\mathbf{x})-\varphi(\psi)+\upsilon(t)) \qquad (4)$$

which transforms system (1) into the error model

$$\dot{\psi}=f(\mathbf{x},\boldsymbol{\theta})-f(\mathbf{x},\hat{\boldsymbol{\theta}})-\varphi(\psi)+\upsilon(t), \qquad (5)$$

where $f(\mathbf{x},\boldsymbol{\theta})=L_\mathbf{f}(\mathbf{x},\boldsymbol{\theta})\psi(\mathbf{x})$, $\varphi\in\mathcal{C}_\varphi\subset C^0:\mathbb{R}\rightarrow\mathbb{R}$ and $\varphi(\psi)\psi>0\ \forall\ \psi\neq 0$, $\upsilon,d:\mathbb{R}_+\rightarrow\mathbb{R}$. The desired dynamics of the system is given by equation $\dot{\psi}=-\varphi(\psi)$, and $\upsilon(t)$ stands for auxiliary control or disturbances depending on the context.

Definition 1. Adaptive control law (4) is called *non-dominating* in class $\mathcal{C}_\varphi$ if for any $\epsilon>0$ there exists such function $\hat{\boldsymbol{\theta}}(\mathbf{x},t,\delta(\epsilon))\in\Omega_\theta$, $\delta(\epsilon):\mathbb{R}_+\rightarrow\mathbb{R}$, and time t^* that $|\psi(\mathbf{x}(t,\mathbf{x}_0,t_0))|<\epsilon$ for any $t\geq t^*$, $\mathbf{x}_0\in\Omega_\mathbf{x}$, $\boldsymbol{\theta}\in\Omega_\theta$, $\varphi\in\mathcal{C}_\varphi$ and $\upsilon(t)\equiv 0$.

In the present study we will restrict class $\mathcal{C}_\varphi$ to the following class of functions:

$$\mathcal{C}_\varphi(k)=\{\varphi:\mathbb{R}\rightarrow\mathbb{R}|\ \varphi\in C^1,\ \varphi(\psi)\psi\geq\psi^2k,\ k\in\mathbb{R}_+\} \qquad (6)$$

The fact that adaptation is non-dominating in this class of functions means that for any arbitrary small gain $k>0$ in feedback $\varphi(\psi)=k\psi$ there exists function $\hat{\boldsymbol{\theta}}(\mathbf{x},t,\delta)$ such that the control goal is reached in finite time. In order to specify the desired performance of the adaptive algorithm itself we require that $\hat{\boldsymbol{\theta}}(\mathbf{x},t,\delta)$ does not change along the manifold $f(\mathbf{x},\boldsymbol{\theta})-f(\mathbf{x},\hat{\boldsymbol{\theta}})=0$ and norm $\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|$ does not grow with time.

In conventional certainty-equivalence adaptive control the problem of adaptation is usually viewed as a problem of design of a function $A(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t)$, such that solutions of

$$\dot{\hat{\boldsymbol{\theta}}}=A(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t) \qquad (7)$$

together with (4) ensure the goal relation (2). Function $A(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t)$ should not depend on unknown $\boldsymbol{\theta}$, or require derivative $\dot{\mathbf{x}}$. The standard way to solve this problem for $\upsilon(t)\equiv 0$ is to design function $A(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t)$ such that $\psi(\mathbf{x})=0$, $\hat{\boldsymbol{\theta}}=\boldsymbol{\theta}$ is a stable manifold in the Lyapunov sense.

