ON QUANTIFYING TOLERABLE UNCERTAINTY FOR A SPECIFIED LEVEL OF CLOSED-LOOP PERFORMANCE [1]

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Abstract: Given a plant and a feedback controller it is natural to ask: *How much uncertainty can be tolerated by the closed-loop, while achieving a specified level of performance*? Here, a characterisation of this question is formulated in terms of an optimisation problem with a cost that reflects the size of weights used to quantify system uncertainty and a structured singular value constraint, which captures the specified level of robust performance. In the case of unstructured uncertainty the problem can be solved as a family of convex problems pointwise in frequency. An iterative algorithm is developed for the case of structured uncertainty.

Keywords: Robust performance, Uncertainty, H-infinity control, Structured Singular Value

1. INTRODUCTION

It is well-known (Zhou et al., 1996; Skogestad and Postlethwaite, 1996) that the structured singular-value (introduced by Doyle (1982)) can be used to determine whether a particular level of closed-loop performance is achieved with all plants in a specified set, when *weighted* $\mathcal{H}_\infty$ norms are employed to quantify performance and the size of the uncertain plant set, which could be structured. Consider, for example, the general interconnection structure shown in Figure 1, where

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix}$$

is a *generalized plant* constructed from a nominal model of the plant and so-called performance and uncertainty weights, so that: [2]

(i) The Upper Linear Fractional Transformation (LFT)

$$\mathcal{F}_u\left(\begin{pmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{pmatrix}, \Delta\right) := G_{33}(s) + G_{31}\Delta(I - G_{11}\Delta)^{-1}G_{13}$$

describes the uncertain plant set as Δ varies over the unit ball

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[2] For notational convenience everything is taken to be square. This is without loss of generality because a square structure can be achieved via the inclusion of dummy inputs and outputs as required.

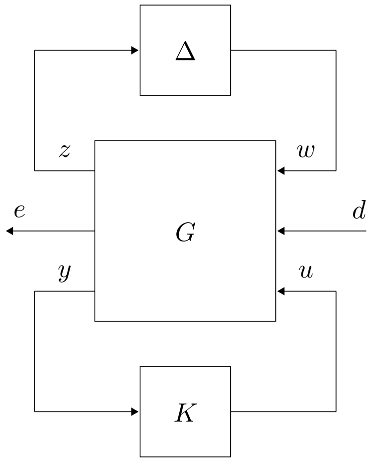

Fig. 1. General LFT Interconnection Structure

$$\mathcal{B}(\boldsymbol{\Delta}) := \{ \Delta \in \mathcal{H}_\infty^{r\times r} : \|\Delta\|_\infty < 1 \text{ and } \Delta(s) \in \boldsymbol{\Delta} \text{ for all } s \in \bar{\mathbb{C}}_+ \}, \quad (1)$$

where the block diagonally structured set

$$\boldsymbol{\Delta} := \{ \text{diag}_{i=1}^{f}(I_{\alpha_i} \otimes \Delta_i) : \Delta_i \in \mathbb{C}^{\beta_i \times \beta_i} \text{ and } \sum_{i=1}^{f} \alpha_i \beta_i = r \} \subset \mathbb{C}^{r\times r}, \qquad (2)$$

the Kroneker matrix product is defined by $A \otimes B := [a_{ij}B]$, $\mathcal{H}_\infty$ denotes the standard Hardy ∞-space (i.e. stable transfer functions) and $\|\cdot\|_\infty$ the associated norm; and

(ii) For a given controller K, the Lower LFT

$$\mathcal{F}_\ell\left(\begin{pmatrix} G_{22} & G_{23} \\ G_{32} & G_{33} \end{pmatrix}, K\right) := G_{22} + G_{23}K(I - G_{33}K)^{-1}G_{32} = \mathcal{F}_u(\mathcal{F}_\ell(G,K), 0)$$

accounts for all *nominal* (weighted) closed-loop transfer-functions to be used for gauging performance.

