A GRAPHICAL METHOD FOR COMPUTATION OF ALL STABILIZING PI CONTROLLERS

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Abstract: In this paper, a new method for the calculation of all stabilizing PI controllers is given. The proposed method is based on plotting the stability boundary locus in the (k_p, k_i)-plane and then computing the stabilizing values of the parameters of a PI controller. The technique presented does not require sweeping over the parameters and also does not need linear programming to solve a set of inequalities. Thus it offers several important advantages over existing results obtained in this direction. Computation of stabilizing PI controllers which achieve user specified gain and phase margins is also studied. Furthermore, the proposed method is used to compute all the parameters of a PI controller which stabilize a control system with an interval plant family. Examples are given to show the benefits of the method presented.

Keywords: Stabilization; PI control; Gain and Phase margins; Time delay; Uncertain systems.

1. INTRODUCTION

There has been a great amount of research work on the tuning of PI, PID and lag/lead controllers since these types of controllers have been widely used in industries for several decades (see Zhuang and Atherton, 1993; Astrom and Hagglund, 1995 and references therein). However, many important results have been recently reported on computation of all stabilizing P, PI and PID controllers after the publication of work by Datta, *et al*., 2000. A new and complete analytical solution which is based on the generalized version of the Hermite-Biehler theorem has been provided in (Datta, *et al*., 2000) for computation of all stabilizing constant gain controllers for a given plant. A linear programming solution for characterizing all stabilizing PI and PID controllers for a given plant has been obtained in (Datta, *et al*., 2000). This approach besides being computationally efficient has revealed important structural properties of PI and PID controllers. For example, it was shown that for a fixed proportional gain, the set of stabilizing integral and derivative gains lie in a convex set. This method is very important since it can cope with systems that are open loop stable or unstable, minimum or nonminimum phase. However, the computation time for this approach increases in an exponential manner with the order of the system being considered. It also needs sweeping over the proportional gain to find all stabilizing PI and PID controllers which is a disadvantage of the method. An alternative fast approach to this problem based on the use of the Nyquist plot has been given in (Söylemez, *et al*., 2003). Computation of the lag/lead controller parameters has been given in (Tan and Atherton, 1999). A parameter space approach using the singular frequency concept has been given in (Ackermann and Kaesbauer, 2001) for design of robust PID controllers. More direct graphical approaches to this problem based on frequency response plots have been given in (Shafiei and Shenton, 1997; Huang and Wang, 2000). However, the requirement for frequency gridding has become the major problem for this approach. Other results related to this subject can be found in (Ho, *et al*., 1998; Tan, *et al*., 2002).

In this paper, a new approach for computation of stabilizing PI controllers in the parameter plane, (k_p, k_i)-plane is given. The result of (Söylemez, *et al.*, 2003) is used to obtain the stability boundary locus over a possible smaller range of frequency. Thus, a very fast way of calculating the stabilizing values of PI controllers for a given single-input single-output (SISO) control system with time delay is given. The proposed method is also used for computation of PI controller parameters for achieving user specified gain and phase margins. The proposed method is finally used for computation of PI controllers for the stabilization of interval systems.

The paper is organized as follows: the proposed method is described in section 2. The design of PI controllers which achieve user specified gain and phase margins is given in section 3. In section 4, the computation of PI controllers for interval plant stabilization is given. Concluding remarks are given in section 5.

2. STABILIZATION USING PI CONTROLLER

Consider the SISO control system of Figure 1 where

$$G_p(s) = G(s)e^{-\tau s} = \frac{N(s)}{D(s)}e^{-\tau s} \qquad (1)$$

is the plant to be controlled and $C(s)$ is a PI controller of the form

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s} \qquad (2)$$

The problem is to compute the parameters of the PI controller of Eq. (2) which stabilize the system of Figure 1.

