

# DISCONTINUOUS EXPONENTIAL STABILIZATION OF DYNAMIC CHAINED FORM SYSTEMS

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Abstract: In this paper, the exponential stabilization of chained form system with inertia is addressed. A discontinuous static time invariant control law is proposed that achieves exponential stability in the Lyapunov's sense. Such a result was previously obtained only by means of time varying feedback or hybrid feedback endowed of memory capabilities. The total number of switching is less or equal to three with a subdivision of the state space into four subsets. *Copyright ©2005 IFAC*

Keywords: Feedback stabilization, nonholonomic systems, discontinuous control, exponential stability

## 1. INTRODUCTION

The stabilization of nonholonomic systems and in particular chained form systems have focused a lot of attention these last years (see e.g. the survey papers by Kolmanovky and McClamroch (1995); Canudas de Witt et al. (1996)). This interest is motivated by two main reasons. The first one is that many mechanical systems can be transformed into chained form which therefore makes up an important class of generic mechanical systems (Murray and Sastry, 1993; Kolmanovky and McClamroch, 1995). The second reason is that, due to the lack of linearizing feedback transformation (Isidori, 1995) and of continuous static stabilizing feedback (Brockett, 1983), these systems are very interesting benchmarks for the development of new control approaches. These works were initially limited to first-order chained systems which infer non-integrable kinematic constraints. Among the techniques developed, one can mention time varying feedback initiated by Coron (1992) and largely explored (Teel et al., 1995; Samson, 1995; Lin, 1996; Morin and Sam-

son, 1997). This approach requires non-smooth periodic (Sørdalen and Egeland, 1995; M'Closkey and Murray, 1997; Godhavn and Egeland, 1997) or aperiodic feedback (Yu and Shihua, 2000) to ensure exponential stability. Time invariant discontinuous feedback can also be used to obtain different kinds of convergence/stabilization with better performances than time-varying approaches. Almost exponential stability, that is exponential stability for any initial condition in an open dense subset of  $\mathbb{R}^n$ , was obtained by many contributors (Astolfi, 1996; Canudas de Witt and Khennouf, 1995; Reyhanoglu et al., 1998; Jiang, 2000). The two main drawbacks of this approach underlined, among others, by Luo and Tsiotras (2000), is first that there are forbidden initial conditions and that the control is most of the time not bounded in a neighborhood of the origin. An improvement has been brought in (Jiang, 2000) who proposed a way to first escape the forbidden initial conditions ensuring an exponential convergence to the origin for any initial condition. Finally, a recent work showed that exponential stability could be achieved by means of a discontinuous static feed-

back that remains bounded in a neighborhood of the origin (Marchand and Alamir, 2003). Let us also mention hybrid approaches that enable robustness results, (Morin and Samson, 2000; Prieur and Astolfi, 2003, see) and the references therein. In all the works cited, the system considered is the kinematic model given by:

$$\begin{aligned} \dot{x}_0 &= u_0 & \dot{x}_1 &= u_0 x_2 \\ & & \vdots & \\ \dot{x}_{n-1} &= u_0 x_n & \dot{x}_n &= u_1 \end{aligned} \quad (1)$$

Beside these works, few works deal with the dynamical model, that is the derivation of a general dynamical model describing the dynamical relations between the configuration coordinates and the torques developed by the embarked motors. Nevertheless, many mechanical systems operating for instance at high speeds or in water do not perform well if we neglect dynamics. A complete analysis and classification of these systems can be found in (Campion et al., 1996), where it is stated that the dynamical model can be obtained from (1) simply by adding

$$\begin{aligned} \dot{u}_0 &= v_0 \\ \dot{u}_1 &= v_1 \end{aligned} \quad (2)$$

where  $v := (v_0, v_1)^T$  becomes the new control. The augmented system also fails to be stabilizable by a static time invariant continuous feedback. To date a limited amount of work has been done in the area of dynamical nonholonomic systems (Bloch and Crouch, 1995; Bloch et al., 1992; Morgansen, 1999; Aguiar and Pascoal, 2000). Another interesting contribution, more in the spirit of this paper, is due to Laiou and Astolfi (1999) extending (Astolfi, 1996) for a larger class of nonholonomic systems including (1-2). The proposed static time invariant discontinuous feedback ensure exponential almost stability. However, it seems that no static time invariant control laws that ensure Lyapunov stability has been proposed for this class of common systems. The aim of this paper is to fill this gap with the unicycle as example. The problem is known to be non trivial since, as mentioned by Dixon et al. (2000), extending the feedback law to incorporate the dynamic model via the standard backstepping procedure is unclear since the feedback is not differentiable. Indeed, if results on nonsmooth backstepping recently appeared (Tanner and Kyriakopoulos, 2003), a Lipschitz Lyapunov function for the system without the integrator stage is then required, which, to our knowledge, has never been exhibited for chained form system with more than three state variable.

