MOTION PLANNING AND ITS FEEDBACK STABILIZATION FOR UNDERACTUATED SHIPS: VIRTUAL CONSTRAINTS APPROACH

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Abstract: The paper suggests a method for motion generation and feedback stabilization of a dynamical model of an underactuated ship with 3 degrees of freedom and 2 control inputs with a presence of lumped enviromental forces acting on the model. If the geometrical path for ship motion is given, the method suggests a description of all feasible motions of the model along this path. It is shown how to desing controller to counteract enviromental forces remaining on path.

1. INTRODUCTION

This paper discusses the motion planning and motion feedback stabilization for dynamical underactuated ship models taken from (Fossen, 2002). The related control problems of orbital stabilization for underactuated mechanical systems have recently been approached and solved based on ideas of *virtual holonomic constraints*. The constraint functions of the configuration states for the mechanical system are not imposed by physical constraints, but are chosen during the design. The function values are then made invariant along the closed loop system solutions by feedback action, see details in (Shiriaev and Canudas de Wit, 2004).

The ship models considered do not enter into the class of systems considered in (Shiriaev and Canudas de Wit, 2004), as they contain the first and second order velocity terms due to the friction forces, which were disregarded in (Shiriaev and Canudas de Wit, 2004). However, the beauty of such ship models is that the path planning for such models results in a dynamical system which could be analyzed by the method suggested in (Shiriaev and Canudas de Wit, 2004) and (Shiriaev *et. al.*, 2004).

This observation opens up several possibilities for motion planning and the feedback stabilization for underactuated ships. It also provides new insights and interpretation to previously developed control schemes reported in (Hauser and Hindman, 1995; Fossen and Strand, 2001; Skjetne *et al.*, 2004) for classes of fully actuated systems. To clarify the paper contribution and to emphasize its differences compared to the results reported in (Fossen and Strand, 2001; Skjetne *et al.*, 2004), we should state that both approaches suggest to choose a particular *path* (geometrical constraint imposed on configuration coordinates), which a feedback controller is to make orbitally stable. If such stabilizing controller is found, the remaining degree of freedom is then the scalar variable θ which parameterizes the motion of the system along the prescribed *path*. In the case of full actuation the dynamics of the θ-variable [1] could be chosen [2] arbitrarily.

[1] that is, the desired velocity $\dot{\theta}$ and acceleration $\ddot{\theta}$
[2] under appropriate technical assumptions

However, for underactuated systems the dynamics of θ cannot be chosen arbitrarily. It is determined primarily by the dynamics of the system itself and the prescribed *path*. Therefore, for an underactuated case a number of questions arise: How to determine what functions $\theta(t)$ are obtained provided that the *path* is given? Could one describe all functions $\{\theta(t)\}$ provided that the *path* is given? Could one stabilize the motion of the system encapsulated by a particularly chosen $\theta(t)$ provided that the *path* is given? In this paper these questions are considered and partly answered. The rest of the paper is organized as follows: Section 2 contains both a description of the ship model and one of the main results stating the stability properties of the desired equilibrium. The proof of Theorem 2 is given in Appendix A. Section 3 describes the controller design and a numerical example with simulations is given in Section 4. Section 5 concludes the paper.

2. MOTION PLANNING FOR UNDERACTUATED SHIP MODEL

In this Section a path planning problem for a ship with 3 degrees of freedom and 2 control inputs is discussed. Feasible motions of the ship model for these paths are computed for an illustrative example [3] taken from (Fossen, 2002, Example 10.1, p. 410): it describes a high speed container ship of length $L = 175$ m and displacement volume 21.222 m^3; it is assumed that it is actuated by one rudder and a forward thrust propeller. Let

$$\eta = [n,\ e,\ \psi]^T \tag{1}$$

denote the North-East positions and the yaw angle, and

$$\nu = [u,\ v,\ r]^T \tag{2}$$

be the velocity vector in the body frame. The kinematics equation is then

$$\frac{d}{dt}\eta = R(\psi)\nu, \quad \nu = R(\psi)^T\frac{d}{dt}\eta \tag{3}$$

with

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

being the rotation matrix in yaw. The surge speed equation and the steering equations (sway and yaw) are assumed to be decoupled. The ship dynamics written in body frame is then

$$M\dot{\nu} + N(\nu)\nu = B(\nu)\tau + R(\psi)^T w \tag{4}$$

where $\tau = [T,\ \delta]^T$ is the control inputs with T being the propeller forward thrust and δ is the angle of the rudder; w is the vector of environmental disturbances; the matrix functions M, N, and B look as