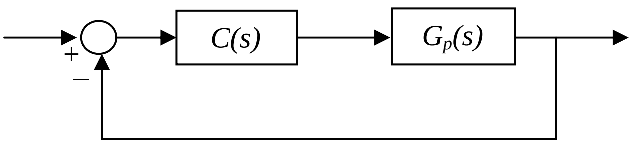

Fig. 1: A SISO control system

The closed loop characteristic polynomial $\Delta(s)$ of the system of Figure 1, i.e. the numerator of $1 + C(s)G_p(s)$, can be written as

$$\Delta(s) = sD(s) + (k_p s + k_i)N(s)e^{-\tau s} \qquad (3)$$

Decomposing the numerator and the denominator polynomials of $G(s)$ in Eq. (1) into their even and odd parts, and substituting $s = j\omega$, gives

$$G(j\omega) = \frac{N_e(-\omega^2) + j\omega N_o(-\omega^2)}{D_e(-\omega^2) + j\omega D_o(-\omega^2)} \qquad (4)$$

For simplicity $(-\omega^2)$ will be dropped in the following equations. Thus, the closed loop characteristic polynomial of Eq. (3) can be written as

$$\begin{aligned}\Delta(j\omega) &= [(k_i N_e - k_p\omega^2 N_o)\cos(\omega\tau) + \omega(k_i N_o + k_p N_e) \\ &\sin(\omega\tau) - \omega^2 D_o] + j[\omega(k_i N_o + k_p N_e)\cos(\omega\tau) - \\ &(k_i N_e - \omega^2 k_p N_o)\sin(\omega\tau) + \omega D_e] = R_\Delta + jI_\Delta = 0\end{aligned} \qquad (5)$$

Then, equating the real and imaginary parts of $\Delta(j\omega)$ to zero, one obtains

$$\begin{aligned}&k_p(-\omega^2 N_o\cos(\omega\tau) + \omega N_e\sin(\omega\tau)) + \\ &k_i(N_e\cos(\omega\tau) + \omega N_o\sin(\omega\tau)) = \omega^2 D_o\end{aligned} \qquad (6)$$

and

$$\begin{aligned}&k_p(\omega N_e\cos(\omega\tau) + \omega^2 N_o\sin(\omega\tau)) + \\ &k_i(\omega N_o\cos(\omega\tau) - N_e\sin(\omega\tau)) = -\omega D_e\end{aligned} \qquad (7)$$

Let

$$\begin{aligned}Q(\omega) &= \omega N_e\sin(\omega\tau) - \omega^2 N_o\cos(\omega\tau) \\ R(\omega) &= N_e\cos(\omega\tau) + \omega N_o\sin(\omega\tau) \\ X(\omega) &= \omega^2 D_o\end{aligned} \qquad (8)$$

and

$$\begin{aligned}S(\omega) &= \omega N_e\cos(\omega\tau) + \omega^2 N_o\sin(\omega\tau) \\ U(\omega) &= \omega N_o\cos(\omega\tau) - N_e\sin(\omega\tau) \\ Y(\omega) &= -\omega D_e\end{aligned} \qquad (9)$$

Then, Eqs. (6) and (7) can be written as

$$\begin{aligned}k_p Q(\omega) + k_i R(\omega) &= X(\omega) \\ k_p S(\omega) + k_i U(\omega) &= Y(\omega)\end{aligned} \qquad (10)$$

From these equations

$$k_p = \frac{X(\omega)U(\omega) - Y(\omega)R(\omega)}{Q(\omega)U(\omega) - R(\omega)S(\omega)} \qquad (11)$$

and

$$k_i = \frac{Y(\omega)Q(\omega) - X(\omega)S(\omega)}{Q(\omega)U(\omega) - R(\omega)S(\omega)} \qquad (12)$$

Substituting Eqs. (8) and (9) into Eqs. (11) and (12), it can be shown that

$$k_p = \frac{(\omega^2 N_o D_o + N_e D_e)\cos(\omega\tau) + \omega(N_o D_e - N_e D_o)\sin(\omega\tau)}{-(N_e^2 + \omega^2 N_o^2)} \qquad (13)$$

and

$$k_i = \frac{\omega^2(N_o D_e - N_e D_o)\cos(\omega\tau) - \omega(N_e D_e + \omega^2 N_o D_o)\sin(\omega\tau)}{-(N_e^2 + \omega^2 N_o^2)} \qquad (14)$$

