CONSERVATION OF FILTERING IN MANUFACTURING SYSTEMS

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Abstract: This paper addresses the issue of reliable satisfaction of customer demand by unreliable production systems. Using a simple Production-Storage-Customer model, we show that this can be accomplished by filtering out production randomness. The filtering of randomness is ensured by finished goods buffers (filtering in space) and shipping periods (filtering in time). The following question is considered: How are filtering in space and in time interrelated? As an answer, we show that there exists a conservation law: In lean manufacturing systems, the amount of filtering in space multiplied by the amount of filtering in time (both measured in appropriate dimensionless units) is practically constant.

Keywords: Manufacturing systems, Filtering, Conservation law.

1. INTRODUCTION

Manufacturing systems with unreliable machines usually contain Finished Goods Buffers (FGB) intended to filter out production randomness and, thereby, ensure reliable satisfaction of customer demand. "Filtering in space", provided by FGBs, is complemented by "filtering in time", offered by shipping periods. For a given shipping period and shipment size, the smallest FGB capacity, which ensures the desired level of customer demand satisfaction, is referred to as *lean*. The question addressed in this paper is: How are filtering in time and space interrelated? As an answer to this question, we show that there exists a conservation law, which, roughly speaking, can be formulated as follows: In manufacturing systems with lean FGBs, *the product of "filtering in time" and "filtering in space" is practically constant.*

Along with theoretical significance and insight into the behavior of production systems, this law offers practitioners a quantitative tool for managing lean FGBs. In particular, it shows how FGB capacity should be adjusted, when the shipping period is changed, so that neither the leanness of FGB nor the level of customer demand satisfaction are sacrificed.

The literature, related to the problem addressed in this paper, consists of publications devoted to production variability and customer demand satisfaction. Due to space limitations, we mention here only a few publications on these topics. More references can be found in (Li *et al.*, 2004*a*). The study of production variability has been initiated in (Miltenburg, 1987), and continued, for example, in (Gershwin, 1993) and (Tan, 2000). Although production variance is an important metric of production variability, it does not, by itself, characterize the level of customer demand satisfaction. Therefore, to address this issue, references (Jacobs and Meerkov, 1995) and (Li *et al.*, 2004*b*), introduced and analyzed the notion of due time performance In the present paper, we use the method developed in (Li *et al.*, 2004*b*) for analysis of filtering of the production randomness in manufacturing systems with unreliable machines and finished goods buffers.

The outline of this paper is as follows: Section 2 introduces the model of the system under consideration. In Section 3, this model is parameterized in terms of three dimensionless parameters, and the problem to be analyzed is formulated. The method of analysis is outlined in Section 4, and the main result, i.e., the conservation law mentioned above, is described in Section 5. The conclusions are given in Section 6.

2. MODEL

2.1 Assumptions

The manufacturing system analyzed in this paper consists of three subsystems: Production, Storage, and Customer. Each of them and their interactions are formalized as follows:

Production Subsystem

- (i) The production subsystem is intended to produce one part during a fixed time interval (referred to as the *cycle time*). Due to machine breakdowns, this may or may not happen, depending on the status of the last machine in the system (up or down) and the buffer occupancy in front of it (empty or not). Therefore, the production subsystem may be in one of two states: *active* or *passive*. When active, a part is produced during each cycle time. When passive, no parts are produced.
- (ii) The time intervals, during which the production subsystem is active or passive, are exponentially distributed random variables defined by parameters α and β, respectively.

Storage Subsystem

- (iii) The storage subsystem consists of a finished goods buffer with capacity $0 < N < \infty$. Parts produced by the production subsystem are immediately transferred to the FGB.

Interaction between the Production and Storage Subsystems

- (iv) The production subsystem is blocked at time t if the FGB is full at time t.

Customer Subsystem

- (v) The customer requires D parts to be shipped during each shipping period. The duration of the shipping period is T cycles of time. To avoid triviality, it is assumed that

$$D < Te. \qquad (1)$$

where e is the average production rate of the production subsystem, i.e.,

$$e = \frac{\beta}{\alpha+\beta} = \frac{T_\alpha}{T_\alpha + T_\beta}. \qquad (2)$$

Here $T_\alpha = 1/\alpha$ and $T_\beta = 1/\beta$ are the average values of the active and passive periods, respectively.