*Notations:* Let  $I_{k \times k}$  be the identity matrix of dimension  $k \times k$  and  $0_{k \times l}$  be the null matrix of dimension  $k \times l$ , let  $z_i$  denote the  $i^{\text{th}}$  coordinate of the vector  $z$ .

## 2. PRELIMINARY DEFINITIONS

For all  $k > 2$ , let  $A_k$  and  $B_k$  be defined by:

$$A_k = \begin{pmatrix} 0_{k-1 \times 1} & I_{k-1 \times k-1} \\ 0 & 0_{1 \times k-1} \end{pmatrix}, B_k = \begin{pmatrix} 0_{k-1 \times 1} \\ 1 \end{pmatrix}$$

Defining:

$$\begin{aligned} y &:= (x_0, u_0)^T \in \mathbb{R}^{2 \times 1} \\ z &:= (x_1, x_2, \dots, x_n, u_1)^T \in \mathbb{R}^{n+1 \times 1} \end{aligned}$$

system (1-2) becomes:

$$\begin{aligned} \dot{y} &= A_2 y + B_2 v_0 \\ \dot{z} &= \begin{pmatrix} y_2 A_n & B_n \\ 0_{1 \times n} & 0 \end{pmatrix} z + B_{n+1} v_1 \end{aligned} \quad (3)$$

The control law is based on a partition of the state space in regions. In each regions the feedback takes different form and is issued from two approaches (see the proof for detail). The first one is based on backstepping: let  $P$  be the symmetric positive definite solution of the Riccati equation:

$$P A_n + A_n^T P - 2 P B_n B_n^T P = -P \quad (4)$$

and  $Q$  be the symmetric positive definite matrix given by:

$$Q = \begin{pmatrix} P & 0_{n \times 1} \\ 0_{1 \times n} & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} P B_n y_2 \\ 1 \end{pmatrix} (y_2 B_n^T P \ 1) \quad (5)$$

Note that  $Q$  depends upon the state  $y_2$  but remains positive for all  $y_2$ . Finally, let the vector  $K = (k_1, \dots, k_n, k_{n+1})$  be defined by:

$$\begin{aligned} (k_1 \dots k_n) &= (v_0 + y_2 + \frac{y_2^2}{2} + 2) B_n^T P + y_2^2 B_n^T P A_n \\ k_{n+1} &= y_2 B_n^T P B_n + \frac{1}{2} y_2 + 1 \end{aligned} \quad (6)$$

Note that the feedback adaptive gain  $K$  depends upon  $y_2$  and the control  $v_0$ . The second approach used is issued from linear optimal control. The change of variables  $\xi = Tz$  with  $T = \text{diag}(1, y_2, y_2^2, \dots, y_2^{n-2}, y_2^{n-1}, y_2^{n-1})$  results in:

$$\dot{\xi} = \left( A_{n+1} + \frac{\dot{y}_2}{y_2} L \right) \xi + B_{n+1} y_2^{n-1} v_1 \quad (7)$$

where  $L = \text{diag}(0, 1, 2, \dots, n-2, n-1, n-1)$ . Hence, if  $R$  is some symmetric positive definite matrix such that for all  $\varepsilon \in [0, \frac{3}{2\alpha}]$  with  $\alpha > 0$ :

$$\begin{aligned} R(A_{n+1} - \varepsilon L) + (A_{n+1} - \varepsilon L)^T R \\ - 2R B_{n+1} B_{n+1}^T R \leq -R \end{aligned} \quad (8)$$

then for all  $y_2$  such that  $\frac{\dot{y}_2}{y_2} \in [-\frac{3}{2\alpha}, 0]$ , the feedback  $-y_2^{1-n} B_{n+1}^T R T z$  will ensure an exponential decrease of  $\xi^T R \xi$ .