It can be seen that if the denominator of Eqs. (13) and (14) $N_e(-\omega^2) + \omega^2 N_o(-\omega^2) \neq 0$ then the stability boundary locus, $l(k_p, k_i, \omega)$, can be constructed in the (k_p, k_i)-plane. Once the stability boundary locus has been obtained then it is necessary to test whether stabilizing controllers exist or not since the stability boundary locus, $l(k_p, k_i, \omega)$, and the line $k_i = 0$ may divide the parameter plane ((k_p, k_i)-plane) into stable and unstable regions. Here, the line $k_i = 0$ is the boundary line obtained from substituting $\omega = 0$ into Eq. (3) and equating it to zero since a real root of $\Delta(s)$ of Eq. (3) can cross over the imaginary axis at $s = 0$.

It can be seen that the stability boundary locus is dependent on the frequency ω which varies from 0 to ∞. However, one can consider the frequency below the critical frequency, ω_c, or the ultimate frequency since the controller operates in this frequency range. Thus, the critical frequency can be used to obtain the stability boundary locus over a possible smaller range of frequency such as $\omega \in [0, \omega_c]$. Since the phase of $G_p(s)$ at $s = j\omega_c$ is equal to -180°, one can write

$$\tan^{-1}\left(\frac{\omega N_o}{N_e}\right) - \tan^{-1}\left(\frac{\omega D_o}{D_e}\right) - \omega\tau = -\pi \quad (15)$$

or

$$\tan(\omega\tau) = \frac{\omega(N_o D_e - N_e D_o)}{N_e D_e + \omega^2 N_o D_o} = f(\omega) \quad (16)$$

Thus, ω_c is the solution of Eq. (16) in the interval $(0, \pi)$. By plotting $\tan(\omega\tau)$ and $f(\omega)$ versus ω, it can be seen that ω_c is the smallest value of ω at which plots of $\tan(\omega\tau)$ and $f(\omega)$ intersect with each other.

Example 1: Consider the control system of Figure 1 with transfer function

$$G_p(s) = \frac{1}{(0.5s+1)(2s-1)} e^{-0.5s} \quad (17)$$

which is an unstable second order plus time delay process transfer function where $\tau = 0.5$. The aim is to compute all the stabilizing values of k_p and k_i which make the characteristic polynomial of Eq. (3) Hurwitz stable. From Eqs. (13) and (14)

$$\begin{aligned} k_p &= (\omega^2 + 1)\cos(\omega) + 1.5\omega\sin(\omega) \\ k_i &= 1.5\omega^2\cos(\omega) + \omega(-\omega^2 - 1)\sin(\omega) \end{aligned} \quad (18)$$

For $\omega \in [0,60]$, the stability boundary locus is shown in Figure 2. However, in order to find the stabilizing values of k_p and k_i, it is only necessary to obtain stability boundary locus for ω below the critical frequency ω_c. From Eq. (16), $\tan(0.5\omega) = 1.5\omega/(\omega^2 + 1) = f(\omega)$ and the plots of $\tan(0.5\omega)$ and $f(\omega) = 1.5\omega/(\omega^2 + 1)$ are shown in Figure 3 where it has been computed that the value of the critical frequency, ω_c, is found to be 1.2615 rad/sec. The region enclosed by the stability boundary locus for $\omega \in [0,1.2615]$ and $k_i = 0$ is the stability region which is the shaded region shown in Figure 4. Figure 5 shows the stability regions for $\tau = 0.3$, $\tau = 0.4$, $\tau = 0.5$ and $\tau = 0.8$ where it can be seen that the time delay has an important affect on the stability region.