In general nonlinear setup, however, this condition is hardly ever met for every $\boldsymbol{\theta}\in\Omega_\theta$ and $\mathbf{x}(t_0)\in\mathbb{R}^n$. To show this consider (1)–(7) as follows

$$\begin{pmatrix} \dot{\psi} \\ \dot{\tilde{\boldsymbol{\theta}}} \end{pmatrix} = \begin{pmatrix} -\phi_\psi & \mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) \\ \mathcal{A}_\psi(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t) & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\boldsymbol{\theta}} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \upsilon(t), \quad \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \tag{8}$$

where functions ϕ_ψ, $\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}})$, $\mathcal{A}_\psi(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t)$ follow from Hadamard lemma[1]. Unlike in the linear parametrization case, explicit dependance of function $\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}})$ on unknown $\boldsymbol{\theta}$ does not allow to compensate for uncertainty by choosing appropriate function $\mathcal{A}_\psi(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t)$ in (8). Therefore, in general, it is necessary to use additional control input $\upsilon(t)$ in order to ensure Lyapunov stability of (8) and that $\psi(\mathbf{x}) \to 0$ as $t \to \infty$. Success of this strategy is reported in (Loh *et al.*, 1999; Lin and Qian, 2002*a*; Lin and Qian, 2002*b*).

One step forward towards obtaining non-dominating adaptation is to reformulate the problem as follows: design function $\hat{\boldsymbol{\theta}}(\mathbf{x},t)$ such that either

$$\lim_{t\to\infty} f(\mathbf{x}(t),\boldsymbol{\theta}) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(\mathbf{x}(t),t)) = 0, \tag{9}$$

or

$$f(\mathbf{x}(t),\boldsymbol{\theta}) - f(\mathbf{x}(t),\hat{\boldsymbol{\theta}}(\mathbf{x}(t),t)) \in L_2 \tag{10}$$

hold[2]. One possible way to achieve goal (9), (10) is to use information about the difference $f(\mathbf{x},\boldsymbol{\theta}) - f(\mathbf{x},\hat{\boldsymbol{\theta}}(\mathbf{x},t))$ explicitly in the adaptation algorithm rather than using external control $\upsilon(t)$. For a class of nonlinear parameterized systems this additional information can be introduced into adjustment schemes by mere structural changes in the adjustment law. In particular it is suggested in (Tyukin, 2003) to use adaptive algorithms in differential-integral or *finite form* instead of differential form. The extended system in this case can be described as follows:

$$\begin{pmatrix} \dot{\psi} \\ \dot{\tilde{\boldsymbol{\theta}}} \end{pmatrix} = \begin{pmatrix} -\phi_\psi & \mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) \\ 0 & -\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}})\alpha(\mathbf{x},t) \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\boldsymbol{\theta}} \end{pmatrix}, \tag{11}$$

where $\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}})\alpha(\mathbf{x},t)$ is positive semi-definite time-varying matrix. Sufficient conditions for existence of such algorithms are given in (Tyukin *et al.*, 2003*a*; Tyukin *et al.*, 2004). While a solution to this problem was shown to exist for a wide range of functions $\alpha(\mathbf{x},t)$ and $\psi(\mathbf{x})$, it is not always possible to guarantee that $\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}})\alpha(\mathbf{x},t)$ is positive semi-definite for arbitrary $f(\mathbf{x},\boldsymbol{\theta})$ and $\boldsymbol{\theta} \in \Omega_\theta$, $\mathbf{x}(t_0) \in \mathbb{R}^n$. As a result Lyapunov stability of the system becomes problematic, if not impossible, for general nonlinear parameterizations.

These observations suggest that, both in conventional (5), (7) and nonconventional problem settings (8), (11), ensuring Lyapunov stability for an adaptive system in case of nonlinear parametrization leads to either domination of the nonlinearity or to additional restrictions on the class of nonlinear parameterizations. In the other words, the problem of Lyapunov stable and non-dominating adaptive control is ill-posed in general. As a candidate for replacement of Lyapunov stability one could think of *set attractivity* (Milnor, 1985) of target set (2). This concept allows to design systems which, while being unstable in Lyapunov sense, have bounded solutions and also are capable of reaching a goal asymptotically. The main problem with this concept, however, is that there is a number of conditions to check which critically depend on precise knowledge of the vector fields of the adaptive system. This knowledge includes the properties of yet unknown function $\hat{\boldsymbol{\theta}}(\mathbf{x},t,\delta)$. Therefore the least demanding concept of the control goals appears to be the notion of *ω-limit set*

Definition 2. A point $p \in \mathbb{R}^n$ is called an ω-limit point $\omega(\mathbf{x}(t,\mathbf{x}_0,t_0))$ of $\mathbf{x}_0 \in \mathbb{R}^n$ if there exists sequence $\{t_i\}$, $t_i \to \infty$, such that $\mathbf{x}(t,\mathbf{x}_0,t_0) \to p$. The set of all limit points $\omega(\mathbf{x}(t,\mathbf{x}_0,t_0))$ is the ω-limit set of $\mathbf{x}_0$.

