ROBUST $\mathcal{H}_2/\mathcal{H}_\infty$ DYNAMIC OUTPUT-FEEDBACK CONTROL SYNTHESIS FOR SYSTEMS WITH POLYTOPE-BOUNDED UNCERTAINTY

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Abstract: This paper presents a strategy for robust $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output-feedback control synthesis, with regional pole placement, applied to linear continuous-time time-invariant systems with polytope-bounded uncertainty. The proposed synthesis approach is based on a multiobjective optimization algorithm applied directly in the space of controller parameters. The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, computed in all polytope vertices and in possible "worst case" interior points are taken as the optimization objectives. A branch-and-bound algorithm based on LMI guaranteed cost formulation is applied to validate the controller design. Examples are presented to show the effectiveness of the proposed strategy, including examples of full order and low order, centralized and decentralized control systems.

Keywords: Robust $\mathcal{H}_2/\mathcal{H}_\infty$ control, dynamic output feedback, regional pole placement, reliable control system, multiobjective optimization.

1. INTRODUCTION

This paper deals with the problem of robust $\mathcal{H}_2/\mathcal{H}_\infty$ dynamic output-feedback control synthesis, with regional pole placement, for systems with polytope-bounded uncertainty, also considering the robust decentralized control case. There is no direct LMI-based formulation to solve this control synthesis problem, unless those that can be characterized as a bilinear matrix inequality (BMI)

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optimization problem. In (Kanev *et al.*, 2004) it is presented an extensive review of previous works in this field, as well as a comparison of five local BMI approaches for dynamic output-feedback control. Several works are based on the strategy of transforming the BMI problem in two coupled LMI optimization problems by fixing one of the variables in each problem, however those approaches are concerned only with the $\mathcal{H}_2$ performance (Iwasaki, 1999; de Oliveira *et al.*, 2000).

In this paper, a two-phase synthesis strategy is also proposed, with a different approach. In the first phase, the controller is computed based on an optimization problem formulated directly in the space of controller parameters; the objective functions and constraints are verified at the polytope vertices of the uncertain domain. In the second phase, the controller performance is verified by means of LMI-based analysis formulations, in order to compute the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ costs and to certificate that the regional pole placement holds. If the controller do not attain the required specification, the controller is re-designed with the inclusion of the worst case polytope interior points. Both the cost computation and the worst case point search are performed by a branch-and-bound LMI-based algorithm for exact guaranteed cost computation. The examples in Section 4 show that the contribution of this paper is to present a computationally tractable algorithm to compute full or low order, centralized or decentralized, multiobjective robust dynamic output-feedback controllers.

2. PROBLEM STATEMENT

Consider the LTI system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) + B_w w(t) \\ z_\infty(t) &= C_{z1}x(t) + D_{zu1}u(t) + D_{zw1}w(t) \\ z_2(t) &= C_{z2}x(t) + D_{zu2}u(t) \\ y(t) &= C_y x(t) + D_{yw}w(t) \end{aligned} \tag{1}$$

in which $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{p_u}$ is the control input, $w \in \mathbb{R}^{p_w}$ is the exogenous input, $z_\infty \in \mathbb{R}^{m_{z\infty}}$ is the controlled output related to the $\mathcal{H}_\infty$ performance, $z_2 \in \mathbb{R}^{m_{z2}}$ is the controlled output related to the $\mathcal{H}_2$ performance, and $y \in \mathbb{R}^{m_y}$ is the measurement output.