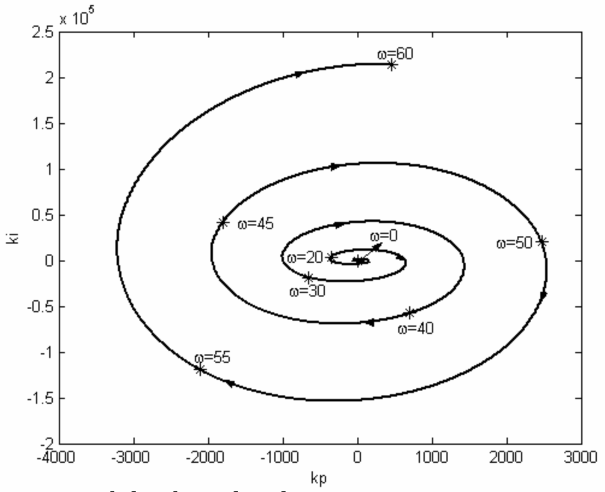

Fig. 2: Stability boundary locus

Using the Pade approximation and other results, the stabilizing k_p and k_i values can be computed. However, the result will not be correct as shown in Figure 4. The stability region shown in Figure 4 is obtained taking first order Pade approximation for $e^{-0.5s}$ in Eq. (17) and then using other methods such as (Datta, *et al*., 2000). From Figure 4, it can be seen that the stability region obtained for the first order Pade approximation encloses the shaded region which is the correct stability region. For example, for $k_p = 2.5$ and $k_i = 0.35$, using the Nyquist plot or Bode diagram it can be seen that the system is not stable. However, from Figure 4, it can be seen that the system after taking the first order Pade approximation is stable for these values of k_p and k_i. When the order of the Pade approximation is increased, the results obtained using other methods approach to the shaded region. However, in this case, the computation of the stability region will be difficult since the order of the system increases and also the result may still not be exact. Thus, the approach presented in this paper have advantages over existing results since it gives exact stability region and it is easy to use for the computation of the stability region.

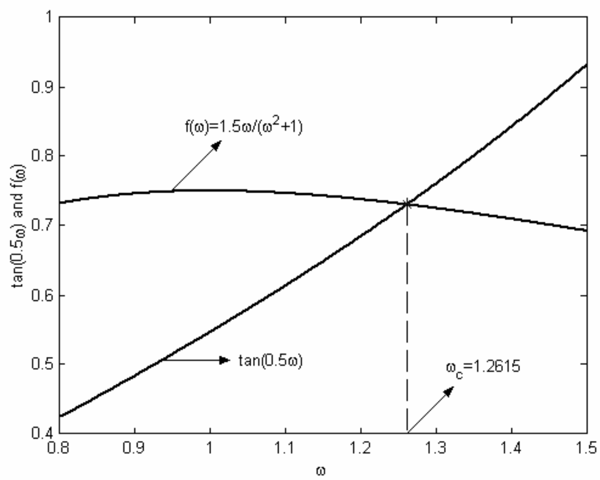

Fig. 3: Computation of ω_c

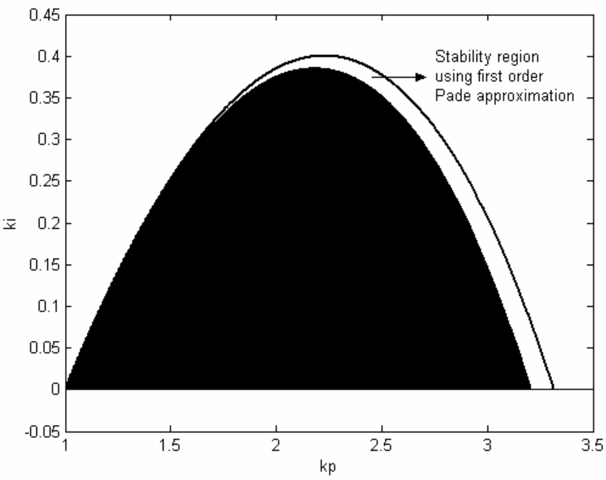

Fig. 4: Stability region

If there is a stability boundary locus for a given system, it does not mean that a stabilizing PI controller exists. Even for zero delay, many systems cannot be stabilized by simple PI controller. For example, the transfer function $G(s)=\frac{(s-1)(s-3)}{(s+1)(s-2)(s-4)}$ cannot be stabilized by a PI controller. When the stability boundary locus for this transfer function has been plotted in the (k_p, k_i)-plane it can be seen that there is not any stable region. So, one can say that both sides of the stability boundary locus may correspond to unstable parameter combinations. Thus, it is necessary to test whether stabilizing controllers exist or not after the stability boundary locus has been constructed.