Let, therefore, the control goal be to ensure that for some positive ε the set

$$\Omega_\psi(\varepsilon) = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} : |\psi(\mathbf{x})| \leq \varepsilon\} \tag{12}$$

contains the *ω-limit set* of $\Omega_{\mathbf{x}}$ for non-autonomous system (1), (4) with $\upsilon(t) \equiv 0$. Hence, the main question of our current study is the following: is there a non-dominating adaptive scheme for a reasonably large class of parameterizations of the uncertainty, such that all $\Omega_\psi(\varepsilon)$ contain the ω-limit set of the adaptive system for any arbitrary small $\varepsilon > 0$, all trajectories of the system are bounded, and the volume of the domain of the uncertainty is decreasing with time?

3. MAIN RESULTS

Let us consider the case where function $f(\cdot,\cdot)$ is parameterized by scalar $\theta \in \Omega_\theta = [\underline{\theta},\overline{\theta}] \subset \mathbb{R}$, $\underline{\theta} < \overline{\theta}$. For each $\theta \in \Omega_\theta$ and nonnegative $\Delta \in \mathbb{R}_{\geq 0}$ we introduce the following equivalence relation

$$\theta \sim \theta' \Leftrightarrow \quad |f(\mathbf{x},\theta) - f(\mathbf{x},\theta')| \leq \Delta \ \forall \ \mathbf{x} \in \mathbb{R}^n$$

and corresponding equivalence classes $[\theta]_\Delta = \{\theta' \in \Omega_\theta | \theta \sim \theta'\}$. For the given functions $\varphi(\psi)$ and $\alpha(t) : \mathbb{R}_+ \to \mathbb{R}$, $\alpha(t) \in C^1$ let us define the following function

$$S_\delta(\varphi(\psi),\alpha(t)) = \begin{cases} 1, & |\varphi(\psi) + \alpha(t)| > \delta \\ 0, & |\varphi(\psi) + \alpha(t)| \leq \delta \end{cases}$$

With the function $S_\delta(\varphi(\psi(\mathbf{x}(t))),\alpha(t))$ we associate the time sequence $\mathcal{T} = \underline{t}_0 \leq \overline{t}_0 < \underline{t}_1 < \overline{t}_1 < \cdots < \underline{t}_i < \overline{t}_i < \underline{t}_{i+1} < \overline{t}_{i+1} < \ldots$, where

[1] In particular, these functions can be calculated as follows $\mathcal{F}(\mathbf{x},\boldsymbol{\theta},\hat{\boldsymbol{\theta}}) = \int_0^1 \frac{\partial f(\mathbf{x},\lambda\boldsymbol{\theta}+(1-\lambda)\hat{\boldsymbol{\theta}})}{\partial\lambda\boldsymbol{\theta}+(1-\lambda)\hat{\boldsymbol{\theta}}} d\lambda$, $\phi_\psi = \int_0^1 \frac{\partial\varphi(\lambda\psi)}{\partial\lambda\psi} d\lambda$, $\mathcal{A}_\psi(\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t) = \int_0^1 \frac{\partial A(\lambda\psi,\mathbf{x},\hat{\boldsymbol{\theta}},t)}{\partial\lambda\psi} d\lambda$

