ADAPTIVE REGULATION TO INVARIANT SETS

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Abstract: A new framework for adaptive regulation of dynamical systems to invariant sets is proposed. The framework allows to adaptively steer the system trajectories to a desired non-equilibrium state without requiring knowledge or even existence of a specific strict Lyapunov function.

Keywords: adaptive systems, non-equilibrium dynamics, invariance, algorithms in finite form

1. INTRODUCTION

In control theory, solutions to adaptive as well as non-adaptive problems are usually pursued in terms of stabilization, be it of an intrinsic or introduced equilibrium, or of the tracking of a reference signal. Stabilization in control theory requires that for a given feedback the Lyapunov function must be known that ensures the asymptotic stability of the target dynamics. While the function ensures asymptotic convergence of the solution without adaptation, in the presence of adaptation it may fail to do so. A breakthrough solution to this problem was reported in (Panteley *et al.*, 2002; Astolfi and Ortega, 2003). The solution applies to equilibria that can be made asymptotically stable by state feedback.

The solution obscures a more fundamental issue; many problems in the sciences as well as in engineering, call for a different strategy. Rather than enforced stability of an arbitrary, often extrinsically designed, equilibrium, they need a solution in terms the natural motions of the system itself. Amongst these, the one which optimally satisfies the control goal is selected and modified gently with small control effort (Kolesnikov, 1994; Fradkov, 2003). An early application of this principle (Ott *et al.*, 1990) has fascinated many theorists and led to practical applications.

The problem of non-equilibrium control is gaining substantial attention in recent year, especially in the framework of output regulation. Byrnes and Isidori (Byrnes and Isidori, 2003) proposed a number of sufficient and necessary conditions assuring the existence of a solution to this problem.

The contribution of our paper is as follows. First, we aim to formulate the problem of *adaptive regulation* to a desired non-equilibrium dynamics. The dynamics should be invariant under system flow. the formulation will require properties such as boundedness of the trajectories and/or partial stability (Vorotnikov, 1998). Lyapunov asymptotic stability should not be required a-priori. Second, under these assumptions we derive adaptation algorithms capable of steering the system trajectories to the desired invariant set. To this purpose we employ the *algorithms in finite form* (Tyukin, 2003). These algorithms guarantee improved performance and can handle nonlinear parametrization of the uncertainty (Tyukin *et al.*, 2003*a*). The main idea of this approach is to introduce the desired invariant set into the system dynamics by means of virtual adaptation algorithms and then realize them using an embedding technique proposed in (Tyukin *et al.*, 2003*b*; Tyukin *et al.*, 2004; Tyukin *et al.*, 2003*a*).

The paper is organized as follows: in Section 2 we provide the necessary notations and formulate the problem. Section 3 contains the main results of the paper given in Theorem 3. The proof of the theorem is provided in subsequent subsections. Each subsection substitutes a step of the proof. Subsection 3.1 addresses the design of the virtual algorithms, Subsection 3.2 provides an auxiliary system which is necessary for the embedding, Subsection 3.3 contains the main arguments of the proof. Section 4 concludes the paper.

Throughout the paper we will use the following notations: symbol $\mathbf{x}(t, \mathbf{x}_0, t_0)$ stands for the flow which maps $\mathbf{x}_0 \in \mathbb{R}^n, t_0, t \in \mathbb{R}_+$ into $\mathbf{x}(t)$. Function $\nu : R_+ \rightarrow R$ is said to belong to L_2 iff $L_2(\nu) = \int_0^\infty \nu^2(\tau)d\tau < \infty$. The value $\sqrt{L_2(\nu)}$ stands for the L_2 norm of $\nu(t)$. Function $\nu : R_+ \rightarrow R$ belongs to L_∞ iff $L_\infty(\nu) = \sup_{t\geq 0} \|\nu(t)\| < \infty$, where $\|\cdot\|$ is the Euclidean norm. The value of $L_\infty(\nu)$ stands for the L_∞ norm of $\nu(t)$.

2. PROBLEM FORMULATION

Definition 1. A point $p \in \mathbb{R}^n$ is called an ω-limit point $\omega(\mathbf{x}(t, \mathbf{x}_0, t_0))$ of $\mathbf{x}_0 \in \mathbb{R}^n$ if there exists a sequence $\{t_i\}$, $t_i \rightarrow \infty$, such that $\mathbf{x}(t, \mathbf{x}_0, t_0) \rightarrow p$. The set of all limit points $\omega(\mathbf{x}(t, \mathbf{x}_0, t_0))$ is the ω-limit set of $\mathbf{x}_0$.

In order to represent explicitly in our notation which particular flow is referred to in the notion of the ω-limit set we use notations $\omega_{\mathbf{f}}(\mathbf{x}_0)$ (and $\mathbf{x}_{\mathbf{f}}(t, \mathbf{x}_0, t))$ to denote the ω-limit set (and flow) of $\mathbf{x}_0$ in the following system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}_0 \in X \subset \mathbb{R}^n$. Symbol $\Omega_{\mathbf{f}}(\mathbf{x})$ denotes the union of all $\omega_{\mathbf{f}}(\mathbf{x}_0)$, $\mathbf{x}_0 \in X$. Throughout the paper we will refer to set $\Omega_{\mathbf{f}}(\mathbf{x})$ as the $\Omega_{\mathbf{f}}$-limit set (or simply Ω-limit set if the corresponding flow is defined from the context) of the system.

Definition 2. Set $S \subset \mathbb{R}^n$ is invariant (forward-invariant) under the flow $\mathbf{x}_{\mathbf{f}}(t, \mathbf{x}_0, t_0)$ iff $\mathbf{x}_{\mathbf{f}}(t, \mathbf{x}_0, t_0) \in S$ for any $\mathbf{x}_0 \in S$ for all $t > t_0$.