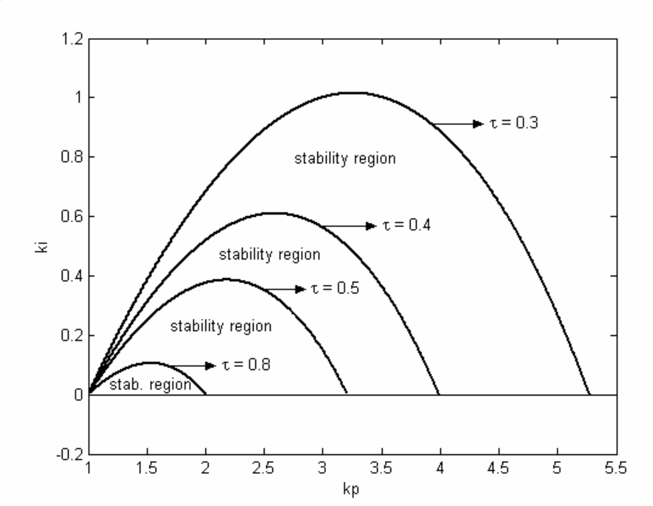

Fig. 5: Stability regions for different values of τ

3. STABILIZATION FOR SPECIFIED GAIN AND PHASE MARGINS

It is known that the phase and gain margins are two important frequency domain performance measures which are widely used in classical control theory for controller design. The approach given above can also be used for achieving user specified gain and phase margins. Consider Figure 1 with a gain-phase margin tester, $G_c(s)=Ae^{-j\phi}$, which is connected in the feed forward path. Then, using Eqs. (8), (9), (11) and (12) it can be found that

$$k_p=\frac{(\omega^2 N_o D_o + N_e D_e)\cos(h)+\omega(N_o D_e - N_e D_o)\sin(h)}{-A(N_e{}^2+\omega^2 N_o{}^2)} \tag{19}$$

and

$$k_i=\frac{\omega^2 (N_o D_e - N_e D_o)\cos(h)-\omega(N_e D_e + \omega^2 N_o D_o)\sin(h)}{-A(N_e{}^2+\omega^2 N_o{}^2)} \tag{20}$$

where $h=\omega\tau+\phi$. To obtain the stability boundary locus for a given value of gain margin A, one needs to set $\phi=0$ in Eqs. (19) and (20). On the other hand, setting $A=1$ in Eqs. (19) and (20), one can obtain the stability boundary locus for a given phase margin ϕ.

Example 2: Consider

$$G_p(s)=\frac{1.37s^2+1.98s+0.68}{3s^5+14s^4+23.75s^3+18.75s^2+7s+1} \tag{21}$$

The aim is to find all stabilizing PI controllers which satisfy the conditions that the phase margin of the system is greater than 45° and the gain margin is greater than 4(12.04 db).