Interaction among the Production, Storage, and Customer Subsystems

- (vi) At the beginning of shipping period i, parts are removed from the FGB in the amount $\min\{H(i-1), D\}$, where $H(i-1)$ is the number of parts in the FGB at the end of the $(i-1)$-st shipping period. If $H(i-1) \geq D$, the shipment is complete; if $H(i-1) < D$, the balance of the shipment, i.e., $D - H(i-1)$ parts, is to be produced by the production subsystem during the shipping period T. The parts produced are immediately removed from the FGB and prepared for shipment, until the shipment is complete, i.e., D parts are available. If the shipment is complete before the end of the shipping period, the production subsystem continues operating, but with the parts being accumulated in the FGB, either until the end of the shipping period or until the production system is blocked by the full FGB, whichever occurs first. If the shipment is not complete by the end of the shipping period, an incomplete shipment is sent to the customer. No backlog is allowed.

Remarks:

- The production system defined by assumption (i) may have an arbitrary topological structure; it could be either a serial line, or an assembly line, or even a re-entry line.
- The exponential distributions of active and passive periods of the production subsystem (assumption (ii)) are introduced to enable an analytical approach to the problem at hand. Similar results, using numerical simulations, may be obtained for non-exponential distributions as well, provided that their coefficients of variation are less than 1.
- A fixed shipping period T (assumption (v)) is typical in the automotive industry: usually, assembly plants interact with the first-tier suppliers on a fixed delivery schedule.
- A fixed shipment size D (assumption (v)) is also a part of standard agreements among

assembly plants and their suppliers. In reality, however, the shipment size may sometimes vary. The results, reported here, can be extended to the case of a random demand (using the analytical technique developed in (Li *et al.*, 2004*b*).

2.2 Demand Satisfaction Metric

The demand satisfaction metric, used in this work, is the probability that D parts are shipped to the customer during a shipping period T. We refer to this metric as the *Due Time Performance* (DTP) (Jacobs and Meerkov, 1995). To formalize this metric in terms of the Production-Storage-Customer system described above, we note that in the time scale of the shipping period T, assumptions (i)-(vi) define a stationary, ergodic Markov chain. Let $\widehat{t}(i)$ be the random variable representing the number of parts produced by the production subsystem during the i-th shipping period in the steady state of this Markov chain. Then DTP can be represented as follows:

$$DTP = \Pr\Big(H(i-1) + \widehat{t}(i) \geq D\Big), \qquad (3)$$

where, as before, $H(i-1)$ is the number of parts in the FGB at the end of the $(i-1)$-st shipping period.

A method for calculating DTP has been developed in (Li *et al.*, 2004*b*). Using this method, it is easy to show that DTP tends to 1 when either N or T tend to infinity. In this sense, N and T provide filtering of the production randomness in space and time, respectively. In this paper, we investigate how T and the smallest N, which results in the desired value of DTP, are interrelated.

3. PARAMETERIZATION AND PROBLEM FORMULATION

The Production-Storage-Customer system, defined by assumptions (i)-(vi), is characterized by five parameters: α, β, N, D, and T. None of them, independently, define the regime, in which the system is operating. For instance, the same D could represent either a high or low demand on the production system (depending on its average production rate). Similarly, the same N (respectively, T) may represent either a high or low level of filtering in space (respectively, in time). Therefore, the five parameters must be normalized so that the regimes of operation become explicit. This situation is similar to that of fluid mechanics, where the Reynold's number and other dimensionless parameters are introduced to quantify flow regimes. Below, we introduce three parameters describing regimes of manufacturing systems operation.

(a) *Relative FGB capacity*:

$$\nu = \frac{N}{D}. \qquad (4)$$

Clearly, ν characterizes regimes of operation from the point of view of filtering in space: $\nu \ll 1$ implies that shipments are practically just-in-time (even if N is large), while $\nu \gg 1$ means that the system operates in the regime with large space filtering (even if N is small).