We consider the following class of systems:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + G_u(\phi(\mathbf{x})\boldsymbol{\theta} + \mathbf{u}), \\ \dot{\boldsymbol{\theta}} &= S(\boldsymbol{\theta}),\ \boldsymbol{\theta}(t_0) \in \Theta \subset \mathbb{R}^d \end{aligned} \tag{1}$$

where $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^{m\times d}$, are C^0-smooth vector-fields, $G_u \in \mathbb{R}^{n\times m}$, $\boldsymbol{\theta}$ is a vector of unknown time-varying parameters, $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $S \in C^1$ is known. The vector of initial conditions $\boldsymbol{\theta}(t_0) \in \Theta$, however, is assumed to be unknown. Without loss of generality we assume that $\Omega_S(\Theta) \subseteq \Theta$, and that Θ is bounded. Our goal is to steer the system to the *target domain*:

$$\Omega^*(\mathbf{x}) \subset \mathbb{R}^n$$

Let us introduce the following set of assumptions related to the choice of domain $\Omega^*(\mathbf{x})$.

Assumption 1. Set $\Omega^*(\mathbf{x}) \subset \mathbb{R}^n$ is bounded and closed in $\mathbb{R}^n$.

Assumption 2. There exists a positive-definite matrix $H = H^T \in \mathbb{R}^{d\times d}$, such that function $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$ in (1) satisfy the following inequality:

$$H\frac{\partial S(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \frac{\partial S(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^T H \leq 0\ \forall\ \boldsymbol{\theta} \in \mathbb{R}^d$$

Assumption 3. For the given $\Omega^*(\mathbf{x})$ and system (1) there exists a control function $\mathbf{u}_0(\mathbf{x})$ such that

$$G_u\mathbf{u}_0(\mathbf{x}) + \mathbf{f}(\mathbf{x}) = \mathbf{f}_0(\mathbf{x})$$

and, furthermore, for any $\mathbf{x}_0 \in \mathbb{R}^n$ the following holds: $\Omega^*(\mathbf{x}) \subset \Omega_{\mathbf{f}_0}(\mathbf{x})$, where the flow $\mathbf{x}(t, \mathbf{x}_0, t)$ is defined by

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) \tag{2}$$

Let us finally introduce two alternative hypotheses. The first is formulated in Assumptions 4, 5, and 6. The second is given by Assumption 7.

Assumption 4. There exist functions $\psi(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$, $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, and induced by function $\psi(\mathbf{x})$ set:

$$\Omega_\psi = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} : \varphi(\psi(\mathbf{x})) = 0\}$$

such that the following holds $\Omega^* \subseteq \Omega_{\mathbf{f}_0}(\Omega_\psi)$, i. e. $\Omega^*(\mathbf{x})$ is the largest invariant set of (2) in Ω_ψ.

Assumption 5. For the given function $\psi(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$, $\psi(\mathbf{x}) \in C^1$ and vector field $\mathbf{f}_0(\mathbf{x})$ defined in (2) there exists function $\beta(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $\beta(\mathbf{x})$ is separated from zero and satisfies the following equality:

$$\begin{aligned} &\psi\frac{\partial \psi(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}_0(\mathbf{x}) \leq -\beta(\mathbf{x})\varphi(\psi)\psi, \\ &\int_0^\psi \varphi(\sigma)d\sigma \geq 0,\ \lim_{\psi\rightarrow\infty} \int_0^\psi \varphi(\sigma)d\sigma = \infty \end{aligned} \tag{3}$$

Assumption 6. For the given function $\psi(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$, $\psi(\mathbf{x}) \in C^1$ the following relation holds:

$$\psi(\mathbf{x}(t)) \in L_\infty \Rightarrow \mathbf{x} \in L_\infty$$

Notice that function $\psi(\mathbf{x})$ in Assumptions 5, 6 should not necessarily be (positive) definite. Neither this is require for function $\varphi(\psi)\psi$.

Assumption 7. Consider system (2) with additive input $\boldsymbol{\varepsilon}_0(t) : \mathbb{R} \rightarrow \mathbb{R}^n$, $\boldsymbol{\varepsilon}_0(t) \in C^1$:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \boldsymbol{\varepsilon}_0(t), \ \boldsymbol{\varepsilon}_0 \in L_2 \tag{4}$$

System (4) has finite $L_2 \rightarrow L_\infty$ gain, and in addition $\Omega^* \subseteq \Omega_{\mathbf{f}_0}$.

The main question of our current study is wether it is possible to design the adaptation algorithm $\hat{\boldsymbol{\theta}}(t)$ for system (1) such that the feedback of the following form

$$\mathbf{u}(\mathbf{x}, \boldsymbol{\xi}) = \mathbf{u}(\mathbf{x}, \boldsymbol{\xi}, \hat{\boldsymbol{\theta}}), \ \dot{\boldsymbol{\xi}} = \mathbf{f}_\xi(\mathbf{x}, \boldsymbol{\xi}, \hat{\boldsymbol{\theta}}), \ \boldsymbol{\xi} \in \mathbb{R}^k$$

ensures boundedness of the trajectories in the closed loop system and that $\mathbf{x}(t) \rightarrow \Omega^*$ as $t \rightarrow \infty$.