To find all stabilizing PI controllers for which the phase margins is greater than 45°, it is required to set $A=1$ and $\phi=45^\circ$ in Eqs. (19) and (20). Using Eqs. (19) and (20) for these values of A and ϕ gives,

$$k_p=\frac{A\cos(\pi/4)+B\sin(\pi/4)}{-1.8838\omega^4-2.0538\omega^2-0.4624} \tag{22}$$

where

$$A=-13.275\omega^6-11.7706\omega^4-0.2625\omega^2+0.68$$
$$B=4.1175\omega^7-6.9169\omega^5-11.3675\omega^3-2.78\omega$$

and

$$k_i=\frac{C\cos(\pi/4)-D\sin(\pi/4)}{-1.8838\omega^4-2.0538\omega^2-0.4624} \tag{23}$$

where

$$C=4.1175\omega^8-6.9169\omega^6-11.3675\omega^4-2.78\omega^2$$
$$D=-13.275\omega^7-11.7706\omega^5-0.2625\omega^3+0.68\omega$$

To find all stabilizing PI controllers for which the gain margin is greater than 4, in this case, it is required to set $A=4$ and $\phi=0$ in Eqs. (19) and (20). Using Eqs. (19) and (20) for these values of A and ϕ gives,

$$k_p=\frac{13.275\omega^6+11.7706\omega^4+0.2625\omega^2-0.68}{4(1.8838\omega^4+2.0538\omega^2+0.4624)} \tag{24}$$

and

$$k_i=\frac{-4.1175\omega^8+6.9169\omega^6+11.3675\omega^4+2.78\omega^2}{4(1.8838\omega^4+2.0538\omega^2+0.4624)} \tag{25}$$

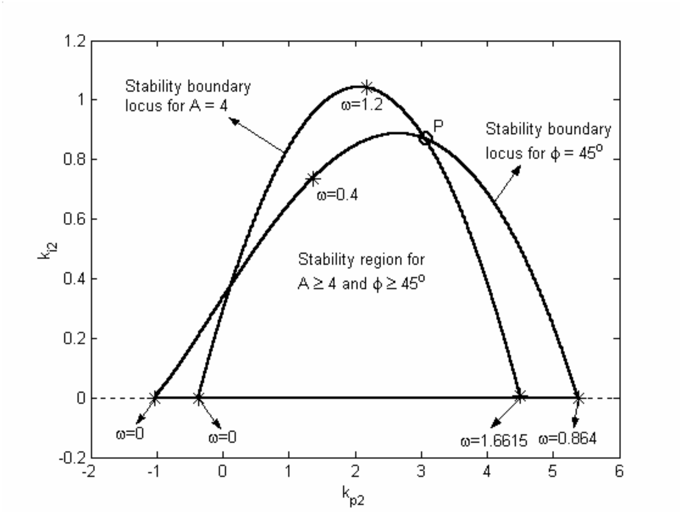

Fig. 6: Stabilizing PI controllers for specified gain and phase margins

Thus, the stabilizing PI controllers for $A\geq 4$ and $\phi\geq 45^\circ$ are shown in Figure 6. From Figure 6, the values of k_p and k_i at point P, which is an intersection point of the stability boundary locus for $\phi=45^\circ$ and the stability boundary locus for $A=4$, are $k_p=2.2716$ and $k_i=0.7782$. For these values of parameters it has been computed that the gain margin is equal to 4 and the phase margin is equal to 45°.

4. INTERVAL PLANT STABILIZATION

Compensator design in classical control engineering is based on a plant with fixed parameters. In the real world, however, most practical system models are not known exactly, meaning that the system contains uncertainties. Much recent work on systems with uncertain parameters has been based on Kharitonov's result (Kharitonov, 1979) on the stability of interval polynomials. Using the Kharitonov theorem, there have been many developments in the field of parametric robust control related to the stability and performance analysis of uncertain control systems represented as interval plant (Bhattacharyya, *et al.*, 1995). The method presented can be applied for the case of uncertain parameters as illustrated in the following example.