[2] Similar idea was proposed in (Ortega *et al.*, 2002) as adaptive "root-searching" strategy

$$
\begin{aligned}
&\underline{t}_0 = t_0 \\
&\overline{t}_i = \inf_{t \geq \underline{t}_i} \{t: \ |\varphi(\psi(\mathbf{x}(t))) + \alpha(t)| < \delta\} \\
&\underline{t}_i = \inf_{t \geq \overline{t}_{i-1}} \{t: \ |\varphi(\psi(\mathbf{x}(t))) + \alpha(t)| > \delta\}
\end{aligned} \quad (13)
$$

The elements of this sequence are time instances $\underline{t}_i$ (or $\overline{t}_i$) at which the sum $\varphi(\psi(\mathbf{x}(t))) + \alpha(t)$ leaves (or enters) domain $|\varphi(\psi(\mathbf{x}(t))) + \alpha(t)| \leq \delta$. We define that $\underline{t}_0 = \overline{t}_0$ if $|\varphi(\psi(\mathbf{x}(t_0))) + \alpha(t_0)| < \delta$. Let us, in addition, introduce function λ with the following properties:

$$
\begin{aligned}
&\lambda: \mathbb{R} \rightarrow [\underline{\theta}, \overline{\theta}], \ \lambda \in C^1, \ \mathcal{I}m(\lambda) \supset [\underline{\theta}, \overline{\theta}] \\
&\forall \ s \in \mathbb{R}, \theta \in \Omega_\theta \ \ \exists \ T, \tau(s) > 0: \\
&\theta = \lambda(s + \tau(s)), \ 0 < \tau(s) < T
\end{aligned} \quad (14)
$$

An example of such a function $\lambda(s) = \underline{\theta} + \overline{\theta}\frac{1}{2}(\sin(s) + 1)$. As a candidate for $\hat{\boldsymbol{\theta}}(\mathbf{x}, t, \delta)$ we choose the following adaptation algorithm:

$$
\begin{aligned}
&\hat{\theta}(\mathbf{x}, t, \delta) = \lambda(\hat{\theta}_0(\mathbf{x}, t, \delta)) \\
&\hat{\theta}_0(\mathbf{x}, t, \delta) = \gamma \left(\hat{\theta}_P(\mathbf{x}, t) + \theta_I(t) + C_\theta(t)\right) \\
&\hat{\theta}_P(\mathbf{x}, t) = \psi(\mathbf{x}) \left(\alpha(t) + \frac{1}{2}\psi(\mathbf{x})\right) \\
&\dot{\hat{\theta}}_I = S_\delta(\varphi(\psi), \alpha(t))(\psi(\mathbf{x})\varphi(\psi) - \\
&\quad \psi(\mathbf{x})(\xi_2 + b_1\psi(\mathbf{x}))) \\
&\alpha(t) = (1, 0)(\xi_1, \xi_2)^T r \\
&\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \psi \\
&\quad b_1 \neq 0, \ a_1, a_2 < 0, \\
&C_\theta(t) = \begin{cases} \frac{1}{\gamma}\hat{\theta}(\overline{t}_{i-1}) - \hat{\theta}_I(\overline{t}_{i-1}) - \\ \quad \hat{\theta}_P(t), \ t \in (\overline{t}_{i-1}, \underline{t}_i) \\ C_\theta(\overline{t}_{i-1}) + \hat{\theta}_P(\overline{t}_{i-1}) - \\ \quad \hat{\theta}_P(\underline{t}_i), \ t \in [\underline{t}_i, \overline{t}_i] \end{cases}
\end{aligned} \quad (15)
$$

Properties of algorithm (15) are summarized in the following theorem:

Theorem 3. Let system (1) with control function (4) and corresponding error model (5) be given. Let function $\upsilon(t), \dot{\upsilon}(t) \in L_\infty$ and $\|\upsilon(t)\|_\infty \leq \Delta$. Let in addition function $f(\mathbf{x}, \boldsymbol{\theta})$ be bounded. Then
1) for any $\varepsilon > 0$ and $\varphi \in \mathcal{C}_\varphi(k)$, $k > 0$ there exist functions $\delta_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\delta_0 \in C^0$, $\delta(0) = 0$, $\delta(\varepsilon, \Delta) = \delta_0(\varepsilon) + \Delta$, and function $\hat{\theta}(\mathbf{x}, t, \delta)$, given by (15) with arbitrary $\gamma \in \mathbb{R}, \gamma > 0$ and initial conditions such that $\Omega_\psi(\varepsilon + \frac{\Delta}{k})$ contains the ω-limit set of system (1);
2) all trajectories of the system are bounded and solutions $\mathbf{x}(t, \mathbf{x}_0, t_0)$ converge into the domain $\Omega_\psi(\varepsilon + \frac{\Delta}{k})$ in finite time;
3) if for any $\theta \in \Omega_\theta$ there exist constants $T_1 > 0$, $M > 2\delta_0(\varepsilon) + \Delta > 0$ and function $\tau(t) : \mathbb{R} \rightarrow (0, T_1)$ such that

$$
\begin{aligned}
&|f(\mathbf{x}(t + \tau(t)), \theta) - f(\mathbf{x}(t + \tau(t)), \hat{\theta})| > M \\
&\forall \ \hat{\theta} \in \ \Omega_\theta \setminus \mathcal{U}_\epsilon([\theta])
\end{aligned} \quad (16)
$$

then $\hat{\theta}$ converges into $\mathcal{U}_\epsilon([\theta])$ in finite time.

Theorem 3[3] states that for arbitrary C^1-smooth and bounded function $f(\mathbf{x}, \theta)$ there exists a non-dominating adaptation algorithm in class $\mathcal{C}_\varphi(k)$ (to see to let $\Delta = 0$). In presence of unknown perturbation $\upsilon(t)$ we guarantee convergence of the trajectories $\mathbf{x}(t, \mathbf{x}_0, t_0)$ to an arbitrary small neighborhood of $\Omega_\psi(\frac{\Delta}{k})$, subject to the choice of $\delta_0 > 0$. Algorithm (15) ensures boundedness of the solutions in the extended system and furthermore, under assumption (16), it guarantees convergence of the estimates $\hat{\theta}$ to arbitrary small neighborhood $\mathcal{U}_\epsilon([\theta]_\Delta)$ of the equivalence class (set) $[\theta]_\Delta$. In case $[\theta]_\Delta = \{\theta\}$ it guarantees convergence to the small neighborhood of the actual value of θ.

Condition (16), which we require for convergence of the parameter $\hat{\theta}$ into $\mathcal{U}_\epsilon$, can be regarded as a new version of *nonlinear persistent excitation* (Cao *et al.*, 2003). Our condition, however, is more easy to verify. In addition, this condition is consistent with *linear persistent excitation* condition (Narendra and Annaswamy, 1989): $\exists \ T > 0, \ \rho > 0 : \ \int_t^{t+T} \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau > \rho I_n$. Indeed $\int_t^{t+T} \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau > \rho I_n \Rightarrow \forall \boldsymbol{\theta} \neq \boldsymbol{\theta}' \ (\boldsymbol{\theta} - \boldsymbol{\theta}')^T \times \int_t^{t+T} \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau (\boldsymbol{\theta} - \boldsymbol{\theta}') > \rho\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|^2 \Rightarrow |(\boldsymbol{\theta} - \boldsymbol{\theta}')^T \mathbf{x}(t_1)| > \frac{\rho}{T}\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|, \ t_1 \in [t, t+T] \Rightarrow |(\boldsymbol{\theta} - \boldsymbol{\theta}')^T \mathbf{x}(t_1)| > M, \ \forall \theta \in \Omega_\theta \backslash \mathcal{U}_\epsilon, \ M = \frac{\rho\epsilon}{T}$.