(b) *Relative shipment period*:

$$\tau = \frac{T}{T_\alpha + T_\beta}. \qquad (5)$$

The parameter τ quantifies the shipping period in units of what is referred to as the *reliability cycle*, $T_\alpha + T_\beta$. When τ is large, the shipping period offers significant filtering in time; small τ implies a regime with insignificant time filtering.

(c) *Load factor*:

$$L = \frac{D}{Te}. \qquad (6)$$

Due to (1), $L < 1$. When L is close to 1, the production system operates in a regime with a heavy load. Often, manufacturing managers view large L as a desirable regime. In Japanese industry, however, small L seems to be preferred.

The smallest ν, which ensures the desired DTP, is referred to as the *lean* relative FGB capacity; it is denoted as ν_{DTP}.

The problem addressed in this paper is: *Given the Production-Storage-Customer system, defined by assumptions (i)-(vi), and the desired DTP, analyze the interrelationship between ν_{DTP} and τ, i.e., investigate how filtering in space can be traded off against filtering in time.*

A solution to this problem is given in Section 5, while the approach of this research is outlined in Section 4.

4. APPROACH

4.1 Calculation of DTP

A method for calculating DTP in manufacturing systems defined by assumptions (i)-(vi) has been developed in (Li *et al.*, 2004*b*). Briefly, it can be described as follows:

Let $t(i)$ denote the number of parts produced during shipping period i if no blocking occurs. Introduce the following quantities:

$$\mathcal{P}(x) = \Pr\Big(t(i) \geq x\Big), \quad x \in \{0, 1, \ldots, T\},$$

$$r_{k,j} = \Pr\Big(t(i) = D + k - j\Big), \quad k = 1, \ldots, N-1,$$
$$j = 0, 1, \ldots, N,$$
$$\widehat{r}_{N,j} = \Pr\Big(t(i) \geq D + N - j\Big), \quad j = 0, 1, \ldots, N.$$

These quantities can be calculated as follows: As it has been shown in (Jacobs and Meerkov, 1995),

$$\mathcal{P}(x) = \frac{\beta e^{-\alpha x}}{\alpha + \beta}\left[1 + \sum_{j=2}^{\infty} \frac{(\alpha x)^{j-1}}{(j-1)!}\Big(1 - e^{-\beta(T-x)} \sum_{k=0}^{j-2} \frac{[\beta(T-x)]^k}{k!}\Big)\right] + \frac{\alpha e^{-\alpha x}}{\alpha + \beta} \sum_{j=1}^{\infty} \frac{(\alpha x)^{j-1}}{(j-1)!} \left[1 - e^{-\beta(T-x)} \sum_{k=0}^{j-1} \frac{[\beta(T-x)]^k}{k!}\right]. \quad (7)$$

To calculate $r_{k,j}$, the following expression can be used:

$$r_{k,j} = \mathcal{P}(D + k - j) - \mathcal{P}(D + k - j + 1). \quad (8)$$

The $\widehat{r}_{N,j}$ can be calculated as

$$\widehat{r}_{N,j} = \mathcal{P}(D + N - j). \quad (9)$$

Introduce matrix $\mathcal{R}$ and vector Z_0 defined by

$$\mathcal{R} = \begin{pmatrix} r_{1,1} - r_{1,0} - 1 & r_{1,2} - r_{1,0} & \ldots & r_{1,N} - r_{1,0} \\ r_{2,1} - r_{2,0} & r_{2,2} - r_{2,0} - 1 & \ldots & r_{2,N} - r_{2,0} \\ \ldots & \ldots & \ldots & \ldots \\ \widehat{r}_{N,1} - \widehat{r}_{N,0} & \widehat{r}_{N,2} - \widehat{r}_{N,0} & \ldots & \widehat{r}_{N,N} - \widehat{r}_{N,0} - 1 \end{pmatrix},$$
$$Z_0 = \begin{pmatrix} r_{1,0} \\ r_{2,0} \\ \ldots \\ \widehat{r}_{N,0} \end{pmatrix}. \quad (10)$$

Matrix $\mathcal{R}$ is nonsingular due to the uniqueness of the stationary probability distribution defined by system (i)-(vi).