Example 3: Consider the position control system with the block diagram of Figure 7, where the motor is assumed to have the transfer function

$$G(s)=\frac{K_m}{s(Js+b)(L_f s+R_f)} = \frac{K_m}{L_f Js^3+(bL_f+JR_f)s^2+bR_f s} \qquad (26)$$

R_f and L_f are the resistance and inductance of the field, K_m is the motor constant, J is the inertia, b is the viscous friction and the controller is a PI controller of the form of Eq. (2). Here, an integrator is needed for elimination of a steady state error to a ramp input or a steady state error to a torque disturbance. The major uncertainty is assumed to be in the parameters K_m, b, L_f and J. Initially assume that all these parameters are fixed and have the nominal values $K_m = 60\times10^{-3}$, $R_f = 1.2$, $b = 2.5\times10^{-3}$, $L_f = 1.3\times10^{-2}$ and $J = 2\times10^{-3}$. For these nominal values, the nominal transfer function of the motor can be written as

$$G_{no}(s)=\frac{60}{s(0.026s^2+2.43s+3)} \qquad (27)$$

Using Eqs. (13) and (14), $k_p = 0.0405\omega^2$ and $k_i = -0.00043\omega^4 + 0.05\omega^2$. All the stabilizing values of k_p and k_i for the nominal transfer function are enclosed by G_{no} in Figure 8.

Assume now that the parameters K_m, b, L_f and J may vary by 10%, 15%, 20% and 40% around their nominal values, respectively. Then,

$K_m \in [\underline{K_m}, \overline{K_m}] = [54\times10^{-3}, 66\times10^{-3}]$

$b \in [\underline{b}, \overline{b}] = [2.1\times10^{-3}, 2.9\times10^{-3}]$

$L_f \in [\underline{L_f}, \overline{L_f}] = [1.04\times10^{-2}, 1.56\times10^{-2}]$

$J \in [\underline{J}, \overline{J}] = [1.2\times10^{-3}, 2.8\times10^{-3}]$

Thus, by overbounding the coefficients of Eq. (26), since the uncertain parameters of Eq. (26) are multilinearly dependent on the above parameters, the transfer function of the motor can be written in the form of the interval transfer function as follows

$$G(s)=\frac{q_0}{p_3 s^3+p_2 s^2+p_1 s+p_0} \qquad (28)$$

where $q_0 \in [\underline{K_m}, \overline{K_m}] = [54\times10^{-3}, 66\times10^{-3}]$, $p_0 = 0$, $p_1 \in [\underline{b}\underline{R_f}, \overline{b}\overline{R_f}] = [2.52\times10^{-3}, 3.48\times10^{-3}]$ $p_2 \in [\underline{b}\underline{L_f} + \underline{J}R_f, \overline{b}\overline{L_f} + \overline{J}R_f] = [1.46\times10^{-3}, 3.4\times10^{-3}]$ $p_3 \in [\underline{L_f}\,\underline{J}, \overline{L_f}\,\overline{J}] = [1.25\times10^{-5}, 4.37\times10^{-5}]$. The stabilizing parameters can be calculated for all eight Kharitonov plants ($i=1$ to 2 and $j=1$ to 4) and their bounds are also shown in Figure 8. Obviously in this simple case all G_{1j} are outside G_{2j} so it is only necessary to plot four loci. The interior of all these, which is bounded by parts of G_{21} and G_{23}, gives the region of all stabilizing parameters for stable control of the uncertain system, and is shown in Figure 9. Step responses of eight Kharitonov plants of position control system for $k_p = 0.08$ and $k_i = 0.001$ are shown in Figure 10.