## 3. MAIN RESULT

The first step of any control law for chained form systems (even when neglecting the inertia) is to

bring the system away from the manifold  $y_2 = 0$  where there is a lack of controllability. Almost stabilizing controller dodge this problem by prohibiting these initial conditions (Laiou and Astolfi, 1999; Reyhanoglu, 1996, 1997). One solution could consist in applying a constant control during a constant time or as long as some norm of  $y$  is lower than a fixed level but such approaches are doomed to fail in obtaining exponential stability. Indeed, this would systematically force  $y$  to go far from the origin which is unacceptable as regards Lyapunov stability. Another approach could be to introduce the time or the initial condition in the controller but, because of its simplicity and its efficiency, the approach adopted here is to find a purely static time-invariant feedback.

Let  $\mathbb{R}^{n+3}$  be divided into the four regions:

$$\begin{aligned}\Omega_0 &= 0_{\mathbb{R}^{n+3}} \\ \Omega_1 &= \left\{ (y, z) \in \mathbb{R}^{n+3}, y \neq 0, \text{ s.t.} \right. \\ &\quad \left. y_1^2 - \frac{\frac{1}{2}y_1^2}{1+y_1^2} \leq -\alpha y_1 y_2 \leq y_1^2 + \frac{\frac{1}{2}y_1^2}{1+y_1^2} \right\} \\ \Omega_2 &= \left\{ (y, z) \in \mathbb{R}^{n+3}, (y, z) \notin \{\Omega_0 \cup \Omega_1\}, \text{ s.t.} \right. \\ &\quad \left. (y_2^2 + 2|y_1|)e^{-y_1} \geq z^T Q z \right\} \\ \Omega_3 &= \mathbb{R}^{n+3} \setminus \{\Omega_0 \cup \Omega_1 \cup \Omega_2\}\end{aligned}$$

where  $\alpha > 3(n-1)$  is some strictly positive real number. The control law will be such that if  $y$  starts in  $\Omega_3$ , it reaches  $\Omega_2$  and then  $\Omega_1$  in finite time and finally converge to the origin. During that time,  $z$  will also converge to the origin implying that  $\Omega_2$  and  $\Omega_3$  will vary. Let the control law be defined by:

$$\begin{aligned}(y, z) \in \Omega_0 &: \begin{cases} v_0 = 0 \\ v_1 = 0 \end{cases} \\ (y, z) \in \Omega_1 &: \begin{cases} v_0 = \frac{y_2^2}{y_1} \\ v_1 = -y_2^{1-n} B_{n+1}^T R T z \end{cases} \quad (9) \\ (y, z) \in \Omega_2 &: \begin{cases} v_0 = -\text{sign}(y_1) \\ v_1 = -K(y_2, v_0)z \end{cases} \\ (y, z) \in \Omega_3 &: \begin{cases} v_0 = 1 \\ v_1 = -K(y_2, v_0)z \end{cases}\end{aligned}$$

With the above notations, our result is the following:

*Theorem 1.* The static discontinuous feedback (9) is such that for any in initial condition, the solution of the closed loop system exists and is unique. Furthermore, the origin is exponentially asymptotically stable.

This result lead to some remarks. First, let us be more specific about the notion of solution considered here. Indeed, various definitions have been proposed for ordinary differential equations with discontinuous right hand side. Here, we implicitly

consider Carathéodory solutions<sup>1</sup>. Note that the proposed control law fail to stabilize Filippov solutions (see Filippov, 1988, for an exact definition). Indeed, in that context of solution, the Brockett's condition still holds<sup>2</sup> (Ryan, 1994). This result was corroborated by Coron and Rosier (1994) who established that if a driftless system admits a discontinuous feedback law that asymptotically stabilizes the solutions in the Filippov sense, then the system is continuously asymptotically stabilizable. Stabilization in the Filippov sense is hence doomed to fail. This is not surprising since Filippov solutions are known to be too numerous in particular for the stabilization problem of nonlinear systems (see Ceragioli, 2002; Clarke et al., 1997). Note that the proposed feedback is not the first discontinuous control that stabilizes (even exponentially) Carathéodory solutions<sup>3</sup> without stabilizing the Filippov solutions of a system. This is in particular the case for the feedback proposed by Bloch and Drakunov (1996) for the nonholonomic integrator or by Canudas de Witt and Sordalen (1992) for the unicycle (see Ceragioli, 2002, for a detailed study) or for the more general feedback proposed in (Marchand and Alami, 2003) for chained systems. Note that the control (9) can be proved to be bounded for bounded states and exponentially decaying to zero, but this will not be done here.