The following Theorem is proved in (Li *et al.*, 2004*b*):

Theorem 1. Under assumptions (i)-(vi),

$$DTP = \sum_{k=0}^{N} \mathcal{P}(D - k) z_k, \quad (11)$$

where $z_k = \Pr\Big(H(i-1) = k\Big)$, $k = 0, 1, \ldots, N$, and vector $Z = [z_1, z_2, \ldots, z_N]^T$ is calculated according to

$$Z = -\mathcal{R}^{-1} Z_0. \quad (12)$$

4.2 Evaluation of ν_{DTP}

For given α, β, D, and T, we first evaluate L and τ. Then, assuming $N = 0$, we use Theorem 1 and calculate DTP, i.e., $DTP(N = 0)$. If it is larger than the desired DTP, we assume that the lean FGB capacity is 0. Otherwise, we assume $N = 1$, calculate $DTP(N = 1)$ and again compare it with the desired DTP. We continue this procedure until we arrive at the smallest N, denoted as N_{DTP}, for which $DTP(N_{DTP})$ is larger than the desired DTP. Then we evaluate ν_{DTP} as follows:

$$\nu_{DTP} = \frac{N_{\nu_{DTP}}}{D}. \quad (13)$$

Thus, for given L and τ, the value of ν_{DTP} is determined.

Results, obtained using this approach, are described below.

5. MAIN RESULTS

5.1 Typical Behavior

The typical behavior of ν_{DTP} as a function of τ, i.e., the function

$$\nu_{DTP} = F_{DTP}(\tau), \quad (14)$$

is illustrated in Figure 1.Three regimes of system operation are presented: heavy load ($L = 0.97$), medium load ($L = 0.92$), and light load ($L = 0.71$). These graphs are calculated, using the approach described above for the production system characterized by $e = 0.825$ and $T_\alpha + T_\beta = 25$, i.e., $T_\alpha = 20.625$, and $T_\beta = 4.375$. Also, the desired DTP is assumed to be 0.99, which is a typical desired level of customer demand satisfaction in the automotive industry.

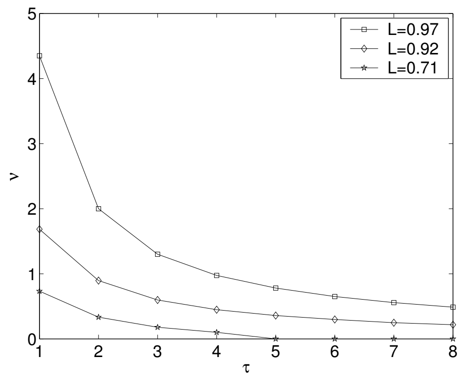

Fig. 1. Typical behavior of ν_{DTP} vs. τ

Clearly, the graphs of Figure 1 exhibit a tradeoff between filtering in time and filtering in space. For instance, in the heavy load regime, $\tau = 5$ requires $\nu_{DTP} \cong 1$, while for $\tau = 1$, the $\nu_{DTP} \cong 4.5$ is necessary. Similar tradeoffs take place in the medium and light load regimes. However, the amount of filtering in space, necessary to achieve the desired DTP, drops significantly when the load is decreased. For example, in the light load regime and $\tau = 5$, no finished good buffer is necessary, and the deliveries can be just-in-time, while this is impossible for the medium and heavy loads. This dramatic improvement in the acceptable leanness, ensured by low loads, may be a justification of the Japanese firms' tendency to operate in light load regimes: it allows them to maintain a high level of customer demand satisfaction with a small (if any) finished goods inventory.

5.2 Conservation Law

The nature of curves in Figure 1 suggests that ν_{DTP} and τ may be related in a hyperbolic manner, i.e.,

$$\nu_{DTP}\tau = const. \tag{15}$$

This hypothesis is substantiated by the graphs of Figure 2, while the value of the constant in the right-hand side of (15) is also indicated.

These graphs correspond to $DTP = 0.99$; similar results have been obtained for other values of DTP as well (see (Li *et al.*, 2004*a*)).