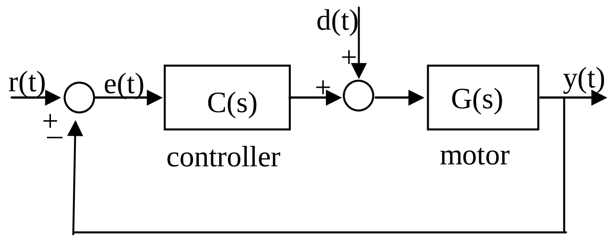

Fig. 7: Position control system

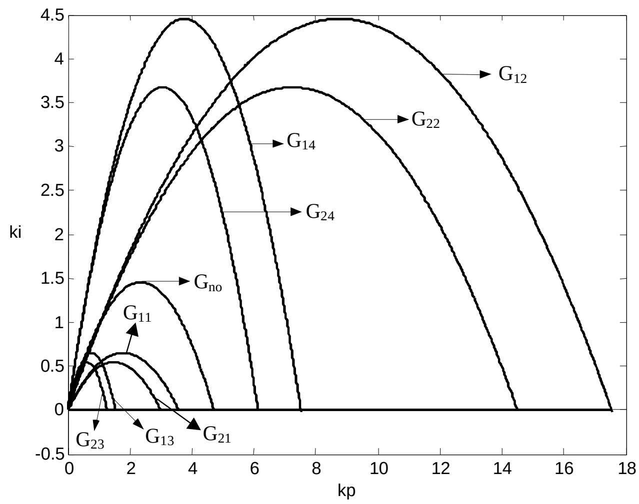

Fig. 8: Stability regions of eight Kharitonov plants

5. CONCLUSION

In this paper, a graphical approach has been presented for the computation of PI controller parameters that guarantee stability. The approach is based on the stability boundary locus which can be easily obtained by equating the real and the imaginary parts of the characteristic equation to zero. The computation of PI compensator parameters for achieving user specified gain and phase margins have also been studied. The proposed method has further been used to find the stabilizing region of PI parameters for the control of a plant with uncertain parameters. The method presented does not require sweeping over the parameters. Also, it does not need linear programming to solve a set of inequalities. Therefore, the method has advantages over existing results. The examples given in the paper clearly show the value of the method presented.

The method proposed in Section 2 can be further developed for computation of stabilizing PID controllers of the form $C(s) = k_p + k_i / s + k_d s$. Using the procedure given in Section 2, the stability boundary locus in the (k_p, k_i) -plane plane can be obtained for a fixed value of k_d or in the (k_p, k_d) -plane for a fixed value of k_i. However, it is not possible to obtain the stability boundary locus in the (k_i, k_d) -plane for a fixed value of k_p since $Q(\omega)U(\omega) - R(\omega)S(\omega)$ will be equal to zero for this case. Extension of the method for computation of all stabilizing PID controllers is the subject of future work.

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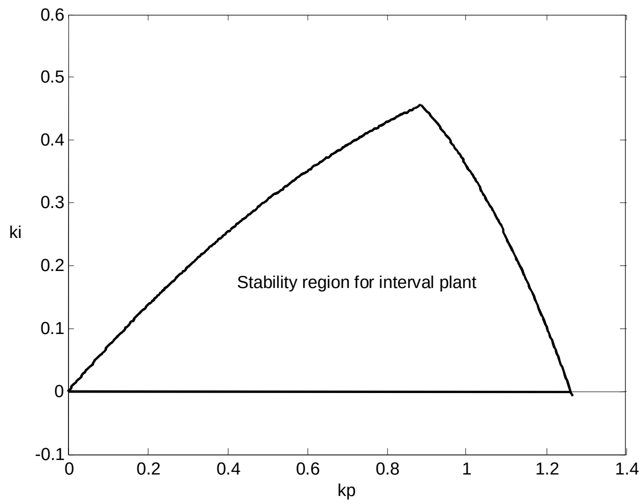

Fig. 9: Stability region for interval plant of motor

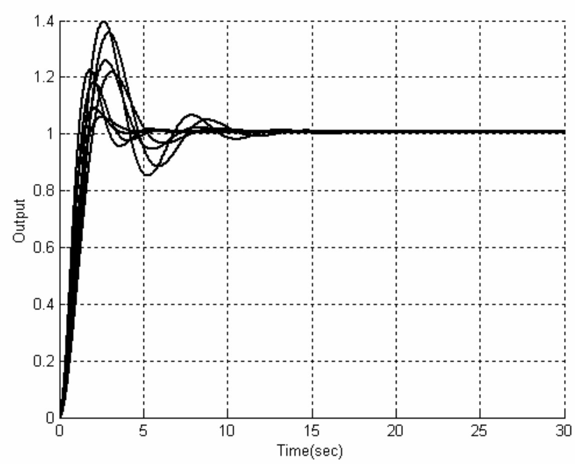

Fig. 10: Step responses of Kharitonov plants