Then (Packard and Doyle, 1993; Zhou et al., 1996) $\mathcal{F}_u(\mathcal{F}_\ell(G,K),\Delta) \in \mathcal{H}_\infty^{m\times m}$ and

$$\mathcal{F}_u(\mathcal{F}_\ell(G,K),\Delta)\|_\infty \leq 1 \text{ for all } \Delta \in \mathcal{B}(\boldsymbol{\Delta})$$

if, and only if, $\mathcal{F}_\ell(G,K) \in \mathcal{H}_\infty^{(m+r)\times(m+r)}$ and

$$\sup_{\omega \in \mathbb{R}\cup\{\infty\}} \mu_{\boldsymbol{\Delta}_\text{T}}\left(\mathcal{F}_\ell(G,K)(j\omega)\right) \leq 1,$$

where $\mu_{\boldsymbol{\Delta}_\text{T}}$ denotes the structured singular value taken with respect to the structured set

$$\boldsymbol{\Delta}_\text{T} := \{\text{diag}(\Delta, \Delta_p) : \Delta \in \boldsymbol{\Delta} \text{ and } \Delta_p \in \mathbb{C}^{m\times m}\}.$$

Now, more specifically, consider the uncertain closed-loop system shown in Figure 2, where P is a nominal model of an irrigation channel, the inputs and outputs of the controller K are, respectively, the water level errors relative to set points and the positions of gates which regulate water flow, the disturbances d and w model water off-takes to farms and gate position uncertainty, respectively. For modelling details see Li et al. (2004) where, for

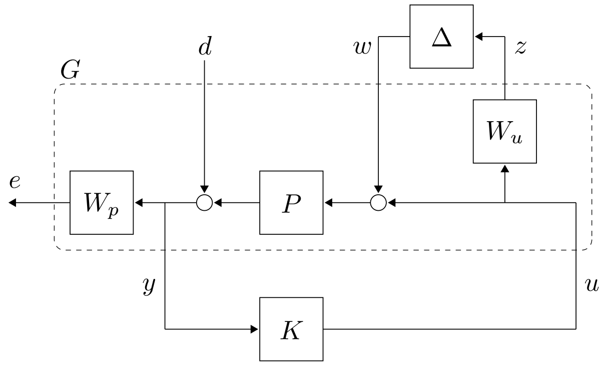

Fig. 2. A Closed-Loop Interconnection

a controller K described therein, the structured singular value result just described is used to verify that $\|W_p(I + P(I + \Delta W_u)K)^{-1}\|_\infty \leq 1$ for all $\|\Delta\|_\infty < 1$, with the particular performance and uncertainty weights W_p and W_u shown in Figure 3. In particular, the corresponding μ-curve, which is shown in Figure 4, is less than unity over all frequencies. Observe, however, that particularly over the lower frequency range, the value of μ is much less than unity. What might one conclude from this? Two possibilities are:

(i) The system could achieve more demanding performance requirements over the low frequency range (i.e. W_p could be larger at

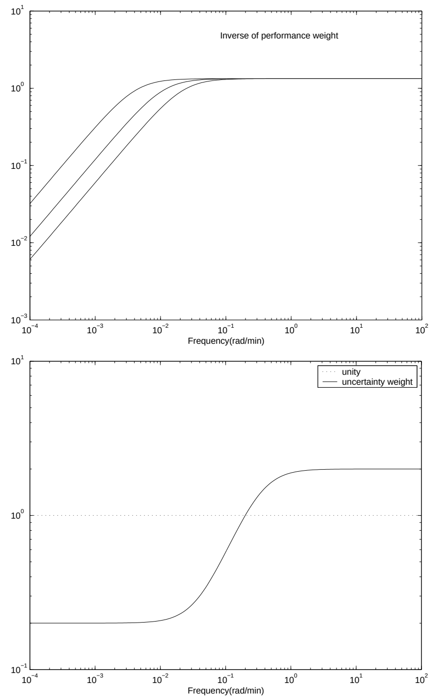

Fig. 3. The inverse of the performance weight W_p and the multiplicative uncertainty weight W_u

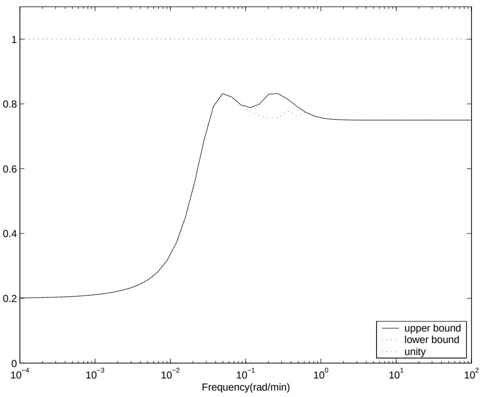

Fig. 4. μ-curve for the performance and uncertainty weights W_p and W_u shown in Figure 3

low frequencies), for the level of robustness characterised by the uncertainty weight W_u;

(ii) The system could tolerate more uncertainty over the low frequency range (i.e. W_u could be larger at low frequencies), for the level of performance characterised by the performance weight W_p.