3. MAIN RESULTS

The main idea of our approach is two-fold. First, we search for the desired dynamics of the closed loop system with feedback $\mathbf{u}(\mathbf{x}, \boldsymbol{\xi}, \hat{\boldsymbol{\theta}})$ and yet unknown $\hat{\boldsymbol{\theta}}(t)$, $\boldsymbol{\xi}(t)$, which ensures the desired properties of the controlled system. These properties should allow us to show that under specific conditions $\mathbf{x}(t) \rightarrow \Omega^*$ as $t \rightarrow \infty$. Derivative of function $\hat{\boldsymbol{\theta}}(t)$ with respect to t, can, at this stage, depend on unknown parameters $\boldsymbol{\theta}$. The family of all such desired subsystems is referred to as *virtual adaptation algorithms.*

The second stage of our method is to render these algorithms into computable and physically realizable form. In particular, these realizations neither should rely on a-priory unknown parameters, nor require measurements of the right-hand side of (1) (i.e. derivatives).

In order to achieve this goal we invoke *algorithms in finite form* (Tyukin, 2003; Tyukin *et al.*, 2003*a*). For for the purpose of physically realizable and computable control the embedding argument was introduced in (Tyukin *et al.*, 2003*b*; Tyukin *et al.*, 2004). In general, finite form realizations of virtual adaptation algorithms require the analytic solution of a partial differential equation, the *explicit realization condition.* with the embedding technique proposed in our earlier publications, however, it is possible to meet this requirement and derive adaptation schemes as known and well-defined functions of $\mathbf{x}, t$. The main result of our current study is formulated in Theorems 3 and 4.

Theorem 3. Let system (1) be given and Assumptions 1–6 hold. Let, in addition, there exist C^1-smooth function $\kappa(\mathbf{x})$ such that the following estimate holds: $\left\| \frac{\partial \psi(\mathbf{x})}{\partial \mathbf{x}} \right\| \leq |\kappa(\mathbf{x})|$. Then there exists an auxiliary system

$$\begin{aligned} &\dot{\boldsymbol{\xi}} = \mathbf{f}_\xi(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\nu}) \\ &\dot{\boldsymbol{\nu}} = \mathbf{f}_\nu(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\nu}), \ \boldsymbol{\xi} \in \mathbb{R}^n, \ \boldsymbol{\nu} \in \mathbb{R}^d \end{aligned} \tag{5}$$

control input $\mathbf{u}(\mathbf{x}, \hat{\boldsymbol{\theta}}) = \mathbf{u}_0(\mathbf{x}) - \phi(\boldsymbol{\xi})\hat{\boldsymbol{\theta}}(t)$, and adaptation algorithm

$$\begin{aligned} &\hat{\boldsymbol{\theta}} = (H^{-1}\Psi(\boldsymbol{\xi})\mathbf{x} + \hat{\boldsymbol{\theta}}_I(t)), \\ &\Psi(\boldsymbol{\xi}) = (\kappa^2(\boldsymbol{\xi}) + 1)(G_u \phi(\boldsymbol{\xi}))^T \\ &\dot{\hat{\boldsymbol{\theta}}}_I = S(\hat{\boldsymbol{\theta}}) - H^{-1}\frac{\partial \Psi(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \mathbf{f}_\xi(\mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\nu})\mathbf{x} - \\ &\quad H^{-1}\Psi(\boldsymbol{\xi})\mathbf{f}_0(\mathbf{x}) \end{aligned} \tag{6}$$

such that the following properties hold:

1) $\hat{\boldsymbol{\theta}}(t), \mathbf{x}(t) \in L_\infty$

2) trajectories $\mathbf{x}(t)$ converge into the domain Ω^* as $t \rightarrow \infty$

3) if $G_u \phi(\boldsymbol{\xi})$ is persistently exciting then $\hat{\boldsymbol{\theta}}(t, \hat{\boldsymbol{\theta}}_0, t_0)$ asymptotically converges to $\boldsymbol{\theta}(t, \boldsymbol{\theta}_0, t_0)$.

Theorem 4. Let system (1) be given and Assumptions 1–3, and 7 hold. Then there exists an auxiliary system of type (5), control input $\mathbf{u}(\mathbf{x}, \hat{\boldsymbol{\theta}}) = \mathbf{u}_0(\mathbf{x}) - \phi(\boldsymbol{\xi})\hat{\boldsymbol{\theta}}(t)$ and adaptation algorithm (6) with $\kappa(\boldsymbol{\xi}) \equiv 0$ such that statements 1)–3) of Theorem 3 hold.

The proof of the theorems is given in the next subsections. In subsection 3.1 we derive virtual adaptation algorithms which satisfy in part the requirement of the theorem. Subsection 3.2 introduces function $\boldsymbol{\xi}(t)$ satisfying the embedding assumption from (Tyukin *et al.*, 2003*b*),(Tyukin *et al.*, 2003*a*). In subsection 3.3 we combine these results to complete the proofs.

3.1 Design of Virtual Adaptive Algorithms

Let us consider the following dynamic state feedback $\mathbf{u}(\mathbf{x}, \hat{\boldsymbol{\theta}}) = \mathbf{u}_0(\mathbf{x}) - \phi(\boldsymbol{\xi})\hat{\boldsymbol{\theta}}(t)$. This feedback renders system (1) into the following form

$$\begin{aligned} \dot{\mathbf{x}} = &\mathbf{f}_0(\mathbf{x}) + G_u \phi(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)) + \\ &G_u(\phi(\mathbf{x}) - \phi(\boldsymbol{\xi}))\boldsymbol{\theta}, \end{aligned} \tag{7}$$

Let us denote $G_u \phi(\mathbf{x}) = \boldsymbol{\alpha}(\mathbf{x})$ and consider the following auxiliary system

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t), \\ &\dot{\boldsymbol{\theta}} = S(\boldsymbol{\theta}) \\ &\dot{\hat{\boldsymbol{\theta}}} = S(\hat{\boldsymbol{\theta}}) + H^{-1}(\kappa^2(\boldsymbol{\xi}) + 1)\boldsymbol{\alpha}(\boldsymbol{\xi})^T \times \\ &\quad (\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)) \\ &\kappa(\boldsymbol{\xi}) : \mathbb{R}^n \rightarrow \mathbb{R}, \ \kappa \in C^1 \end{aligned} \tag{8}$$

Lemma 5. (Virtual Adaptation Algorithm). Let system (8) be given and Assumptions 1–3, 6 hold. Furthermore, let $\kappa(\boldsymbol{\xi}(t))\boldsymbol{\varepsilon}(t) \in L_2$, and $\boldsymbol{\varepsilon} \in L_2$.