[3] This does not reduce the generality or difficulty of the problem, but rather gives the reader a taste of a real practical example.

$$M = \begin{bmatrix} m - X_{\dot{u}} & 0_{1\times 2} \\ 0_{2\times 1} & I_{2\times 2} \end{bmatrix} \tag{5}$$

$$N(\nu) = \begin{bmatrix} -X_u - |u|X_{|u|u} & 0_{1\times 2} \\ 0_{2\times 1} & -\dfrac{U}{L}\begin{bmatrix} a_{11} & La_{12} \\ \dfrac{a_{21}}{L} & a_{22} \end{bmatrix} \end{bmatrix} \tag{6}$$

$$B(\nu) = \begin{bmatrix} (1-t_d) & 0 \\ 0 & \dfrac{U^2}{L}b_{11} \\ 0 & \dfrac{U^2}{L^2}b_{21} \end{bmatrix} \tag{7}$$

Here $X_{(\cdot)}$, a_{ij}, and b_{ij} are the hydrodynamic coefficients [4] ; t_d is the thrust deduction number, $t_d \in (0,1)$ and $U = \sqrt{u^2+v^2}$ is the total speed.

It will be assumed that the vector w of environmental disturbance in (4) is constant. Next statement shows the way how all motions of the ship could be found provided its control inputs T and δ are chosen to preserve a given path invariant for ship movements.

Theorem 1. Given a path and the yaw angle as C^2-smooth functions of the new independent variable θ

$$n = \phi_1(\theta), \quad e = \phi_2(\theta), \quad \psi = \phi_3(\theta), \tag{8}$$

suppose that there exists a control input $\tau^* = [T^*,\ \delta^*]^T$, which makes the relations (8) invariant for the ship dynamics (4), then the variable θ is one of the solutions of the dynamical system

$$\alpha(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \beta_2(\theta)\dot{\theta}|\dot{\theta}| + \gamma(\theta) = 0 \tag{9}$$

The explicit forms of the functions $\alpha(\theta)$, $\beta_1(\theta)$, $\beta_2(\theta)$, and $\gamma(\theta)$ are given in the proof. ■

Proof. The relations (8) imply that

$$\frac{dn}{dt} = \phi_1'(\theta)\frac{d\theta}{dt}, \ \frac{de}{dt} = \phi_2'(\theta)\frac{d\theta}{dt}, \ \frac{d\psi}{dt} = \phi_3'(\theta)\frac{d\theta}{dt}$$

and

$$\begin{aligned} \frac{d^2n}{dt^2} &= \phi_1'(\theta)\frac{d^2\theta}{dt^2} + \phi_1''(\theta)\dot{\theta}^2, \\ \frac{d^2e}{dt^2} &= \phi_2'(\theta)\frac{d^2\theta}{dt^2} + \phi_2''(\theta)\dot{\theta}^2, \\ \frac{d^2\psi}{dt^2} &= \phi_3'(\theta)\frac{d^2\theta}{dt^2} + \phi_3''(\theta)\dot{\theta}^2. \end{aligned}$$

The last differential relations in matrix form look as

$$\dot{\eta} = \Phi'(\theta)\dot{\theta}, \quad \ddot{\eta} = \Phi'(\theta)\ddot{\theta} + \Phi''(\theta)\dot{\theta}^2 \tag{10}$$

where the vector functions Φ' and Φ'' are given by

$$\begin{aligned} \Phi'(\theta) &= [\phi_1'(\theta), \phi_2'(\theta), \phi_3'(\theta)]^T \\ \Phi''(\theta) &= [\phi_1''(\theta), \phi_2''(\theta), \phi_3''(\theta)]^T \end{aligned}$$

The dynamics of the ship (4) takes the form

$$\begin{aligned} M\left[R(\psi)^T\ddot{\eta} + \dot{R}(\psi)^T\dot{\eta}\right] + N(R(\psi)\dot{\eta})R(\psi)\dot{\eta} = \\ = B(R(\psi)\dot{\eta})\tau^* + R(\psi)^T w \end{aligned}$$

[4] The numerical values for the example are $m = 21.2\cdot 10^6$, $X_{\dot{u}} = -6.38\cdot 10^5$, $a_{11} = -0.7072$, $a_{12} = -0.286$, $a_{21} = -4.1078$, $a_{22} = -2.6619$, $b_{11} = -0.2081$, $b_{21} = -1.5238$.