In view of these possible conclusions, it is natural to ask:

(i) For a specific uncertain plant set, what level of performance (measured in terms of the ∞-norm) can be achieved in closed-loop?

(ii) How much uncertainty can be tolerated by a given closed-loop, while achieving a specified level of performance (measured in terms of the ∞-norm)?

The first question is addressed in Lanzon and Cantoni (2003), where the idea of skewed-μ introduced by Fan and Tits (1992) is extended to accommodate variation of the required scaling over frequency and performance channels. A corresponding algorithm for the synthesis of a controller and a weight function which reflects the achievable level of robust performance is also presented in Lanzon and Cantoni (2003). The second question is investigated along a similar line in this paper. In particular, the question is formulated in terms of an optimisation problem with a cost that reflects the size of weights used to quantify system uncertainty and a structured singular value constraint, which captures the specified level of robust performance. It is shown that solving the optimisation problem can yield a μ-curve which is close to unity across frequency, as is necessary to answer the second question.

2. FORMULATION OF THE OPTIMISATION PROBLEM

Towards quantifying the level of uncertainty that can be tolerated by a given closed-loop, while achieving a specified level of performance, consider the LFT interconnection structure shown in Figure 5. Here the uncertainty weight W has been purposefully omitted from the construction of the generalised plant G, which includes a nominal plant model and performance weights. Furthermore, it is assumed that K achieves nominal closed-loop stability, in that $\mathcal{F}_\ell(G,K) \in \mathcal{H}_\infty^{(m+r)\times(m+r)}$.

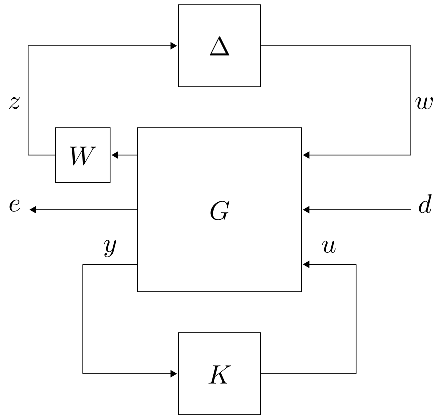

Fig. 5. LFT configuration for quantifying tolerable uncertainty

Now, let the structured sets $\mathbf{\Delta}$ and $\mathcal{B}(\mathbf{\Delta})$ be as defined in (1) and (2). Then from the preceding discussion, $\mathcal{F}_u\left(\mathcal{F}_\ell(G,K),\Delta\right) \in \mathcal{H}_\infty^{m\times m}$ and

$$\|\mathcal{F}_u\left(\mathcal{F}_\ell(G,K),\Delta\right)\|_\infty \le 1 \text{ for all } \Delta \in \mathcal{B}(\mathbf{\Delta}) \quad (3)$$

if, and only if,

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \mu_{\mathbf{\Delta}_\mathrm{T}}\left(\begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)(j\omega)\right) \le 1, \quad (4)$$

where

$$\mathbf{\Delta}_\mathrm{T} := \{\mathrm{diag}(\Delta,\Delta_p) : \Delta\in\mathbf{\Delta} \text{ and } \Delta_p\in\mathbb{C}^{m\times m}\}. \quad (5)$$

The level of tolerable uncertainty could, as such, be quantitatively determined by appropriately maximising some measure of the size of the uncertainty weight W, while ensuring that (4) is satisfied. To this end, consider the following optimisation problem:

Problem 1. Given an optimisation directionality function

$$V \in \mathcal{V} := \{\mathrm{diag}_{i=1}^{r}(v_i) \ : \ v_i \in \mathcal{H}_2\},$$

where $\mathcal{H}_2$ denotes the standard Hardy 2-space,

$$\begin{gathered} \min_{W\in\mathcal{W}} \|VW^{-1}\|_2^2 \\ \text{subject to} \\ \sup_{\omega\in\mathbb{R}\cup\{\infty\}} \mu_{\mathbf{\Delta}_\mathrm{T}}\left(\begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_l(G,K)(j\omega)\right) \le 1, \end{gathered} \quad (6)$$

where the set of permissible uncertainty weights is defined by

$$\mathcal{W} := \{\mathrm{diag}_{i=1}^{r}(w_i) \ : \ w_i, \frac{1}{w_i} \in \mathcal{H}_\infty\}$$

and $\|\cdot\|_2$ denotes the $\mathcal{H}_2$-norm.