Then the following statements hold:

1) $\hat{\boldsymbol{\theta}}(t)$ is bounded for every $\theta(t_0) \in \Theta$, $\hat{\boldsymbol{\theta}}(t_0) \in \mathbb{R}^d$

2) $\kappa(\boldsymbol{\xi})\boldsymbol{\alpha}(\boldsymbol{\xi})(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}(t))$, $\boldsymbol{\alpha}(\boldsymbol{\xi})(\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}(t)) \in L_2$

3) Let, in addition, $\left\|\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}}\right\| \leq |\kappa(\mathbf{x})|$, $\mathbf{x}-\boldsymbol{\xi} \in L_\infty$ then $\mathbf{x} \in L_\infty$

4) if, independently on the conditions of statement 3), $\boldsymbol{\varepsilon}(t) \equiv 0$ and the function $\alpha(\boldsymbol{\xi})$ is persistently exciting, i. e. there exist constants $\delta, T > 0$ such that $\int_t^{t+T} \alpha(\boldsymbol{\xi}(\tau))^T\alpha(\boldsymbol{\xi}(\tau)) \geq \delta I_d$ then trajectory $\hat{\boldsymbol{\theta}}(t)$ converges to $\boldsymbol{\theta}(t, \boldsymbol{\theta}_0, t_0)$ exponentially fast.

Lemma 5 proof. Let us show that statements 1) and 2) hold. Consider the following positive-definite function:

$V_\theta(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, t) = \|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|_H^2 + \epsilon = (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T H(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \epsilon$, where $\epsilon(t) = \frac{1}{2}\int_t^\infty (\kappa^2(\boldsymbol{\xi}(\tau))+1)\boldsymbol{\varepsilon}^T(\tau)\boldsymbol{\varepsilon}(\tau)d\tau \geq 0$. According to the lemma assumptions function $\kappa(\boldsymbol{\xi}(t))\boldsymbol{\varepsilon}(t) \in L_2$. This implies that $\epsilon(t)$ is bounded for every $t > t_0$ and therefore function V_θ is well-defined. Let us consider derivative $\dot{V}_\theta$:

$$
\begin{aligned}
\dot{V}_\theta = &(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T H(S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}})) + (S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}))^T \times \\
&H(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) - 2(\kappa^2(\boldsymbol{\xi})+1)((\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi}) \times \\
&\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T\boldsymbol{\varepsilon}(t) + \frac{\|\boldsymbol{\varepsilon}(t)\|^2}{4}) \\
&= (\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T(S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}})) + (S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}))^T \times \\
&(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) - 2(\kappa^2(\boldsymbol{\xi})+1) \times \\
&\|(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi}) + 0.5\boldsymbol{\varepsilon}(t)\|^2 \qquad (9)
\end{aligned}
$$

Function $S(\cdot)$ is continuous, therefore, applying Hadamard lemma we can write the difference $S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}})$ as follows: $S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) = \int_0^1 \frac{\partial S(\mathbf{z}(\lambda))}{\partial\mathbf{z}(\lambda)}d\lambda(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})$, $\mathbf{z}(\lambda) = \boldsymbol{\theta}\lambda + \hat{\boldsymbol{\theta}}(1-\lambda)$. Hence applying the Mean Value Theorem we derive that $S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) = \frac{\partial S(\mathbf{z}(\lambda'))}{\partial\mathbf{z}(\lambda')}(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})$ for some $\lambda' \in [0,1]$. The last equation leads to the following:

$$
\begin{aligned}
\dot{V}_\theta = &(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T(\frac{\partial S(\mathbf{z}(\lambda'))}{\partial\mathbf{z}(\lambda')}^T H + H\frac{\partial S(\mathbf{z}(\lambda'))}{\partial\mathbf{z}(\lambda')})(\boldsymbol{\theta} - \\
&\hat{\boldsymbol{\theta}}) - 2(\kappa^2(\boldsymbol{\xi})+1)\|(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi}) + 0.5\boldsymbol{\varepsilon}(t)\|^2 \\
&\leq -2(\kappa^2(\boldsymbol{\xi})+1)\|(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi}) + \\
&0.5\boldsymbol{\varepsilon}(t)\|^2 \leq 0 \qquad (10)
\end{aligned}
$$

Inequality (10) ensures that $(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) \in L_\infty$. Taking into account that for every $\boldsymbol{\theta}_0 \in \Theta$ solutions $\boldsymbol{\theta}(t, \boldsymbol{\theta}_0, t_0) \subset \Omega(\Theta) \subseteq \Theta$, where Θ is the bounded set, we can conclude that trajectories $\hat{\boldsymbol{\theta}}(t)$ are bounded, i.e. $\hat{\boldsymbol{\theta}}(t) \in L_\infty$. Thus, 1) is proven. Let us prove statement 2) of the lemma. Notice that function $V(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}, t)$ is non-increasing and bounded from below. Therefore $\kappa(\boldsymbol{\xi})((\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi}) + 0.5\boldsymbol{\varepsilon}(t)) \in L_2$. Hence function $\kappa(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi})$ belongs to L_2 as a sum of two functions from L_2. The fact that $\kappa^2(\boldsymbol{\xi})+1$ is separated from zero implies that $(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}})^T\boldsymbol{\alpha}^T(\boldsymbol{\xi}) \in L_2$. This proves 2).