Expression (15) can be interpreted as a conservation law: For a manufacturing system defined by assumptions (i)-(vi), the amount of filtering in space (in units of ν_{DTP}) multiplied by the amount of filtering in time (in units of τ), necessary to attain the desired DTP, is constant. In other words, filtering in space and filtering in time tradeoff one-to-one.

The conservation law (15) can be written in the form

$$\nu_{DTP}\tau = \Phi_{DTP}(L, e), \tag{16}$$

which specifies the dependency of the constant in (15) on the system parameters. For several values of its arguments, function $\Phi_{DTP}(L, e)$ is characterized by the constants included in Figure 2. How can this function be evaluated for other values of its arguments? An answer to this question is as follows:

For a given manufacturing system and given DTP, calculate ν_{DTP}, using the approach of Section 4, for $\tau = 1$. Then, according to (15), function $\Phi_{DTP}(L, e)$ can be defined as

$$\Phi_{DTP}(L, e) = F_{DTP}(1). \tag{17}$$

In other words, an approximation of the conservation law (15) can be given as follows:

$$\nu_{DTP}\tau = F_{DTP}(1). \tag{18}$$

The behavior of this approximation is illustrated in Figure 3. Similar graphs have been obtained for other values of DTP as well (Li *et al.*, 2004*a*). Since the accuracy of this approximation is relatively high, we conclude that the constant in the right hand side of (15) can be defined as shown in (18).

Relationships (15) and (16) are the main results of this paper. In particular, they imply the following:

1) Changing the shipping period by a factor of k requires a change in the lean FGB capacity by a factor of $1/k$.

2) The lean FGB capacity can be calculated as

$$\begin{aligned} N_{DTP} &= \nu_{DTP} D = \frac{F_{DTP}(1)}{\tau} D \\ &= \frac{F_{DTP}(1)}{T}(T_\alpha + T_\beta)D, \end{aligned} \tag{19}$$

where T_α, T_β, T and D are the manufacturing system parameters and $F_{DTP}(1)$ is a function, which is calculated using the procedure of Subsection 4.1.

6. CONCLUSIONS

For a simple Production-Storage-Customer system, it is shown in this paper that there exists a conservation law of filtering in space and time. Along with providing an insight into the nature of manufacturing systems, this law permits one to evaluate the smallest, i.e., lean, finished goods buffer capacity, which is necessary and sufficient to ensure a reliable satisfaction of customer demand by an unreliable production system.

The results reported here can be generalized in at least two directions. First, random customer demand can be considered. Preliminary results, obtained in this direction, indicate that the conservation law still holds, however, its precise expression is a topic of future work.

The second generalization is in the direction of more realistic assumptions on the distribution of active and passive periods of the production subsystem. Although there are no analytical tools for calculating DTP for the non-exponential case, the problem can be approached using discrete event simulations. Preliminary results, derived in this direction, also indicate that the conservation law still holds. Finalizing these results is another topic of future work.