The differential relations (10) allow us to rewrite the last equations as equations with respect to the θ-variable

$$\begin{aligned}
&MR(\phi_3(\theta))^T\left[\Phi'(\theta)\ddot\theta+\Phi''(\theta)\dot\theta^2\right]+\\
&+M\frac{d}{d\theta}\left[R(\phi_3(\theta))^T\right]\Phi'(\theta)\dot\theta^2+\\
&+N\left[R(\phi_3(\theta))^T\Phi'(\theta)\dot\theta\right]R(\phi_3(\theta))^T\Phi'(\theta)\,\dot\theta=\\
&\quad=B\left[R(\phi_3(\theta))^T\Phi'(\theta)\dot\theta\right]\tau^*+R(\phi_3(\theta))^T w
\end{aligned}\tag{11}$$

It is readily seen that the matrix

$$B^\perp=\left[0,\,-\frac{1}{L}b_{21},\,b_{11}\right]\tag{12}$$

is the orthogonal complement of $B(\cdot)$, that is

$$B^\perp B(\nu)=0_{1\times 2}$$

Hence $B^\perp B(\nu)\tau^*=0$ irrespective of the choice of control input τ^*. Premultiplying both sides of Eq. (11) by $B^\perp$ gives us system (9) with

$$\alpha(\theta)=B^\perp MR(\phi_3(\theta))^T\Phi'(\theta)\tag{13}$$

$$\begin{aligned}\beta_1(\theta)=&\,B^\perp MR(\phi_3(\theta))^T\Phi''(\theta)+\\&+B^\perp M\frac{d}{d\theta}\left[R(\phi_3(\theta))^T\right]\Phi'(\theta)\end{aligned}\tag{14}$$

$$\beta_2(\theta)=B^\perp N\left[R(\phi_3(\theta))^T\Phi'(\theta)\right]R(\phi_3(\theta))^T\Phi'(\theta)\tag{15}$$

$$\gamma(\theta)=-B^\perp R(\phi_3(\theta))^T w\tag{16}$$

The computation of the function β_2 is of special interest. As seen, the friction forces in surge do not contribute at all to the dynamics of the θ-variable. This is due to the fact that the vector $B^\perp$ is orthogonal to the surge direction and that the surge speed equation and the steering equations (sway and yaw) are assumed to be decoupled. This finishes the proof. ∎

The dynamics of (9) does not possess a term linear in velocity, furthermore it has a switching line $\left\{[\theta,\dot\theta]:\dot\theta=0\right\}$ that could lead to presence of sliding modes and non uniqueness of its solutions. These features make the stability analysis of its equilibria nontrivial. The statement below suggests a simple test of asymptotic stability and instability of its equilibria.

Theorem 2. Let θ_0 be an equilibrium of the system (9), that is the point where $\gamma(\theta_0)=0$. Suppose that the functions $\alpha(\theta)$, $\beta_1(\theta)$, $\beta_2(\theta)$, and $\gamma(\theta)$ are so that the following constant

$$\omega_0=\frac{d}{dt}\left[\frac{\gamma(\theta)}{\alpha(\theta)}\right]\Bigg|_{\theta=\theta_0}\tag{17}$$

is positive, i.e., $\omega_0>0$, and that the inequality

$$\frac{\beta_1(\theta_0)+\beta_2(\theta_0)}{\alpha(\theta_0)}>\frac{\beta_1(\theta_0)-\beta_2(\theta_0)}{\alpha(\theta_0)}\tag{18}$$

holds. Then the equilibrium θ_0 of (9) is asymptotically stable. In turn, if the sign of inequality (18) is opposite, then the equilibrium θ_0 of (9) is unstable. ∎

3. CONTROLLER DESIGN

3.1 Partial Feedback Linearization

Given 3 scalar C^2-smooth functions $\phi_1(\theta)$–$\phi_3(\theta)$ describing the desired path specification, introduce new coordinates

$$y_1=n-\phi_1(\theta),\ y_2=e-\phi_2(\theta),\ y_3=\psi-\phi_3(\theta)\tag{19}$$