As it stands, this problem is not computationally straightforward, since the μ-constraint is difficult to handle. Before discussing this aspect of the problem, however, it is instructive to briefly discuss the role of the optimisation directionality function V. First observe that

$$\|VW^{-1}\|_2^2 = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \left| \frac{v_i(j\omega)}{w_i(j\omega)} \right|^2 d\omega, \qquad (7)$$

where $w_i(j\omega)$ (resp. $v_i(j\omega)$) is the i-th diagonal element of $W(j\omega)$ (resp. $V(j\omega)$). From this decomposition, it can be seen that the cost function $1/\|VW^{-1}\|_2^2$ is a cumulative measure of the frequency-dependent size of the uncertainty weights $w_i(j\omega)$. Each uncertainty weight $w_i(j\omega)$ is itself weighted across frequency by an optimisation directionality $v_i(j\omega)$. This can be used to steer the optimisation by choosing $v_i(j\omega)$ to be large (resp. small) where it is expected that the corresponding uncertainty weight $w_i(j\omega)$ should be large (resp. small).[3] Moreover, note that any possible inconsistency between the directionality functions, and the level of robust performance specified by the performance weights within the generalised plant G, is resolved through the optimisation over the uncertainty weights.

3. SOLVING THE OPTIMISATION PROBLEM

As mentioned above, the optimisation Problem 1 is difficult to solve because of the μ-constraint. In this section, it is argued that the μ-constraint can be replaced by one that is more amenable to computation. This may or may not introduce some conservatism as discussed below.

For a given matrix $M \in \mathbb{C}^{r\times r}$, it can be shown that (Packard and Doyle, 1993; Zhou et al., 1996; Skogestad and Postlethwaite, 1996)

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D\in\mathcal{C}(\mathbf{\Delta})} \bar{\sigma}(DMD^{-1}),$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value, $\mathbf{\Delta}$ is the structured set defined in (2) and[4]

$$\mathcal{C}(\mathbf{\Delta}) := \{D \in \mathbb{C}^{r\times r} : \det(D) \neq 0 \text{ and } D\Delta = \Delta D \text{ for all } \Delta \in \mathbf{\Delta}\}. \qquad (8)$$

In general, equality does not hold, but there are situations in which it does (Packard and Doyle, 1993; Zhou et al., 1996). Similarly, with $\mathbf{\Delta}_{\mathrm{T}}$ as defined in (5), it follows that

$$\begin{aligned}
&\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \mu_{\mathbf{\Delta}_{\mathrm{T}}}\left(\begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_\ell(G,K)(j\omega)\right) \\
&\leq \inf_{\substack{D_1\in\mathcal{D}(\mathbf{\Delta}) \\ d_2, \frac{1}{d_2}\in\mathcal{H}_\infty}} \left\| \begin{pmatrix} D_1 W & 0 \\ 0 & d_2 I_m \end{pmatrix} \mathcal{F}_\ell(G,K) \begin{pmatrix} D_1^{-1} & 0 \\ 0 & \frac{1}{d_2} I_m \end{pmatrix} \right\|_\infty \\
&= \inf_{D\in\mathcal{D}(\mathbf{\Delta})} \left\| \begin{pmatrix} DW & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_\ell(G,K) \begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix} \right\|_\infty, \qquad (9)
\end{aligned}$$

where

$$\mathcal{D}(\mathbf{\Delta}) := \{\ D \in \mathcal{H}_\infty^{r\times r} \ : \ D^{-1} \in \mathcal{H}_\infty^{r\times r} \text{ and } D(s) \in \mathcal{C}(\mathbf{\Delta}) \text{ for all } s \in \bar{\mathbb{C}}_+\}. \qquad (10)$$

Note that the upper bound in (9) is more amenable to computation than the structured singular value itself. As such, the question of how much uncertainty can be tolerated by a given closed-loop system, while achieving a specified level of performance, could be addressed in terms of the optimisation problem:

Problem 2. Given an optimisation directionality function $V \in \mathcal{V}$,

$$\min_{W\in\mathcal{W}} \|VW^{-1}\|_2^2$$

$$\text{subject to} \qquad (11)$$

$$\inf_{D\in\mathcal{D}(\mathbf{\Delta})} \left\| \begin{pmatrix} DW & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_\ell(G,K) \begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix} \right\|_\infty \leq 1.$$

In general, the constraint is not convex. However, as shown in the following subsection, the problem can be reformulated in terms of a problem that is convex pointwise in frequency, when the uncertainty set is unstructured. An iterative algorithm is developed in a subsequent subsection for the more general case of structured uncertainty.