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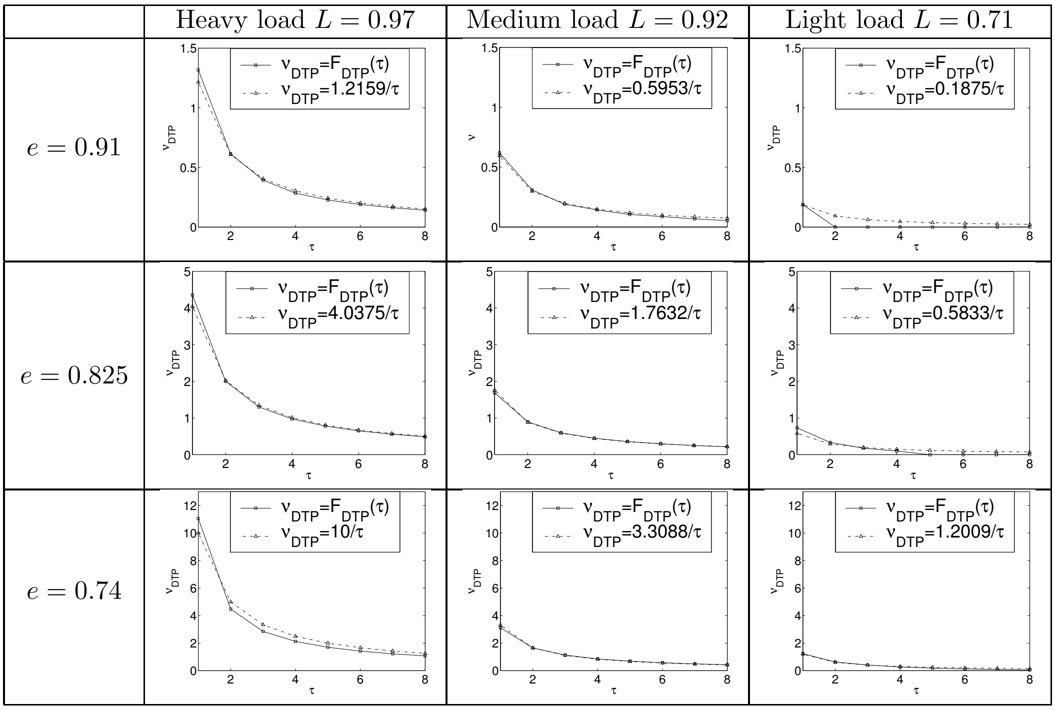

Fig. 2. Illustration of the conservation law: Functions $\nu_{DTP} = F_{DTP}(\tau)$ and $\nu_{DTP} = const/\tau$

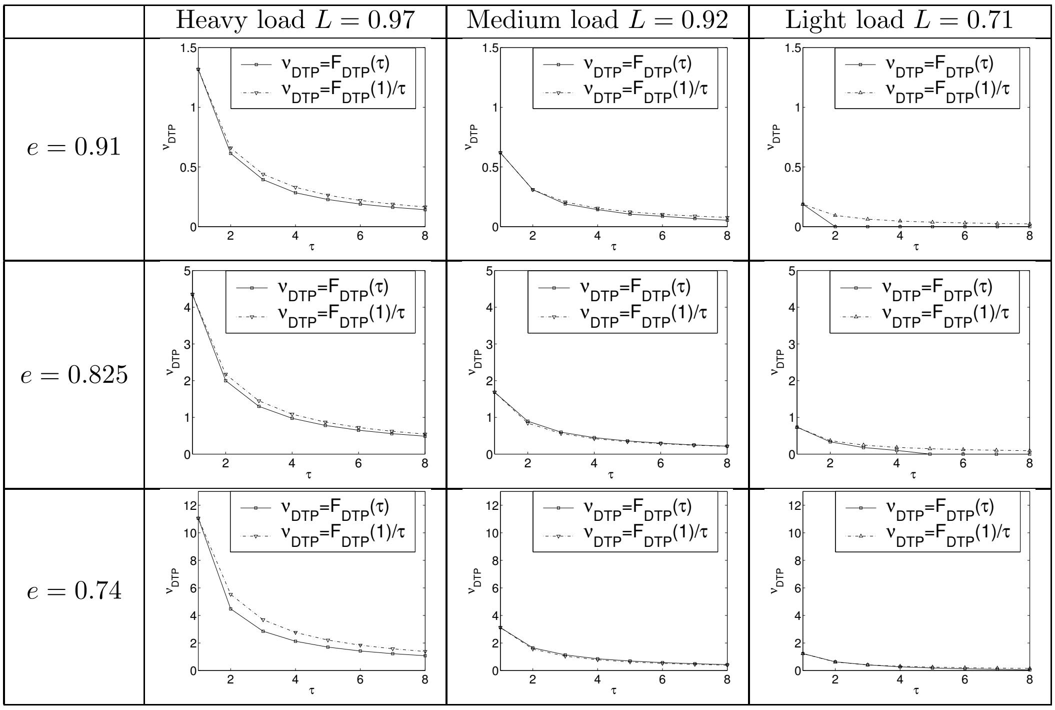

Fig. 3. Approximation of the conservation law: Functions $\nu_{DTP} = F_{DTP}(\tau)$ and $\nu_{DTP} = F_{DTP}(1)/\tau$

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