The system matrices in (1) gathered in the matrix

$$\mathcal{S} \triangleq \begin{bmatrix} A & B_u & B_w \\ C_{z1} & D_{zu1} & D_{zw1} \\ C_{z2} & D_{zu2} & 0 \\ C_y & 0 & D_{yw} \end{bmatrix} \tag{2}$$

can include uncertain parameters belonging to a known polytope, defined by its vertices:

$$\mathcal{P}(\alpha) \triangleq \left\{ \mathcal{S} : \mathcal{S} = \sum_{i=1}^{N} \alpha_i \mathcal{S}_i; \quad \alpha \in \Omega \right\} \tag{3}$$

$$\Omega \triangleq \left\{ \alpha \ : \ \alpha_i \geq 0, \ \sum_{i=1}^{N} \alpha_i = 1 \right\} \tag{4}$$

where $\mathcal{S}_i$, $i = 1, \ldots, N$, are the polytope vertices and the vector $\alpha = \left[\alpha_1 \ \ldots \ \alpha_N \right]'$ parameterizes the polytope.

The system is controlled by a dynamic output-feedback controller, $\mathcal{K}$:

$$\mathcal{K} : \begin{cases} \dot{x_c}(t) = A_c x_c(t) + B_c y(t) \\ u(t) = C_c x_c(t) + D_c y(t) \end{cases} \tag{5}$$

The closed-loop system with $\bar{x}^T(t) = [x^T(t) \ \ x_c^T(t)]$ is:

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}w(t) \\ z_\infty(t) &= \bar{C}_1\bar{x}(t) + \bar{D}_1 w(t) \\ z_2(t) &= \bar{C}_2\bar{x}(t) \end{aligned} \tag{6}$$

where

$$\bar{A} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \tag{7}$$

$$\bar{B} = \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix} \tag{8}$$

$$\bar{C}_1 = \left[C_{z1} + D_{zu1} D_c C_y \quad D_{zu1} C_c \right] \tag{9}$$

$$\bar{D}_1 = \left[D_{zw1} + D_{zu1} D_c D_{yw} \right] \tag{10}$$

$$\bar{C}_2 = \left[C_{z2} + D_{zu2} D_c C_y \quad D_{zu2} C_c \right] \tag{11}$$

Let $T_\infty(\alpha, \mathcal{K})$ denote the closed-loop transfer function matrix from w to z_∞ and $T_2(\alpha, \mathcal{K})$ denote the closed-loop transfer function matrix from w to z_2. Consider the vector of control objectives to be minimized:

$$J(\mathcal{K}) = \begin{bmatrix} \max\limits_{\alpha} ||T_\infty(\alpha, \mathcal{K})||_\infty \\ \max\limits_{\alpha} ||T_2(\alpha, \mathcal{K})||_2 \end{bmatrix} \tag{12}$$

with $||\cdot||_\infty$ and $||\cdot||_2$ respectively the $\mathcal{H}_\infty$ norm and the $\mathcal{H}_2$ norm of the argument. Let Γ denote the set of all controllers that satisfy the regional pole placement constraints for the closed-loop system:

$$\Gamma \triangleq \left\{ \mathcal{K} \ : \ \sigma(\bar{A}(\alpha, \mathcal{K})) \subset \mathcal{D} \ \forall \ \alpha \in \Omega \right\} \tag{13}$$

with $\mathcal{D} \subset \mathbb{C}_-$ and $\sigma(\cdot)$ the spectrum of the argument.

The conceptual problem that is addressed here is stated as:

Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ Guaranteed Cost Problem: *Find controllers $\mathcal{K}^*$ that belong to the Pareto-optimal set Γ^*:*

$$\begin{aligned} \Gamma^* \triangleq \{ & \mathcal{K}^* \in \Gamma \ : \ \nexists \ \mathcal{K} \in \Gamma \ | \\ & J(\mathcal{K}) \leq J(\mathcal{K}^*) \ \textit{and} \ J(\mathcal{K}) \neq J(\mathcal{K}^*) \} \end{aligned} \tag{14}$$

The vector comparison operators $\leq$ and $\neq$ are employed here with the meaning usually adopted in vector optimization (Chankong and Haimes, 1983): consider x_i an entry of the vector $x \in \mathbb{R}^n$ and y_i an entry of the vector $y \in \mathbb{R}^n$, than $x \leq \ y \Rightarrow x_i \leq y_i, \forall i = 1, \ldots, n$; and $x \neq \ y \Rightarrow \exists i \mid x_i \neq y_i$.