These coordinates, y_1–y_3, together with the scalar variable θ constitute excessive coordinates for (4). Therefore, one of them could always be locally resolved as a function of the others. Suppose that this is the case for y_3, that is, we have found a smooth function h so that

$$y_3=h(y_1,y_2,\theta).$$

Then the vector of velocities in body frame for the new coordinates looks as follows

$$\begin{aligned}\nu=R(\psi)^T\dot\eta=&\,R\Big(\phi_3(\theta)+h(y_1,y_2,\theta)\Big)^T\times\\
&\times\begin{bmatrix}1&0&\phi_1'(\theta)\\0&1&\phi_2'(\theta)\\\dfrac{\partial h}{\partial y_1}&\dfrac{\partial h}{\partial y_2}&\left\{\dfrac{\partial h}{\partial\theta}+\phi_3'(\theta)\right\}\end{bmatrix}\begin{bmatrix}\dot y_1\\\dot y_2\\\dot\theta\end{bmatrix}\\
=&\,L(y_1,y_2,\theta)\begin{bmatrix}\dot y_1\\\dot y_2\\\dot\theta\end{bmatrix}\end{aligned}\tag{20}$$

where L in (20) is a 3×3 matrix function. In turn, the time derivative of ν takes the form

$$\dot\nu=L(y_1,y_2,\theta)\begin{bmatrix}\ddot y_1\\\ddot y_2\\\ddot\theta\end{bmatrix}+S(y_1,y_2,\theta,\dot y_1,\dot y_2,\dot\theta),\tag{21}$$

where S is the vector function quadratically dependent on velocities. Substituting expressions (19)-(21) into the dynamics (4) gives us the ship model written in y_1, y_2, and θ as follows

$$\begin{aligned}&M\left[L\begin{bmatrix}\ddot y_1\\\ddot y_2\\\ddot\theta\end{bmatrix}+S\right]+N\left(L\begin{bmatrix}\dot y_1\\\dot y_2\\\dot\theta\end{bmatrix}\right)L\begin{bmatrix}\dot y_1\\\dot y_2\\\dot\theta\end{bmatrix}=\\
&=B\left(L\begin{bmatrix}\dot y_1\\\dot y_2\\\dot\theta\end{bmatrix}\right)\begin{bmatrix}T\\\delta\end{bmatrix}+R\left(\phi_3(\theta)+h\right)^T w\end{aligned}\tag{22}$$

where the arguments of the L, S, and h are suppressed.

If one assumes that the 3×3 matrix

$$ML(y_1,y_2,\theta)\tag{23}$$

has full rank in a vicinity of (8), then premultiplying both sides of the system (22) by the 2×3 matrix

$$G(y_1,y_2,\theta)=\left[I_2,\,0_{2\times 1}\right]\Big(ML(y_1,y_2,\theta)\Big)^{-1}$$

results in equations resolved w.r.t. $\ddot y_1$ and $\ddot y_2$

$$\begin{bmatrix}\ddot y_1\\\ddot y_2\end{bmatrix}=K\begin{bmatrix}T\\\delta\end{bmatrix}+Q\tag{24}$$

where Q is a function independent on τ and

$$K = G(y_1, y_2, \theta) B \left(L(y_1, y_2, \theta) \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta} \end{bmatrix} \right) \quad (25)$$

Let us assume, in addition to invertability of the 3×3 matrix in (23), the full rank of the 2×2 matrix K in some vicinity of the path (8). Such assumptions immediately allow to rewrite the nonlinear equations (24) into a linear one. Indeed, the feedback transform

$$\begin{bmatrix} T \\ \delta \end{bmatrix} = K^{-1} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - Q \right) \quad (26)$$

from the control inputs T and δ to the new one v will then give the linear system

$$\ddot{y}_1 = v_1, \quad \ddot{y}_2 = v_2 \quad (27)$$

of two double integrators.