3.1 The case of unstructured uncertainty

Consider the case in which the system uncertainty is modelled to be unstructured (i.e. in the definition (2) of $\mathbf{\Delta}$, $f = \alpha_1 = 1$ and $\beta_1 = r$). Then the relationship (9) becomes (Packard and Doyle, 1993; Zhou et al., 1996)

[3] As also pointed out in Lanzon and Cantoni (2003), the direction of steepest descent of $\sum_{i=1}^{r} |\frac{v_i}{w_i}|^2$ is dominated by the smallest ratio $\left|\frac{w_i}{v_i}\right|$.

[4] In the definition of $\mathcal{C}(\mathbf{\Delta})$ it is possible to replace the constraint $\det(D) \neq 0$ with the constraint $D = D^* > 0$, without loss of generality (Zhou et al., 1996).

$$\sup_{\omega\in\mathbb{R}\cup\{\infty\}} \mu_{\mathbf{\Delta}_\mathrm{T}}\left(\begin{pmatrix} W & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)(j\omega)\right)$$
$$= \inf_{d_1, \frac{1}{d_1}\in\mathcal{H}_\infty} \left\| \begin{pmatrix} d_1 W & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)\begin{pmatrix} \frac{1}{d_1} I_r & 0 \\ 0 & I_m \end{pmatrix}\right\|_\infty$$
$$= \inf_{d, \frac{1}{d}\in\mathcal{H}_\infty} \left\| \begin{pmatrix} W & 0 \\ 0 & dI_m \end{pmatrix}\mathcal{F}_\ell(G,K)\begin{pmatrix} I_r & 0 \\ 0 & \frac{1}{d} I_m \end{pmatrix}\right\|_\infty . \quad (12)$$

As such, in this case, the optimisation Problems 1 and 2 are equivalent and, furthermore, they can be reformulated in terms of a problem that is convex pointwise in frequency:

Problem 3. Given an optimisation directionality function $V = \mathrm{diag}_{i=1}^r(v_i) \in \mathcal{V}$,

$$\min_{W=\mathrm{diag}_{i=1}^r(w_i)\in\mathcal{W}} \int_{-\infty}^{\infty} \sum_{i=1}^{r} \frac{|v_i(j\omega)|^2}{|w_i(j\omega)|^2}\, d\omega$$
subject to (13)
$$\forall\, \omega \in \mathbb{R}\ \exists\, \delta_\omega \text{ so that}$$
$$\mathcal{F}_\ell(G(j\omega),K(j\omega))\begin{pmatrix} I_r & 0 \\ 0 & \delta_\omega I_m \end{pmatrix}\mathcal{F}_\ell(G(j\omega),K(j\omega))^* \leq \begin{pmatrix} \mathrm{diag}_{i=1}^r(1/|w_i(j\omega)|^2) & 0 \\ 0 & \delta_\omega I_m \end{pmatrix}.$$

Note that at each frequency the constraint in Problem 3 is convex in δ_ω and $1/|w_i(j\omega)|^2$, $i = 1,\ldots,r$. Indeed, since the sum in the cost is always non-negative, Problem 3 can be approximately solved pointwise in frequency using standard LMI tools (Gahinet et al., 1995; Boyd et al., 1993). If required, transfer function characterisations of W and D can be obtained via interpolation of the pointwise solutions, followed by spectral factorisation. State-space methods, similar to those described in Lanzon and Cantoni (2003) can then be used to refine the pointwise solution.

3.2 An iterative algorithm for the case of structured uncertainty

In the case that $\mathbf{\Delta}$ is structured, the following iterative approach could be employed to obtain a local solution the optimisation Problem 2:

(i) Set $i = 0$ and $\lambda_0^\star = \|VW_0^{-1}\|_2^2$, with $W_0 \in \mathcal{W}$ taken to satisfy
$$\inf_{D\in\mathcal{D}(\mathbf{\Delta})} \left\| \begin{pmatrix} DW_0 & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)\begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix}\right\|_\infty < 1,$$
where $\mathcal{D}(\mathbf{\Delta})$ is defined (10).