Let us show that $\mathbf{x}(t) \in L_\infty$ under conditions formulated in statement 3) of the lemma. Consider

$$
\begin{aligned}
\dot{\psi} = &\frac{\partial\psi}{\partial\mathbf{x}}\mathbf{f}_0(\mathbf{x}) + \frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}}\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \frac{\partial\psi}{\partial\mathbf{x}}\boldsymbol{\varepsilon}(t) \\
&= \frac{\partial\psi}{\partial\mathbf{x}}\mathbf{f}_0(\mathbf{x}) + (\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}} - \frac{\partial\psi(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}})(\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \\
&\boldsymbol{\varepsilon}(t)) + \frac{\partial\psi(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}(\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)) \qquad (11)
\end{aligned}
$$

Notice that $\psi \in C^1$, $\mathbf{x}-\boldsymbol{\xi} \in L_\infty$ imply that the norm: $\|\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}} - \frac{\partial\psi(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\|$ is bounded. Moreover, $\|\frac{\partial\psi(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\| \leq \kappa(\boldsymbol{\xi})$. Hence we can rewrite (11) as:

$$
\dot{\psi} = \frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}}\mathbf{f}_0(\mathbf{x}) + \mu(t), \ \mu(t) \in L_2 \qquad (12)
$$

Function $\beta(\mathbf{x})$ is separated from zero, i.e. $\exists\delta > 0 : \beta(\mathbf{x}) > 2\delta \ \forall\mathbf{x} \in \mathbb{R}^n$. Let us consider the following positive-definite function:

$$
V_\psi = \int_0^\psi \varphi(\sigma)d\sigma + \frac{1}{4\delta}\int_t^\infty \mu^2(\tau)d\tau \qquad (13)
$$

Taking into account Assumption 5 and equality (12), derivative $\dot{V}_\psi$ can be estimated as follows: $\dot{V}_\psi \leq -\beta(\mathbf{x})\varphi^2(\psi) + \varphi(\psi)\mu(t) - \frac{1}{4\delta}\mu^2(t) \leq -2\delta\varphi^2(\psi) + \varphi(\psi)\mu(t) - \frac{1}{4\delta}\mu^2(t) = -\delta\varphi^2(\psi) - \delta(\varphi(\psi) - \frac{1}{2}\mu(t))^2 \leq 0$. Boundedness of $\mathbf{x}$ then follows explicitly from Assumption 6. This proves statement 3).

Let us prove that estimate $\hat{\boldsymbol{\theta}}(t)$ converges to $\hat{\boldsymbol{\theta}}$ exponentially fast under assumption of persistent excitation, assuming that $\boldsymbol{\varepsilon} \equiv 0$. Consider the following subsystem

$$
\begin{aligned}
\dot{\hat{\boldsymbol{\theta}}} = &S(\boldsymbol{\theta}) - S(\hat{\boldsymbol{\theta}}) - H^{-1}(\kappa^2(\boldsymbol{\xi})+1)\times \\
&\boldsymbol{\alpha}(\boldsymbol{\xi})^T\boldsymbol{\alpha}(\boldsymbol{\xi})\tilde{\boldsymbol{\theta}} = (\int_0^1 \frac{\partial S(\mathbf{z}(\lambda))}{\partial\mathbf{z}(\lambda)}d\lambda - \qquad (14) \\
&H^{-1}(\kappa^2(\boldsymbol{\xi})+1)\boldsymbol{\alpha}(\boldsymbol{\xi})^T\boldsymbol{\alpha}(\boldsymbol{\xi}))\tilde{\boldsymbol{\theta}}
\end{aligned}
$$

where $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. According to equations (8) system (14) describes the dynamics of $\hat{\boldsymbol{\theta}}(t) - \hat{\boldsymbol{\theta}}(t)$. Solution of (14) can be derived as: $\tilde{\boldsymbol{\theta}}(t) = e^{\int_0^t \frac{\partial S(\boldsymbol{\theta}'(\tau))}{\partial\boldsymbol{\theta}'}}d\tau \ e^{-H^{-1}\int_0^t(\kappa^2(\boldsymbol{\xi}(\tau))+1)\boldsymbol{\alpha}^T(\boldsymbol{\xi}(\tau))\boldsymbol{\alpha}(\boldsymbol{\xi}(\tau))d\tau}$ $\times\tilde{\boldsymbol{\theta}}(t_0)$, where $\boldsymbol{\theta}'(\tau) = \boldsymbol{\theta}(\tau)\lambda' - \hat{\boldsymbol{\theta}}(\tau)(1-\lambda')$ for some $\lambda \in [0,1]$. It follows from Assumption 2 that the induced matrix norm of $e^{\int_0^t \frac{\partial S(\boldsymbol{\theta}'(\tau))}{\partial\boldsymbol{\theta}'}}d\tau$ is bounded, i. e. there exists some positive $D_0 > 0$ such that $\|e^{\int_0^t \frac{\partial S(\boldsymbol{\theta}'(\tau))}{\partial\boldsymbol{\theta}'}}d\tau\| \leq D_0$ for all $t \geq 0$. On the other hand, for every $t > T$ there exists an integer $n > 0$ such that $t = nT + r$, $r \in \mathbb{R}_+ < T$, and the following estimation holds: $\|e^{-H^{-1}\int_0^t(\kappa^2(\boldsymbol{\xi}(\tau))+1)\boldsymbol{\alpha}^T(\boldsymbol{\xi}(\tau))\boldsymbol{\alpha}(\boldsymbol{\xi}(\tau))d\tau}\| \leq D_0\|e^{-H^{-1}\delta I_d n}\| \leq \|e^{-H^{-1}\frac{\delta}{T}I_d t + I}\|$. Hence we can bound the norm $\|\tilde{\boldsymbol{\theta}}(t)\|$ as follows: $\|\tilde{\boldsymbol{\theta}}(t)\| \leq D_0\|e^{-H^{-1}\frac{\delta}{T}I_d t + I}\|\|\tilde{\boldsymbol{\theta}}(t_0)\|$