**PROOF.** First note that figure 1 illustrates the following proof by showing the typical behavior in the  $y$ -plane.

• Let us start with some initial condition in  $\Omega_3$  at initial time  $t = t_3 = 0$  and prove that the states must join  $\Omega_2$  (if it does not join  $\Omega_1$  before) in finite time. In  $\Omega_3$ , one has:

$$\begin{cases} y_1(t) = y_1(0) + y_2(0)t + \frac{1}{2}t^2 \\ y_2(t) = y_2(0) + t \end{cases} \quad (10)$$

Therefore,  $y_2^2 + 2|y_1|$  is increasing after some finite time. For the  $z$  part of (3),  $v_1$  can be designed using backstepping. If one considers the system:

$$\dot{\chi} = y_2 A_n \chi + B_n u_\chi$$

with control  $u_\chi = -y_2 B_n^T P \chi$ , then the Lyapunov function  $V_\chi = \chi^T P \chi$  is such that  $\dot{V}_\chi = -y_2 V_\chi$  resulting in:

$$V_\chi(t) = V_\chi(0)e^{y_1(0) - y_1(t)}$$

<sup>1</sup> A function  $\phi(t) : I \rightarrow \mathbb{R}^n$  is a Carathéodory solution if it is differentiable and satisfies  $\dot{x} = f(x, k(x))$  almost everywhere on  $I$

<sup>2</sup> At least for affine-input systems, for general systems, the result holds if one further has  $A \subseteq \mathbb{R}^m$  convex  $\Rightarrow f(x, A) \subseteq \mathbb{R}^n$  convex

<sup>3</sup> that can be proved to exist in these specific cases

Posing  $\chi = (z_1, \dots, z_n)$  and  $\zeta = z_{n+1} + y_2 B_n^T P \chi$ , the second part of (3) can be transformed into:

$$\begin{aligned}\dot{\chi} &= y_2 (A_n - B_n B_n^T P) \chi + B_n \zeta \\ \dot{\zeta} &= v_1 + u_0 B_n^T P \chi + y_2 B_n^T P B_n \zeta \\ &\quad + y_2^2 B_n^T P (A_n - B_n B_n^T P) \chi\end{aligned}$$

Taking  $v'_1 = v_1 + u_0 B_n^T P \chi + y_2^2 B_n^T P (A_n - B_n B_n^T P) \chi + y_2 B_n^T P B_n \zeta$  directly as control and defining the Lyapunov function  $V = \chi^T P \chi + \frac{1}{2} \zeta^2 = z^T Q z$ , with  $Q$  as in (5), it follows:

$$\dot{V} = -y_2 \chi^T P \chi + 2\zeta B_n^T P \chi + \zeta v'_1$$

Hence, taking  $v'_1 = -2B_n^T P \chi - (1 + \frac{1}{2} y_2) \zeta$ , that is  $v_1 = -Kz$  with  $K$  as in (6),  $V$  will be such that  $\dot{V} = -y_2 V - \zeta^2 \leq -y_2 V$  giving:

$$V(t) \leq V(0) e^{y_1(0) - y_1(t)}$$

It follows that  $z^T Q z e^{y_1}$  remains bounded by  $z^T(0) Q z(0) e^{y_1(0)}$ . This means that if the system does not join a configuration in  $\Omega_1$  or the origin before, it necessarily enter  $\Omega_2$  in finite time. Note that the solution of the closed-loop system is well defined and unique.

- Consider now some starting point in  $\Omega_2$  at time  $t = t_2$  and prove that the state joins  $\Omega_1$  after some finite time and can not (re)join  $\Omega_3$ . In  $\Omega_2$ ,  $y$  turns round  $\{y = 0\}$  following a trajectory such that:

$$\frac{d(y_2^2 + 2|y_1|)}{dt} = -2 \text{sign}(y_1) y_2 + 2 \text{sign}(y_1) \dot{y}_1 = 0$$