3. PROPOSED DESIGN PROCEDURE

Consider the set of points initialized as the polytope vertices $\tilde{\Omega}$:

$$\tilde{\Omega} \triangleq \left\{ \begin{array}{c} \alpha \ : \ \alpha_i = 1 \ \ , \ \ \alpha_j = 0 \ \forall \ j \neq i, \\ i = 1, \ldots, N \end{array} \right\} \quad (15)$$

and the "worst case" $\mathcal{H}_2$ norm in this set, with a given controller $\mathcal{K}$, as:

$$\bar{\delta}_{w.c.} \triangleq \max_{\alpha \in \tilde{\Omega}} \|T_2(\alpha, \mathcal{K})\|_2 \quad (16)$$

Consider also the "worst case" points in the polytope, with a given controller $\mathcal{K}$, as:

$$\begin{array}{l} \alpha_{(2)} \triangleq \arg\max_{\alpha} \|T_2(\alpha, \mathcal{K})\|_2 \\ \alpha_{(\infty)} \triangleq \arg\max_{\alpha} \|T_\infty(\alpha, \mathcal{K})\|_\infty \end{array} \quad (17)$$

Define now the set:

$$\tilde{\Gamma} \triangleq \left\{ \mathcal{K} \ : \ \sigma(\bar{A}(\alpha, \mathcal{K})) \subset \mathcal{D} \ \forall \ \alpha \in \tilde{\Omega} \right\} \quad (18)$$

The following auxiliary problem will be used:

Auxiliary problem: *Given a $\gamma > 0$, find the controller $\tilde{\mathcal{K}}^*$ such that:*

$$\begin{array}{l} \tilde{\mathcal{K}}^* = \arg\min_{\mathcal{K}} \max_{\alpha \in \tilde{\Omega}} ||T_2(\alpha, \mathcal{K})||_2 \\ \text{subject to:} \ \left\{ \begin{array}{l} \mathcal{K} \in \tilde{\Gamma} \\ ||T_\infty(\alpha, \mathcal{K})||_\infty \leq \gamma \end{array} \right. \end{array} \quad (19)$$

The design procedure proposed here is stated as:

Design Procedure

Step 1. Initialize
$i \leftarrow 0$,
$\tilde{\Omega}_0 \leftarrow$ set of polytope vertices,

Step 2. $i \leftarrow i + 1$,
$\tilde{\Omega}_i \leftarrow \tilde{\Omega}_{i-1}$.

Step 3. Solve the Auxiliary Problem, finding $\tilde{\mathcal{K}}^*$.

Step 4. Compute the exact $\mathcal{H}_2$ cost, δ_c, and find $\alpha_{(2)}$ for $\tilde{\mathcal{K}}^*$.
If $\alpha_{(2)} \notin \tilde{\Omega}_i$ and $(\|T_2(\alpha_{(2)}, \tilde{\mathcal{K}}^*)\|_2 - \bar{\delta}_{w.c.})/\bar{\delta}_{w.c.} > \varepsilon_\delta$, then $\tilde{\Omega}_i \leftarrow \tilde{\Omega}_i \cup \alpha_{(2)}$.

Step 5. Compute the exact $\mathcal{H}_\infty$ cost, γ_c, and find $\alpha_{(\infty)}$ for $\tilde{\mathcal{K}}^*$.
If $\alpha_{(\infty)} \notin \tilde{\Omega}_i$ and $\gamma_c > \gamma$, then $\tilde{\Omega}_i \leftarrow \tilde{\Omega}_i \cup \alpha_{(\infty)}$.

Step 6. Check the regional pole placement constraints, if it is found an $\alpha_{(p)}$ that violates these constraints then $\tilde{\Omega}_i \leftarrow \tilde{\Omega}_i \cup \alpha_{(p)}$.