3.2 Embedding (design of the extension)

In this section we show that for the class of systems given by (1) with locally Lipshitz $\phi_i(\mathbf{x})$:

$$\phi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^{d\times m},$$

$$\phi(\mathbf{x}) = \begin{pmatrix} \phi_{1,1}(\mathbf{x}), & \dots, & \phi_{1,d}(\mathbf{x}) \\ \dots, & & \dots, \dots \\ \phi_{m,1}(\mathbf{x}), & \dots, & \phi_{m,d}(\mathbf{x}) \end{pmatrix},$$

$$\phi_i(\mathbf{x}) = (\phi_{i,1}(\mathbf{x}), \dots, \phi_{i,d}(\mathbf{x}))$$

one can design C^1-smooth function $\boldsymbol{\xi}(t)$ such that $(\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta}(t)$, $\kappa(\boldsymbol{\xi})(\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta}(t) \in L_2$.

Lemma 6. Let system (1) be given and functions $\phi_i(\mathbf{x})$ defined as in (15) be locally Lipshitz:

$$\|\phi_i(\mathbf{x}) - \phi_i(\boldsymbol{\xi})\| \le \lambda_i(\mathbf{x}, \boldsymbol{\xi})\|\mathbf{x} - \boldsymbol{\xi}\|,$$

where $\lambda(\mathbf{x}, \boldsymbol{\xi}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $\lambda(\mathbf{x}, \boldsymbol{\xi})$ is locally bounded w.r.t. $\mathbf{x}$, $\boldsymbol{\xi}$. Let, furthermore, Assumption 2 hold. Then there exists system

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{f}(\mathbf{x}) + G_u\mathbf{u} + \lambda(\mathbf{x}, \boldsymbol{\xi})(\mathbf{x} - \boldsymbol{\xi}) + G_u\phi(\mathbf{x})\boldsymbol{\nu} \\ \dot{\boldsymbol{\nu}} &= S(\boldsymbol{\nu}) + H^{-1}(G_u\phi(\mathbf{x}))^T(\mathbf{x} - \boldsymbol{\xi})^T, \\ \lambda(\mathbf{x}, \boldsymbol{\xi}) &= 1 + \sum_{i=1}^{m} \lambda_i^2(\mathbf{x}, \boldsymbol{\xi})(1 + \kappa^2(\boldsymbol{\xi})) \end{aligned} \quad (15)$$

such that the following hold:

1) $\|(\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta}\| \in L_2$, $\|\kappa(\boldsymbol{\xi})(\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta}\| \in L_2$ for every bounded $\boldsymbol{\theta}$;

2) $\mathbf{x} \in L_\infty \Rightarrow \boldsymbol{\xi} \in L_\infty$, $\lim_{t\to\infty} \mathbf{x}(t) - \boldsymbol{\xi}(t) = 0$

Proof of Lemma 6. To prove the lemma it is enough to consider the following positive definite function V_ξ: $V_\xi = 0.5\|\mathbf{x} - \boldsymbol{\xi}\|^2 + 0.5\|\boldsymbol{\theta} - \boldsymbol{\nu}\|_H^2$. Its time-derivative can be written as follows:

$$\dot{V}_\xi \le -\lambda(\mathbf{x}, \boldsymbol{\xi})\|\mathbf{x} - \boldsymbol{\xi}\|^2 + (\mathbf{x} - \boldsymbol{\xi})^T G_u\phi(\mathbf{x})(\boldsymbol{\theta} - \boldsymbol{\nu}) + (\boldsymbol{\theta} - \boldsymbol{\nu})^T(G_u\phi(\mathbf{x}))^T(\mathbf{x} - \boldsymbol{\xi}) \le -\lambda(\mathbf{x}, \boldsymbol{\xi})\|\mathbf{x} - \boldsymbol{\xi}\|^2$$

The last inequality implies that $\lambda_i(\mathbf{x}, \boldsymbol{\xi})\|\mathbf{x} - \boldsymbol{\xi}\|$, $\kappa(\boldsymbol{\xi})\lambda_i(\mathbf{x}, \boldsymbol{\xi})\|\mathbf{x}-\boldsymbol{\xi}\| \in L_2$. Hence $\|\phi_i(\mathbf{x})-\phi_i(\boldsymbol{\xi})\| \le \lambda_i(\mathbf{x}, \boldsymbol{\xi})\|\mathbf{x} - \boldsymbol{\xi}\| \Rightarrow \|\phi_i(\mathbf{x}) - \phi_i(\boldsymbol{\xi})\|, \kappa(\boldsymbol{\xi})\|\phi_i(\mathbf{x}) - \phi_i(\boldsymbol{\xi})\| \in L_2$. Therefore, boundedness of $\boldsymbol{\theta}(t)$ and finiteness of the induced norm G_u ensure that $\|G_u(\phi_i(\mathbf{x}) - \phi_i(\boldsymbol{\xi}))\boldsymbol{\theta}(t)\|$, $\|\kappa(\boldsymbol{\xi})G_u(\phi_i(\mathbf{x}) - \phi_i(\boldsymbol{\xi}))\boldsymbol{\theta}(t)\| \in L_2$. In order to complete the proof we notice that function V_ξ is nonincreasing and radially unbounded. This guarantees that $\boldsymbol{\xi}$ is bounded as long as $\mathbf{x}$ remains bounded. The fact that $\lambda(\mathbf{x}, \boldsymbol{\xi}) > 1$ implies that $\mathbf{x} - \boldsymbol{\xi} \in L_2$. Under the assumptions of the lemma, the right-hand side of the system is locally bounded. This leads to uniform continuity of $\|\mathbf{x} - \boldsymbol{\xi}\|^2$, which guarantees that $\lim_{t\to\infty}(\mathbf{x} - \boldsymbol{\xi}) = 0$. *The lemma is proven*