As expected not all dynamical equations of an underactuated ship model could be feedback linearized. It is readily to check that the rest (nonlinear equation that complements the linear part (27) to an equivalent ship model) is represented by the equation

$$\begin{aligned} B^\perp M \left[L \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\theta} \end{bmatrix} + S \right] + B^\perp N \left(L \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta} \end{bmatrix} \right) L \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta} \end{bmatrix} = \\ = B^\perp B \left(L \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta} \end{bmatrix} \right) \begin{bmatrix} T \\ \delta \end{bmatrix} + B^\perp R \left(\phi_3(\theta) + h \right)^T w \\ = B^\perp R \left(\phi_3(\theta) + h \right)^T w \quad (28) \end{aligned}$$

Here $B^\perp$ is the orthogonal complement matrix from (12). The Eqn. (27) allows to eliminate $\ddot{y}_1$ and $\ddot{y}_2$ from (28)

$$\begin{aligned} B^\perp M \left[L \begin{bmatrix} v_1 \\ v_2 \\ \ddot{\theta} \end{bmatrix} + S \right] + B^\perp N \left(L \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta} \end{bmatrix} \right) L \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\theta} \end{bmatrix} = \\ = B^\perp R \left(\phi_3(\theta) + h \right)^T w \quad (29) \end{aligned}$$

so that (29) is a scalar differential equation w.r.t. $\ddot{\theta}$.

The reader could easily check that Equation (29) becomes system (9) provided the values of

$$y_1, \; \dot{y}_1, \; y_2, \; \dot{y}_2, \; v_1, \; v_2$$

are all taken as zeros. The next statement summarizes the arguments.

Theorem 3. Given the path (8), suppose that the matrices (23) and (25) both have full rank in some vicinity of this path, then the ship dynamical model could be rewritten in some vicinity of the path as follows

$$\begin{aligned} \alpha(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \beta_2(\theta)\dot{\theta}|\dot{\theta}| + \gamma(\theta) = \\ = g_y \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + g_{\dot{y}} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + g_v \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (30) \\ \ddot{y}_1 = v_1 \quad (31) \\ \ddot{y}_2 = v_2 \quad (32) \end{aligned}$$

where

$$\begin{aligned} g_y &= g_y \left(\theta, y_1, y_2, \dot{\theta}, \dot{y}_1, \dot{y}_2, \ddot{\theta}, w \right) \\ g_{\dot{y}} &= g_{\dot{y}} \left(\theta, y_1, y_2, \dot{\theta}, \dot{y}_1, \dot{y}_2, \ddot{\theta}, w \right) \quad (33) \\ g_v &= g_v \left(\theta, y_1, y_2, \dot{\theta}, \dot{y}_1, \dot{y}_2, w \right) \end{aligned}$$

are bounded vector functions of their arguments. ■

3.2 Control Design for the Case of Friction Dominance in Dynamics of Eq. (9)

As shown above, the motion of the underactuated ship along the prescribed path (8) is determined by the dynamics of the scalar non-smooth differential equation (9). Each term in the dynamics — $\beta_1(\theta)\dot{\theta}^2$, $\beta_2(\theta)\dot{\theta}|\dot{\theta}|$, $\gamma(\theta)$ — has clear physical interpretation: $\gamma(\theta)$ comes due to an effect of environmental disturbances; $\beta_2(\theta)\dot{\theta}|\dot{\theta}|$ represents friction effect; $\beta_1(\theta)\dot{\theta}^2$ is due to chosen path profile, see (13)–(16).

Let us consider the practically important case when the dynamics of (9) is damped due to the friction to one of equilibria formed by the environmental disturbances, i.e., that there is an equilibrium θ_0 of the system (9)[5] and that this equilibrium is asymptotically stable. As shown below for this case the nonlinear PD-controller stabilizes such equilibrium[6].

Theorem 4. Given geometrical constraints (8) describing a pre-described path for the ship in the inertial frame, suppose that

(1) The system (9) has an asymptotically stable equilibrium at θ_0[7];
(2) The matrices (23) and (25) both have full rank in some vicinity of the path (8).

Then the feedback controller

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -K_p \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - K_d \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \quad (34)$$

with 2×2 matrices K_p, K_d that stabilizes the double integrator (31)-(32), renders the equilibrium

$$n_0 = \phi_1(\theta_0), \; e_0 = \phi_2(\theta_0), \; \psi_0 = \phi_3(\theta_0) \quad (35)$$

of the closed loop system (3), (4), (19), (26) and (34) asymptotically stable. ■

Proof follows from the fact that the zero dynamics of the closed loop system is asymptotically stable. Therefore, a feedback stabilization of regulated outputs y_1 and y_2 implies local asymptotic stability of the equilibrium (35).