Step 7. If $\tilde{\Omega}_i \neq \tilde{\Omega}_{i-1}$ go to Step 2. Otherwise stop.

These steps are further explained in the next subsections. The idea behind this algorithm is simple. The controller is computed with the optimization algorithm, considering initially only a set that initially contains only the polytope vertices. The design is validated "a posteriori" by means of LMI-based branch-and-bound exact analysis formulations. If the required specifications are not met, then the worst-case polytope interior points, relative to these specifications, are included in the set and a new iteration is processed. To avoid unnecessary iterations, a sufficiently small constant ε_δ is used as a decision tolerance, for including or not interior points that lead to greater worst case $\mathcal{H}_2$ norms than previous points.

3.1 Solution of the Auxiliary Problem

The auxiliar scalar optimization problem is solved by the cone-ellipsoid algorithm, described by the recursive equations presented in (Gonçalves *et al.*, 2005). The optimization algorithm stops when $(f_{max} - f_{min})/f_{min} \leq \epsilon$, where f_{max} and f_{min} are the maximum and minimum objective function values in the last N_ϵ iterations and ϵ is the relative accuracy required.

3.2 Guaranteed Costs Computation

The "exact" $\mathcal{H}_2$ ($\mathcal{H}_\infty$) cost computation, δ_c (γ_c), is based on a branch-and-bound algorithm where the upper bound function is the $\mathcal{H}_2$ ($\mathcal{H}_\infty$) guaranteed cost, δ_{gc} (γ_{gc}), and the lower bound function is the worst case $\mathcal{H}_2$ ($\mathcal{H}_\infty$) norm, $\delta_{w.c.}$ ($\gamma_{w.c.}$), at the vertices of the polytope and subpolytopes. The difference between the upper and lower bound functions tends to zero when the polytope is subdivided in smaller subpolytopes and the algorithm stops when the "exact" cost, δ_c (γ_c), achieves the specified relative accuracy, ϵ_c. The guaranteed cost computations can be based on any LMI-based formulation. In this work, the $\mathcal{H}_2$ guaranteed cost computation is based on the combination of Lemma 1 and Lemma 2 in (de Oliveira *et al.*, 2004*a*) and the $\mathcal{H}_\infty$ guaranteed cost computation is based on Lemma 1 in (de Oliveira *et al.*, 2004*b*).

The polytope subdivision technique applied in the branch operation is performed in two steps: first the polytope is split in simplices (generalizations of triangles in any dimension space) then the simplices are split with an edgewise simplex subdivision.

3.3 Regional Pole Placement Verification

The robust regional pole placement in LMI-regions are verified by the conventional LMI-based analysis formulation as presented in (Chilali and Gahinet, 1999). If the problem is not feasible in the whole uncertain parameter set, the polytope is partitioned until all subpolytopes result in feasible problems indicating that the regional pole placement constraints are satisfied in the whole polytope. If it is found a subpolytope vertex $\alpha_{(p)}$ that violates one of the regional pole placement constraints, it is necessary a new iteration in the controller synthesis algorithm.

4. ILLUSTRATIVE EXAMPLES

The proposed design procedure was implemented with the MATLAB® and a Pentium IV 2.8GHz, 512 MBytes RAM computer.

Example 1

The system is a satellite consisting of two rigid bodies (main module and sensor module) connected by an elastic link that is modelled as a spring with torque constant k and viscous damping f that have uncertainty ranges (Gahinet *et al.*, 1995):

$$0.09 \leq k \leq 0.4 \quad \text{and} \quad 0.0038 \leq f \leq 0.04$$

The state-space description for the satellite system is

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k}{J_1} & \dfrac{k}{J_1} & -\dfrac{f}{J_1} & \dfrac{f}{J_1} \\ \dfrac{k}{J_2} & -\dfrac{k}{J_2} & \dfrac{f}{J_2} & -\dfrac{f}{J_2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{J_1} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{J_1} \\ 0 \end{bmatrix} w \tag{20}$$

$$z_\infty = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x \tag{21}$$

$$z_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{22}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x \tag{23}$$

where θ_1 and θ_2 are the yaw angles for the main body and the sensor module, u is the control torque, and w is a torque disturbance on the main body. It is considered $J_1 = 1$ and $J_2 = 1$.