3.3 Embedding (proof of Theorems 3, 4)

In this section we provide the proof of the main results of our paper.

Proof of Theorem 3. According to Lemma 6 there exist system (15):

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{f}(\mathbf{x}) + G_u\mathbf{u} + \lambda(\mathbf{x}, \boldsymbol{\xi})(\mathbf{x} - \boldsymbol{\xi}) + \\ &\quad G_u\phi(\mathbf{x})\boldsymbol{\nu} \\ \dot{\boldsymbol{\nu}} &= S(\boldsymbol{\nu}) + H^{-1}(G_u\phi(\mathbf{x}))^T(\mathbf{x} - \boldsymbol{\xi})^T, \\ \lambda(\mathbf{x}, \boldsymbol{\xi}) &= 1 + \sum_{i=1}^{m} \lambda_i^2(\mathbf{x}, \boldsymbol{\xi})(1 + \kappa^2(\boldsymbol{\xi})) \end{aligned} \quad (16)$$

such that $\|G_u(\phi(\mathbf{x}) - \phi(\boldsymbol{\xi}))\boldsymbol{\theta}\|$, $\|\kappa(\boldsymbol{\xi})G_u(\phi(\mathbf{x}) - \phi(\boldsymbol{\xi}))\boldsymbol{\theta}\| \in L_2$ for every bounded $\boldsymbol{\theta}(t)$ and trajectory $\mathbf{x}(t)$ generated by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + G_u(\phi(\mathbf{x})\boldsymbol{\theta} + \mathbf{u});\ \dot{\boldsymbol{\theta}} = S(\boldsymbol{\theta}) \quad (17)$$

Using the notation introduced in the previous subsections: $\boldsymbol{\alpha}(\boldsymbol{\xi}) = G_u\phi(\boldsymbol{\xi})$, taking into account that $\mathbf{u}(\mathbf{x}, \hat{\boldsymbol{\theta}}) = \mathbf{u}_0(\mathbf{x}) - \phi(\boldsymbol{\xi})\hat{\boldsymbol{\theta}}(t)$, and denoting $\boldsymbol{\varepsilon}(t) = (\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta}(t)$ we rewrite (17) as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}_0(\mathbf{x}) + \boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)) + \boldsymbol{\varepsilon}(t) \\ \dot{\boldsymbol{\theta}} &= S(\boldsymbol{\theta}),\ \boldsymbol{\varepsilon}(t) \in L_2,\ \kappa(\boldsymbol{\xi})\boldsymbol{\varepsilon}(\boldsymbol{\xi}) \in L_2 \end{aligned} \quad (18)$$

Taking into account equation (18) and expression (6) specifying the function $\hat{\boldsymbol{\theta}}(t)$ we can derive the time-derivative $\dot{\hat{\boldsymbol{\theta}}}$:

$$\dot{\hat{\boldsymbol{\theta}}} = S(\hat{\boldsymbol{\theta}}) + H^{-1}(\kappa^2(\boldsymbol{\xi}) + 1)\boldsymbol{\alpha}^T(\boldsymbol{\xi})(\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)) \quad (19)$$

Then, applying Lemma 5 we can conclude that both $\mathbf{x}(t)$ and $\hat{\boldsymbol{\theta}}$ are bounded, i.e. $\mathbf{x}(t), \hat{\boldsymbol{\theta}}(t) \in L_\infty$. On the other hand, according to Lemma 6, boundedness of $\mathbf{x}$ implies boundedness of $\boldsymbol{\xi}(t)$. Hence statement 1) of the theorem is proven.

Notice also that according to Lemma 6 the following limit holds: $\mathbf{x}(t) - \boldsymbol{\xi}(t) \to 0$ as $t \to \infty$. This fact together with the uniform asymptotic stability of the unperturbed system (19) (i. e. when $\boldsymbol{\varepsilon}(t) \equiv 0$) imply that $\hat{\boldsymbol{\theta}}(t, \hat{\boldsymbol{\theta}}_0, t_0) \to \boldsymbol{\theta}(t, \boldsymbol{\theta}_0, t_0)$ as $t \to \infty$.This proves statement 3) of the theorem.