Note that this also insure that the relay control action will not cause chattering. Beside this, as previously, the evolution of  $V$  still satisfies

$$V(z(t_2 + t)) \leq V(z(t_2)) e^{y_1(t_2) - y_1(t)}$$

Therefore, if  $y_2^2(t_2) + 2|y_1(t_2)| \geq V(z(t_2)) e^{y_1(t_2)}$ , then for all  $t \geq t_2$ :

$$y_2^2(t_2 + t) + 2|y_1(t_2 + t)| \geq V(z(t_2 + t)) e^{y_1(t_2 + t)}$$

The solution of the closed loop being clearly well defined for any initial solution in  $\Omega_2$ , the system thus remains in  $\Omega_2$  and can not (re)join  $\Omega_3$ . Finally, with a simple geometrical interpretation of  $\Omega_1$ , one can see that the projection of  $\Omega_1$  on the  $y$ -plane corresponds to the sectors held between the functions  $y_2 = \frac{1}{\alpha}(-y_1 - \frac{1}{2} \frac{y_1}{1+y_1^2})$  and  $y_2 = \frac{1}{\alpha}(-y_1 + \frac{1}{2} \frac{y_1}{1+y_1^2})$  (without the origin). Hence,  $y$  being turning round the origin of this plane, the system necessarily joins  $\Omega_1$  in finite time.

- Finally, in  $\Omega_1$ , one has  $v_0 = \frac{y_2}{y_1}$  that gives, using a relation between an open-loop trajectory property and the closed-loop stability underlined in (Marchand and Alamir, 1998):

$$\begin{cases} y_1(t_1 + t) = y_1(t_1) e^{\frac{y_2(t_1)}{y_1(t_1)}(t-t_1)} \\ y_2(t_1 + t) = y_2(t_1) e^{\frac{y_2(t_1)}{y_1(t_1)}(t-t_1)} \end{cases} \quad (11)$$

where  $t_1$  is the starting time in  $\Omega_1$ . Note that in  $\Omega_1$ ,  $y_1$  and  $y_2$  are of opposite sign, insuring that  $y$  is exponentially converging. Furthermore,  $\frac{y_1(t)}{y_2(t)}$  remains constant meaning that in the  $y$ -plane,  $y$  goes straight line to the origin.  $\Omega_1 \cup 0_{\mathbb{R}^{n+2}}$  being convex, this implies that the only possibility for  $(y, z)$  to leave  $\Omega_1$  is to join the origin  $0_{\mathbb{R}^{n+2}}$ . Beside this, by (11) and the definition of  $\Omega_1$ , it follows that  $\frac{y_2}{y_1} \in [-\frac{3}{2\alpha}, -\frac{1}{2\alpha}]$ . Hence, thanks to (8),  $\xi = Tz$  is such that  $\frac{d\xi^T R \xi}{dt} \leq -\xi^T R \xi$ . For any matrix  $M$ , let  $\lambda_{\min}^M$  and  $\lambda_{\max}^M$  denote respectively the smallest and the largest singular values of  $M$ . With this notation, it then follows:

$$\begin{aligned} \|z(t_1 + t)\| \lambda_{\min}^{T(t_1+t)} \sqrt{\lambda_{\min}^R} &\leq \\ \|z(t_1)\| \lambda_{\max}^{T(t_1)} \sqrt{\lambda_{\max}^R} e^{-\frac{1}{2}(t-t_1)} &\quad (12) \end{aligned}$$

But, with (11), one obtains

$$\lambda_{\min}^{T(t_1+t)} \geq \min(1, |y_2(t_1)|^{n-1}) e^{(n-1) \frac{y_2(t_1)}{y_1(t_1)}(t-t_1)}$$

and hence

$$\begin{aligned} \frac{\lambda_{\max}^{T(t_1)}}{\lambda_{\min}^{T(t_1+t)}} &\leq \max(|y_2(t_1)|^{n-1}, |y_2(t_1)|^{1-n}) \\ &\quad \times e^{-(n-1) \frac{y_2(t_1)}{y_1(t_1)}(t-t_1)} \end{aligned}$$

With (11) and the definition of  $\Omega_1$ , one has thanks to the choice  $\alpha > 3(n-1)$ :

$$\frac{n-1}{2\alpha} \leq -\frac{y_2(t_1)}{y_1(t_1)}(n-1) \leq \frac{3(n-1)}{2\alpha} < \frac{1}{2} \quad (13)$$

which ensures that  $z(t_1 + t)$  is exponentially decreasing to the origin without leaving  $\Omega_1$ .