Let us generalize the statements of Theorem 3 to the case where $\boldsymbol{\theta} \in \Omega_\theta \subset \mathbb{R}^d$. To this purpose we introduce the following assumption:

Assumption 1. Let Ω_θ be bounded and there exist C^1-smooth function $\boldsymbol{\eta} : [\underline{\theta}, \overline{\theta}] \rightarrow \mathbb{R}^d$ such that that for any $\boldsymbol{\theta} \in \Omega_\theta$ there exists $\lambda^*(\boldsymbol{\theta}) \in [\underline{\theta}, \overline{\theta}]$: $|f(\mathbf{x}, \boldsymbol{\theta}) - f(\mathbf{x}, \boldsymbol{\eta}(\lambda^*)| \leq \Delta, \ \forall \ \mathbf{x} \in \mathbb{R}^n$

Applicability of algorithms (15) to multi-dimensional $\boldsymbol{\theta}$ then follows explicitly from Theorem 3.

Theorem 4. Let system (1) with control function (4) and corresponding error model (5) be given, function $f(\mathbf{x}, \boldsymbol{\theta})$ be bounded, and Assumption 1 hold. Then statements 1) – 3) of Theorem 3 follow.

Adaptation algorithm (15) can be considered as a nonlinear dynamical system which, however, is not internally globally stable in the Lyapunov sense. Nonetheless its internal state is bounded and, furthermore, it ensures reaching of the control goal for arbitrary initial conditions $\mathbf{x}(t_0)$, $\hat{\theta}_I(t_0)$, $\boldsymbol{\xi}(t_0)$. The properties of this algorithm essentially rely on two ideas: *monotonic evolution* of $\hat{\theta}_0(t)$, and *multiple equilibria* in the system[4].

[3] Proof of the theorems are available on-line in (Tyukin and van Leeuwen, 2004).

[4] See the proofs in (Tyukin and van Leeuwen, 2004) for details

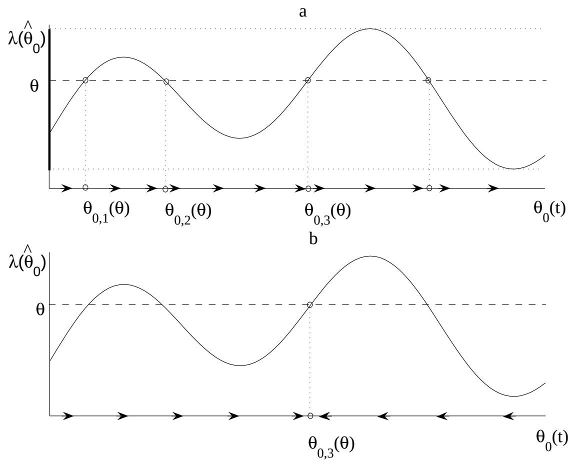

Fig. 1. Adaptation with multiple equivalent equilibriums (plot a) vs. adaptation with single asymptotically stable equilibrium (plot b)

Multiple equilibria are guaranteed by function $\lambda(\cdot)$ defined by (14) and invariance of $\hat{\theta}_0(t)$ on the following set $\{\mathbf{x}, \hat{\theta}_0 | \ \mathbf{x}, \hat{\theta}_0 : \ f(\mathbf{x}, \theta) - f(\mathbf{x}, \lambda(\hat{\theta}_0)) = 0\}$. Monotonicity of $\hat{\theta}_0(t)$ in time and multiple equilibriums ensure existence of $\lim_{t\rightarrow\infty} \hat{\theta}_0(t) = \hat{\theta}_{0,\infty}$ for any initial conditions $\mathbf{x}(t_0)$, $\hat{\theta}_I(t_0)$, $\boldsymbol{\xi}(t_0)$. This fact is used to show convergence of the trajectories $\mathbf{x}(t, \mathbf{x}_0, t_0)$ to the set specified by (12). Notice also that existence of these multiple equilibriums in case of persistent excitation for $v(t) \equiv 0$ ensure that the set $\mathcal{U}_\epsilon([\theta])$ becomes globally attractive w.r.t. function $\hat{\theta}(t, \mathbf{x}, \delta)$.