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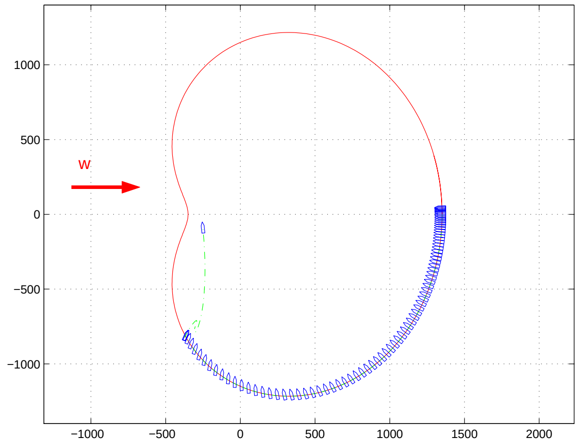
[5] θ_0 is determined as a solution of the equation $\gamma(\theta_0) = 0$
[6] Such result is closely related to the Weather Optimal Control algorithm suggested in (Fossen and Strand, 2001) for the case of fully actuated ship model, but the found controller and the closed-loop stability analysis are different.
[7] Conditions for this are discussed in Theorem 2.

4. EXAMPLE

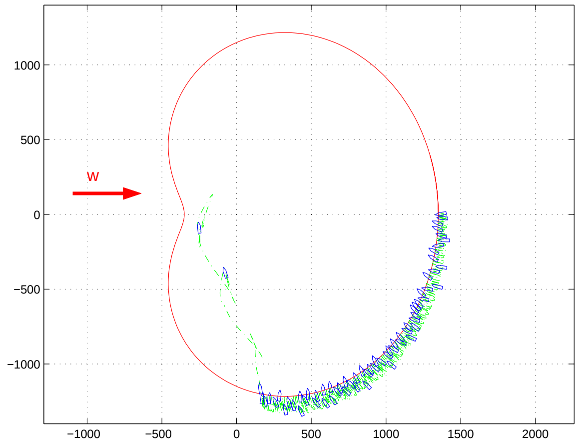
Consider the following desired path given in the inertial frame $\psi(\theta) = -\theta - \pi/2$ and

$$n(\theta) = 1000(1 + 0.5\cos(\theta) - 0.15\cos(2\theta))\sin(\theta),$$
$$e(\theta) = 1000(1 + 0.5\cos(\theta) - 0.15\cos(2\theta))\cos(\theta)$$

The results of the computer simulations are depicted on Fig. 1. In both plots the desired path in the North-



(a)



(b)

Fig. 1. (a) The solution of the closed loop system with the vector of the environmental disturbance w (37), i.e., to east, which originates in the point (36). (b) The same simulation with a substantial level of white noise is added to all signal measurements.

East (n-e) coordinates is shown in red. The true motion of the center of mass of the ship is shown in green, and the ship frame is depicted in blue.

The PD-controller gains K_d and K_p, see (34), were chosen in the simulations as indentity matrices. The initial conditions were

$$\begin{array}{ll} n(0) = -100,\ e(0) = -250, & \psi(0) = -0.1 \\ u(0) = 0.1, \quad v(0) = -0.01, & r(0) = 0 \end{array} \tag{36}$$

The vector w of the enviromental disturbance was

$$w = [0,\, 1,\, 0]^T, \tag{37}$$

of direction to east. It is shown in red on both plots.

5. CONCLUSIONS

In this paper we discuss the problem of motion planning and feedback control design for an underactuated ship model taken from (Fossen, 2002). It is shown that underactuation in the ship dynamics naturally leads to investigation of a particular dynamical system of second order. The detailed investigation of such system is done. This allows to provide some modifications and insights for the weather optimal positioning control design introduced in (Fossen and Strand, 2001) for a fully actuated ship model.