The control law avoids chattering by guaranteeing that once the state has entered one of the regions, it can not go back to the region from which it came. Now to obtain the exponential stability, it must be established that the excursions of the states reduces as the initial condition goes closer to the origin. This directly follows from the fact that the time  $t_1$  needed by the system to go from some initial condition to a state in  $\Omega_1$  goes to the origin with the initial condition. Hence, the norm of the starting point  $(y(t_1), z(t_1))$  in  $\Omega_1$  also goes to zero with the initial condition. Hence, from (11) and (12), one can conclude that there exists some positive definite  $N(y, z)$  and some  $\varepsilon$  such that

$$0 < \varepsilon \leq \min\left(\frac{3}{2\alpha}, \frac{1}{2} - \frac{3(n-1)}{2\alpha}\right)$$

so that the following inequality holds:

$$\|(y(t), z(t))\| \leq N(y(0), z(0)) e^{-\varepsilon t}$$

#### 4. THE UNICYCLE

The unicycle, also known as the Brockett non-holonomic integrator or the Heisenberg system (Brockett, 1981, 1983) is probably the most stud-

ied chained form system. It encompasses many wheeled and water robots and is given by:

$$\begin{aligned}\dot{x} &= \nu_1 \cos \theta \\ \dot{y} &= \nu_1 \sin \theta \\ \dot{\theta} &= \nu_2 \\ m\dot{\nu}_1 &= \tau_1 \\ I\dot{\nu}_2 &= \tau_2\end{aligned}$$

where  $m$  is the mass of the unicycle,  $I$  the inertia around the vertical axis at contact point,  $\nu_1$  is the driving force and  $\nu_2$  the steering torque. This system with drift clearly fails to be convertible into chained but can be transformed into (3) by taking:

$$\begin{aligned}v_0 &= \frac{\tau_2}{I} & v_1 &= \frac{\tau_1}{m} - \frac{\tau_2}{I} z_1 - \nu_1^2 z_2 \\ y_1 &= \theta & z_1 &= x \sin \theta - y \cos \theta \\ y_2 &= \nu_2 & z_2 &= x \cos \theta + y \sin \theta \\ & & z_3 &= \nu_1 - \nu_2 z_1\end{aligned}$$

For illustration purpose, the system parameters have been set to  $m = I = 1$  and  $\alpha = 3(n - 1) + 1 = 7$ . We consider the manoeuvre starting at  $(x(0), y(0), \theta(0), \nu_1(0), \nu_2(0)) = (1, -1, \frac{3\pi}{2}, 0, 0)$ . The resulting paths are shown on Figures 2. Note that the approach in (Laiou and Astolfi, 1999) forbids initial conditions on the line  $y_2 = -\varsigma y_1$  where  $\varsigma$  is some positive controller parameter. This impossibility does not exist in the proposed approach.

## 5. CONCLUSION

In this paper, a discontinuous control law that exponentially stabilizes dynamical chained systems was proposed. The control law is based on a subdivision of the state space into four subsets. Simulations on the unicycle with inertia are given.

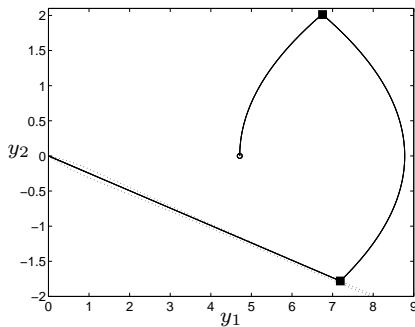


Fig. 1. The unicycle: trajectory in the  $y$  plane of the unicycle. The circle corresponds to the initial condition in  $\Omega_3$  and the two next boxes on the trajectory shows the discontinuities arising when the system enters  $\Omega_2$  and  $\Omega_1$ . The dotted lines shows the frontier of  $\Omega_1$ .

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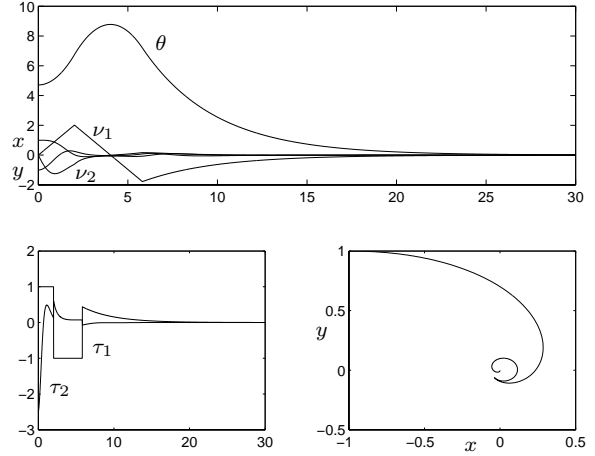


Fig. 2. The unicycle: time evolution of the states and the control and trajectory in the  $(x, y)$ -plane

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