Let us prove that $\mathbf{x}(t) \to \Omega^*$ as $t \to \infty$. In order to do this let us rewrite the closed-loop system in the following form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}_0(\mathbf{x}) + \boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)) + \boldsymbol{\varepsilon}(t) \\ \dot{\boldsymbol{\theta}} &= S(\boldsymbol{\theta}) \\ \dot{\hat{\boldsymbol{\theta}}} &= S(\hat{\boldsymbol{\theta}}) + H^{-1}(\kappa^2(\boldsymbol{\xi}) + 1)\boldsymbol{\alpha}(\boldsymbol{\xi})^T \times \\ &\quad (\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)) \\ \dot{\boldsymbol{\xi}} &= \mathbf{f}(\mathbf{x}) + G_u\mathbf{u} + \lambda(\mathbf{x}, \boldsymbol{\xi})(\mathbf{x} - \boldsymbol{\xi}) + \\ &\quad G_u\phi(\mathbf{x})\boldsymbol{\nu} \\ \dot{\boldsymbol{\nu}} &= S(\boldsymbol{\nu}) + H^{-1}(G_u\phi(\mathbf{x}))^T(\mathbf{x} - \boldsymbol{\xi})^T, \\ \boldsymbol{\varepsilon}(t) &= (\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta},\ \boldsymbol{\varepsilon} \in L_2 \\ \dot{\epsilon}_0 &= -\|\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}}(\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon})\|^2 \\ \dot{\epsilon}_1 &= -\|(\boldsymbol{\alpha}(\mathbf{x}) - \boldsymbol{\alpha}(\boldsymbol{\xi}))\boldsymbol{\theta}\|^2 \\ \dot{\epsilon}_2 &= -\|\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})\|^2 \end{aligned} \quad (20)$$

I has been shown earlier that trajectories of system (20) are bounded except for the function ϵ_0.

Boundedness of $\epsilon_0(t)$, however, follows immediately from the fact that $\frac{\partial\psi(\mathbf{x})}{\partial\mathbf{x}}$ is bounded and that $\boldsymbol{\varepsilon}, \boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) \in L_2$. Let us consider the following function: $V = \int_0^{\psi}\varphi(\sigma)d\sigma + \|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|_H^2 + \frac{1}{4\delta}\epsilon_0(t) + \frac{1}{4}\epsilon_1(t) + \epsilon_2(t)$. Its time-derivative satisfies the following inequality: $\dot{V} \leq -\delta\varphi^2(\psi) - \|\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + 0.5\boldsymbol{\varepsilon}(t)\|^2 \ -\|\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)\|^2 \leq -\delta\varphi^2(\psi) - \|\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)\|^2$ Therefore, applying the LaSalle invariance principle (LaSalle, 1976) we can conclude that $(\mathbf{x}(t), \hat{\boldsymbol{\theta}}(t))$ converge (as $t \rightarrow \infty$) to the largest invariant set in $\Omega_\psi \times \Omega_\theta$, where $\Omega_\psi = \{\mathbf{x} \in \mathbb{R}^n | \ \mathbf{x} : \varphi(\psi(\mathbf{x})) = 0\}$, and $\Omega_\theta : \{\hat{\boldsymbol{\theta}} \in \mathbb{R}^d \mid \hat{\boldsymbol{\theta}} : \boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t) = 0\}$. For the trajectory $\mathbf{x}(t)$ this set is defined as the largest invariant set of system

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) \tag{21}$$

under the restriction that $\mathbf{x}(t) \in \Omega_\psi$. According to Assumption 4 the largest invariant set of (21) in Ω_ψ is Ω^*. Q.E.D.

Proof of Theorem 4. Consider system (16). It follows from Lemma 6 and Assumption 2 that $G_u(\phi(\mathbf{x}) - \phi(\boldsymbol{\xi}))\boldsymbol{\theta} \in L_2$. Then boundedness of $\hat{\boldsymbol{\theta}}(t)$ follows explicitly from the proof of Theorem 3 (let $\kappa(\boldsymbol{\xi}) \equiv 0$ in (10)). Furthermore, Lemma 5 ensures that $G_u\phi(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) \in L_2$. Hence denoting $\boldsymbol{\varepsilon}_0(t) = G_u\phi(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + G_u(\phi(\mathbf{x}) - \phi(\boldsymbol{\xi}))\boldsymbol{\theta}$ we obtain the result that trajectories $\mathbf{x}(t)$ in system (1) satisfy the following equation:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \boldsymbol{\varepsilon}(t), \tag{22}$$

where $\boldsymbol{\varepsilon}(t) \in L_2$. System (22), however, has finite $L_2 \rightarrow L_\infty$ gain and therefore $\mathbf{x}(t)$ is bounded. Therefore, statement 1) of the theorem is proven. Statement 3) follows explicitly from Lemma 5. Let us show that $\mathbf{x}(t) \rightarrow \Omega^*$ as $t \rightarrow \infty$. In order to do so let us consider system (20) excluding the equation for ϵ_0. We have already shown that solutions of system (20) are bounded. Define $V = \|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\|_H^2 + \frac{1}{4}\epsilon_1(t) + \epsilon_2(t)$. Its time-derivative satisfies the following inequality: $\dot{V} \leq -\|\boldsymbol{\alpha}(\boldsymbol{\xi})(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}) + \boldsymbol{\varepsilon}(t)\|^2$ and therefore, by the LaSalle invariance principle, $\mathbf{x}(t) \rightarrow \Omega^*$ as $t \rightarrow \infty$. *Q.E.D.*

4. CONCLUSION

A new framework for adaptive regulation to invariant sets was proposed. The main advantage of the approach is that it does not require knowledge of a strict Lyapunov functions for design of the adaptation schemes. Our method can steer the systems to non-equilibrium Lyapunov instable target dynamics.

The number of additional equations required for implementation of our method is $(n + 2d)$ which compares favorably with $(nd + d + n)$ in (Panteley *et al.*, 2002). Though the conditions we require differ from theirs, our method naturally complements their result.

In the present study we considered linear parameterized of the uncertainties. On the other hand, the machinery we use in the proofs allows us to extend the results to nonlinear parameterized systems (Tyukin *et al.*, 2003*c*; Tyukin *et al.*, 2003*a*). This, together with a robustness analysis, are currently the the topics of our future studies.

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