The difference between our method and conventional Lyapunov-based design is illustrated with Fig. 1. In Fig. 1 the upper plot depicts the solution curve $\hat{\theta}_0(t, \hat{\theta}_0(t_0), t_0)$ of (15), were the arrows point towards increase of the independent variable along the curve. For the given value of θ and initial condition $\hat{\theta}_0(t_0)$ function $\lambda(\hat{\theta}_0)$ generates infinitely many equilibria $\hat{\theta}_{0,i}$, $i \in \mathbb{N}$. If perturbation is applied to the system the state will eventually escape its current equilibrium (for instance, $\hat{\theta}_0 = \hat{\theta}_{0,1}$) and move along the axis $\hat{\theta}_0$. Due to the monotonicity of $\hat{\theta}_0(t, \hat{\theta}_0(t_0), t)$ with respect to t it eventually reaches a neighborhood of the point $\hat{\theta}_0 = \hat{\theta}_{0,2}$ and stops there if the perturbation is released. In order to prevent unbounded growth of $\hat{\theta}_0$ under persistent perturbations it may be necessary to change the sign of γ in (15) upon solution $\hat{\theta}_0(t)$ reaches certain bounds.

In the lower plot we show a hypothetical solution curves of the Lyapunov (asymptotically) stable estimator. The problem, however, is that it is not always possible to ensure such behavior without knowing the value of θ or using domination in the control.

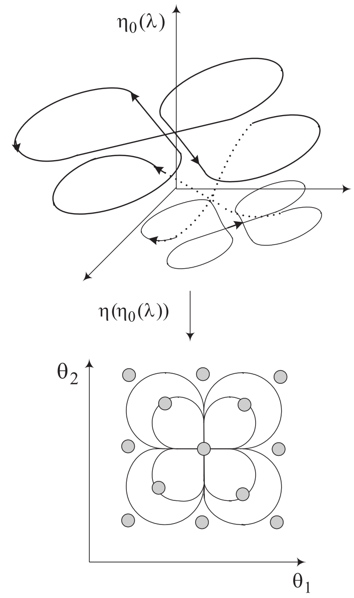

Fig. 2. Trajectory $\boldsymbol{\eta}(\lambda)$ (the bottom panel) as projection of the smooth curve. Gray circles form the gird in the parameters space which induces the curves satisfying Assumption 1.

Algorithms (15) can, in principle, take into account information (available a-priori) about the distribution of the unknown quasi-stationary $\boldsymbol{\theta}$ as a function of time. This information is accounted for by the choice of functions $\boldsymbol{\eta}(\cdot)$ and $\lambda(\cdot)$. In Fig. 2, for example, the curve $\boldsymbol{\eta}(\lambda)$ is designed to visit the neighborhood of the center more frequently (16 times per period) than other points (only 2 times per period) of the domain. Even though the problem of choosing these curves $\boldsymbol{\eta}(\lambda)$ (which fit given distributions of $\boldsymbol{\theta}$) is not trivial, such optimal choice, if successful, can provide room for further enhancements of performance in the system. The next step would be to adjust or tune functions $\boldsymbol{\eta}(\lambda)$ adaptively, enabling self-tuning of the adaptive algorithm. These topics, however, are beyond the goals of our current study.

4. CONCLUSION

We proposed a new technique for adaptive control of nonlinear dynamical systems with nonlinear parametrization. In contrast to conventional concept of Lyapunov stable adaptive control, and as a result domination of the nonlinearity by high-gain feedbacks, we use adaptation schemes which are not stable in Lyapunov sense. Yet, these algorithms guarantee reaching of arbitrary small neighborhood of the desired target set. Moreover the resulting control function is non-dominating. Last but not least is that the method can be used to identify nonlinear systems of rather general class without requesting for linearization of the nonlinearities [5].

Robustness of the system to unmodeled dynamics with known L_∞ norm can be easily ensured by enlarging the value of δ in our algorithms. Whether or not robust behavior can be achieved by choice of another parameters like functions $\boldsymbol{\eta}(\cdot)$ or $\lambda(\cdot)$ will be the topics of our future study.

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[5] Although our method might seem similar to (Ilchman, 1997; Pomet, 1992), they results are substantially different. First, we do not require exponential stability nor asymptotic stability of the target dynamics. Second, the speed of adaptation in our case is not to be slowed down with time.