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Appendix A. PROOF OF THEOREM 2

The sliding surface of (9) is $\Gamma = \left\{[\theta, \dot\theta] : \dot\theta = 0\right\}$. The vector field of (9) on the sliding surface Γ is transversal to Γ except for points where $\gamma(\theta) = 0$. However, these points are isolated equilibria of (9), see (16). Hence, any solution of (9), if exists, is unique. Introduce two dynamical systems

$$\alpha(\theta)\ddot\theta + \Big[\beta_1(\theta) + \beta_2(\theta)\Big]\dot\theta^2 + \gamma(\theta) = 0 \tag{A.1}$$

$$\alpha(\theta)\ddot\theta + \Big[\beta_1(\theta) - \beta_2(\theta)\Big]\dot\theta^2 + \gamma(\theta) = 0 \tag{A.2}$$

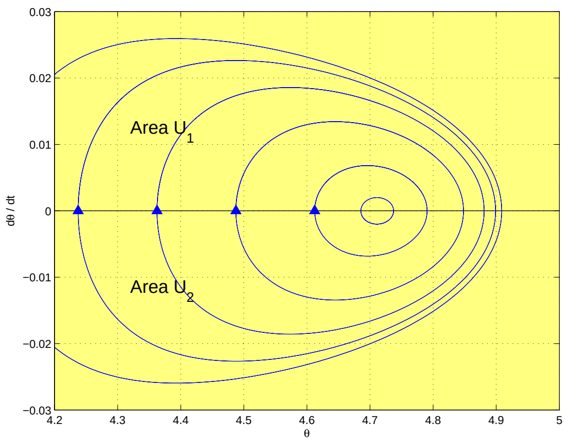
The lack of nontrivial sliding motions on Γ allows us to state that the phase plane is a union of two sets $\mathcal{U}_1$ and $\mathcal{U}_2$ defined by

$$\mathcal{U}_1 = \Big\{[\theta, \dot\theta] : \text{ a solution of (9) with origin at}[\theta_0, \dot\theta_0] \text{ locally coincides with a solution of (A.1) originated from}[\theta_0, \dot\theta_0]\Big\}$$

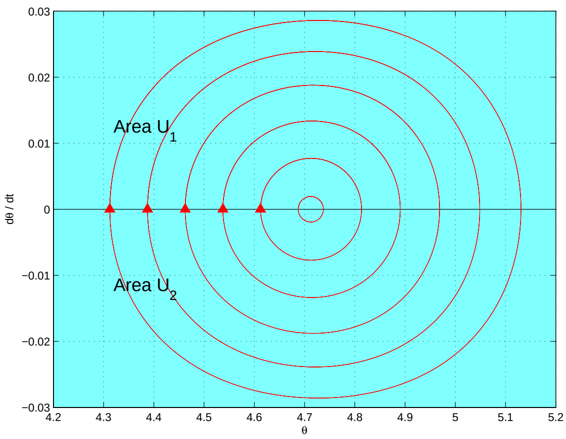
$$\mathcal{U}_2 = \Big\{[\theta, \dot\theta] : \text{ a solution of (9) with origin at}[\theta_0, \dot\theta_0] \text{ locally coincides with a solution of (A.2) originated from}[\theta_0, \dot\theta_0]\Big\}$$

Roughly speaking the set $\mathcal{U}_1$ is a union of an open upper half plane of the phase plane, i.e., $\Big\{[\theta, \dot\theta] : \dot\theta > 0\Big\}$, and some subintervals of the sliding line Γ, while the set $\mathcal{U}_2$, in opposite, consists of an open lower half plane, i.e. $\Big\{[\theta, \dot\theta] : \dot\theta < 0\Big\}$, and complementary parts of the sliding surface Γ.

The condition (17) implies that both systems (A.1) and (A.2) have centers at the equilibrium point θ_0, see (Shiriaev *et. al.*, 2004). Figure A.1(a) and (b) give examples of the phase portraits of (A.1) and (A.2) around their corresponding equilibrium.