(ii) Solve
$$\theta_i^\star := \mathrm{argmin}_{0\leq\theta\leq 1}\theta$$
subject to (14)
$$\inf_{D\in\mathcal{D}(\mathbf{\Delta})} \left\| \begin{pmatrix} DW_i & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)\begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix}\right\|_\infty < \theta,$$
and select $D_i \in \mathcal{D}(\mathbf{\Delta})$ so that
$$\left\| \begin{pmatrix} D_iW_i & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)\begin{pmatrix} D_i^{-1} & 0 \\ 0 & I_m \end{pmatrix}\right\|_\infty \leq \theta_i^\star.$$

(iii) Set $i = i + 1$ and solve
$$W_i := \mathrm{argmin}_{W\in\mathcal{W}}\|VW^{-1}\|_2^2$$
subject to (15)
$$\left\| \begin{pmatrix} D_{i-1}W & 0 \\ 0 & I_m \end{pmatrix}\mathcal{F}_\ell(G,K)\begin{pmatrix} D_{i-1}^{-1} & 0 \\ 0 & I_m \end{pmatrix}\right\|_\infty \leq 1,$$
and let $\lambda_i^\star := \|VW_i^{-1}\|_2^2$.

(iv) If $|\lambda_i^\star - \lambda_{i-1}^\star|$ is sufficiently small then stop, else return to Step *(ii)*.

Observe that the intermediate optimisation problems (14) and (15) can both be reformulated as convex problems, either pointwise in frequency, as in Problem 3 above, or using state-space techniques similar to those described in Lanzon and Cantoni (2003). Furthermore, note that since W_{i-1} is always feasible for Step *(iii)*, $\lambda_i^\star \leq \lambda_{i-1}^\star$ at each iteration.

4. CONCLUDING EXAMPLE

Consider the robust performance problem described in the introduction and illustrated in Figure 2. Solving the corresponding optimisation Problem 2, with the optimisation directionality function
$$V = \mathrm{diag}_{i=1}^3\left(\frac{2\times 10^5(s+0.0354)}{(s+0.3536)(s+10^5)}\right)$$
which is the uncertainty weight W_u shown in Figure 3, with additional roll-off beyond 10^5 rad/sec to ensure that $V \in \mathcal{H}_2$, yields the results shown in Figure 6. Observe that in both the structured (3 scalar blocks) and unstructured cases, the μ-curves with the optimised uncertainty weights are closer to unity than their nominal counterparts. Indeed, one could conclude that significantly more uncertainty can be tolerated at low frequencies, than reflected by the nominal uncertainty weight W_u shown in Figure 3. Finally, it is evident that the iterative algorithm proposed for the case of structured uncertainty has NOT yielded a global minimum, since the μ-curve is not unity across frequency as one would expect. By contrast for the unstructured uncertainty case, for which standard convex programming techniques can be used, the μ-curve with optimised weights is essentially unity across frequency.

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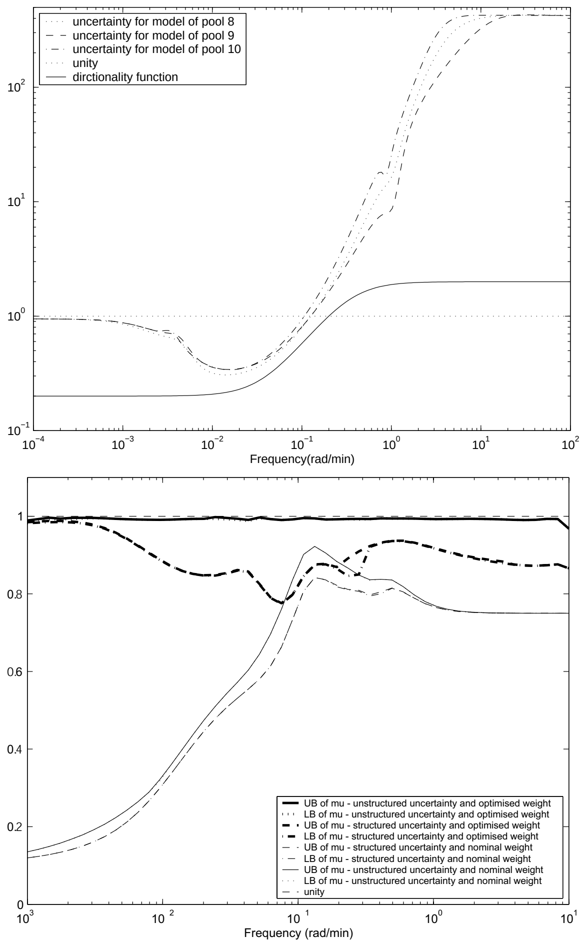

Fig. 6. Quantifying tolerable uncertainty for the closed-loop shown in Figure 2

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