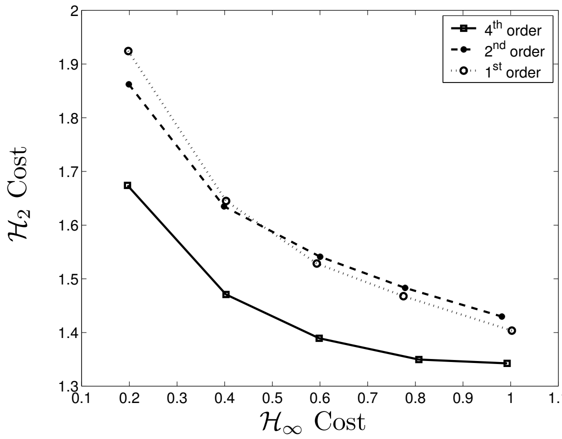

Fig. 1. Trade-off between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ costs

The problem is to design a robust dynamic output-feedback controller that achieves a trade-off between the $||T_2||_2$ and $||T_\infty||_\infty$ norms, with $||T_\infty||_\infty \leq \gamma$, and places the closed-loop eigenvalues $\lambda(\bar{A})$ into the intersection of the half-plane $Real(\lambda(\bar{A})) \leq -0.1$, and the conic sector centered at the origin $\angle\lambda(\bar{A}) \geq 2\pi/3$, for all possible values of the uncertain parameters k and f.

Tables 1, 2, and 3 present the results achieved with the proposed approach for the full order, 2^{nd} order, and 1^{st} order control synthesis with $\gamma = 0.2, 0.4, \ldots, 1.0$, the initial solution as $A_c = -I$, B_c and C_c as matrices with 1's, and $D_c = 0$, the initial ellipsoid defined by $Q_o = 10^4 I$, the optimization stop criteria $\epsilon = 0.01$ and $N_\epsilon = 10$, the accuracy specifications $\varepsilon_\delta = 0.01$ and $\epsilon_c = 0.01$. It is necessary only one iteration for all values of γ.

Table 1. Results achieved with the proposed approach in Example 1 for full order controller.

γ	0.2	0.4	0.6	0.8	1.0
H_∞ cost, γ_c	0.20	0.40	0.60	0.80	0.99
H_2 cost, δ_c	1.67	1.47	1.39	1.35	1.34
CPU time (min)	28.3	25.3	27.7	26.9	27.0

Table 2. Results achieved with the proposed approach in Example 1 for 2^{nd} order controller.

γ	0.2	0.4	0.6	0.8	1.0
H_∞ cost, γ_c	0.20	0.40	0.60	0.78	0.98
H_2 cost, δ_c	1.86	1.63	1.54	1.48	1.43
CPU time (s)	183.9	185.5	162.8	144.5	183.6

Table 3. Results achieved with the proposed approach in Example 1 for 1^{st} order controller.

γ	0.2	0.4	0.6	0.8	1.0
H_∞ cost, γ_c	0.20	0.40	0.59	0.78	1.00
H_2 cost, δ_c	1.92	1.65	1.53	1.47	1.40
CPU time (s)	62.6	50.1	51.6	55.4	165.4

Fig. 1 shows the trade-off between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ costs for controllers of different orders. As