(a)

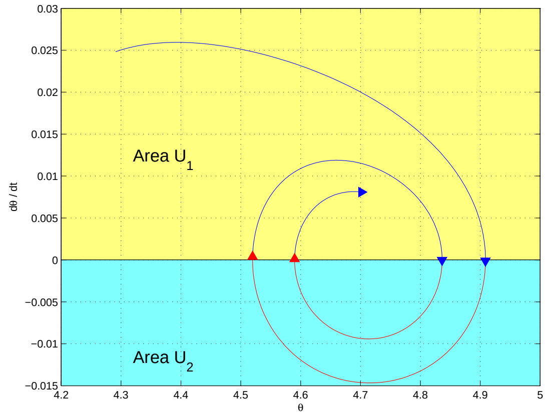


(b)

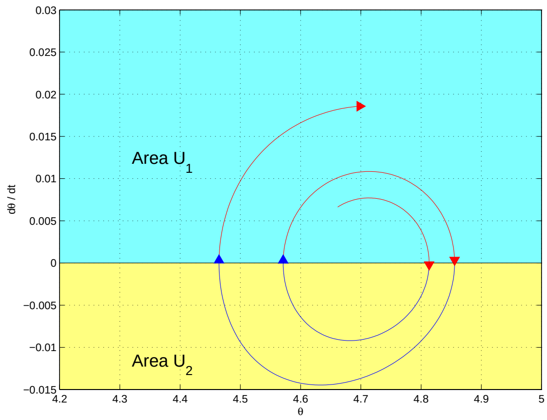
Fig. A.1. (a) An example of the phase portrait of the system (A.1); (b) An example of the phase portrait of the system (A.2). Both systems have the centers around the equilibrium $\theta_0 \approx 4.7$.

Knowing the solutions of the systems (A.1) and (A.2), one could obtain the solution of the original non-smooth system (9). For example, if a solution starts in $\mathcal{U}_1$ then it coincides with the solution of (A.1) with the same origin until it attains the switching surface Γ. Then it enters into the area $\mathcal{U}_2$ and follows the solution of the system (A.1), and so on. This behaviour is illustrated in Figure A.2. As seen in Figure A.2, even if systems (A.1) and (A.2) have centers at the equilibrium θ_0, the non-smooth system (9) would have this equilibrium either asymptotically stable or unstable dependent on the character of the periodic motions of (A.1) and (A.2).

Figure A.2(a) shows an example where the dynamics of (9) in the area $\mathcal{U}_1$ coincides with the one shown on Figure A.1(a); and in



(a)



(b)

Fig. A.2. Examples of phase portraits of the non-smooth system (9): (a) Here the dynamics on $\mathcal{U}_1$ coincides with shown on Figure A.1(a), and on $\mathcal{U}_2$ coincides with shown on Figure A.1(b); (b) An opposite situation: the dynamics on $\mathcal{U}_1$ coincides with shown on Figure A.1(b), and on $\mathcal{U}_2$ coincides with shown on Figure A.1(a). As seen, the plots show two qualitatively different behaviours: asymptotic stability of the equilibrium for the case (a) and instability for the case (b).

the area $\mathcal{U}_2$ coincides with the one shown on Figure A.1(b). As seen, for such a non-smooth dynamical system the equilibrium at θ_0 is asymptotically stable. If one changes the dynamics on $\mathcal{U}_1$ to coincide with the one shown on Figure A.1(b) and on $\mathcal{U}_2$ to coincide with the one shown on Figure A.1(a), then the equilibrium becomes unstable.

To prove Theorem 2 we need to show that the validity of condition (18) implies that the non-smooth system (9) has the phase portrait of the form shown on Figure A.2(a). It is easy to check that the case shown on Figure A.2 takes place only if the next statement is true.

Statement 1. Let θ_0 be an equilibrium of (9) and the inequality (17) be valid. Consider a solution $\theta_+(t)$ of (A.1) and a solution $\theta_-(t)$ of (A.2) both with origin at

$$\Big[\theta_\pm(0),\, \dot\theta_\pm(0)\Big] = [\theta_0 - \varepsilon,\, 0]$$

where $\varepsilon > 0$. Denote T_+ the smallest positive time instant when $\dot\theta_+(T_+) = 0$ and T_- the smallest positive time instant when $\dot\theta_-(T_)$ $= 0$. Both T_+ and T_- are the functions of the parameter ε, $T_+ = T_+(\varepsilon)$, $T_- = T_-(\varepsilon)$. Then the equilibrium θ_0 is asymptotically stable if and only if for all sufficiently small positive ε the inequality

$$\theta_+(T_+(\varepsilon)) < \theta_-(T_-(\varepsilon)) \tag{A.3}$$

is valid. Furthermore, the inequality (A.3) takes place for all sufficiently small positive ε provided that the inequality (18) holds. ■