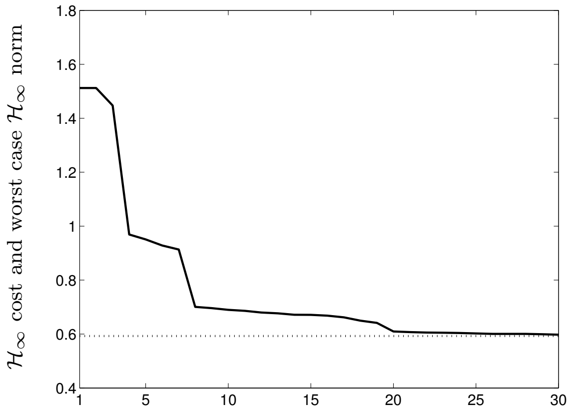

Fig. 2. The upper and lower bound functions when computing the $\mathcal{H}_\infty$ cost in example 1 with $\gamma = 0.6$

an example, the controller matrices achieved, with $\gamma = 0.6$, for full and reduced order are:

- Full order (4^{th} order):

$$A_c = \begin{bmatrix} -8.5350 & 7.8020 & 3.5284 & 0.1101 \\ -8.7362 & -6.8625 & 4.6327 & 0.3778 \\ -2.0578 & -2.9389 & -6.8692 & 2.5733 \\ 0.4084 & -2.1166 & 1.3411 & -0.5612 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1.9385 & -5.6352 \\ 13.4991 & 5.3192 \\ 7.1978 & 5.6589 \\ -0.3369 & 2.2821 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 15.8997 & 6.5053 & -1.2410 & -2.2183 \end{bmatrix}$$

$$D_c = \begin{bmatrix} -24.8033 & -8.1367 \end{bmatrix}$$

- Reduced order (2^{nd} order):

$$A_c = \begin{bmatrix} -17.3132 & 14.6594 \\ -5.3360 & -8.0353 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 12.0920 & -1.2458 \\ 11.7582 & 31.7465 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 28.9127 & -8.4016 \end{bmatrix}$$

$$D_c = \begin{bmatrix} -33.6685 - 38.2308 \end{bmatrix}$$

- Reduced order (1^{st} order):

$$A_c = \begin{bmatrix} -16.7875 \end{bmatrix}, \quad B_c = \begin{bmatrix} 24.0555 & 55.4678 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 30.2365 \end{bmatrix}, \quad D_c = \begin{bmatrix} -46.4142 & -99.1818 \end{bmatrix}$$

The Fig. 2 presents the upper and lower bound function evolutions, in terms of the iteration number, in the "exact" $\mathcal{H}_\infty$ cost computation for the presented full order controller. This figure shows the necessity of applying a branch-and-bound algorithm in the validation phase, since with the LMI-based guaranteed cost formulation alone, it would be erroneously considered that the designed controller did not attain the required specifications.

Example 2

Consider now the reliable control problem based on (Veillette *et al.*, 1992). The nominal system matrices are:

$$A = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -3 \\ -2 & -1 & 2 & -1 \end{bmatrix},$$

$$B_u = \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 0 & b_2 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_{z1} = C_{z2} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_{zu1} = D_{zu2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{zw1} = 0_{3\times 3}$$

$$C_y = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 \end{bmatrix}, \quad D_{yw} = \begin{bmatrix} 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{bmatrix}$$

with $b_1 = b_2 = c_1 = c_2 = 1$.

In this example it is considered the design of a centralized reliable control to deal with three different scenarios: the nominal plant ($c_1 = 1, c_2 = 1$), fail of the first sensor ($c_1 = 0, c_2 = 1$), and fail of the second sensor ($c_1 = 1, c_2 = 0$). The system is modelled by a polytope of matrices with three vertices in the 2-D space corresponding to these scenarios.

Fixing the constraint $\gamma = 3$, the proposed synthesis approach generates the following controller matrices in one external iteration:

$$A_c = \begin{bmatrix} -1.2752 & -0.0071 & -0.2247 & -0.5492 \\ -0.1787 & -1.0717 & -0.7339 & -1.7033 \\ -0.1780 & -0.0479 & -1.1440 & 0.5562 \\ 0.4806 & 1.7213 & 0.7167 & -1.9041 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0.4020 & 0.2442 \\ -0.0574 & 0.0847 \\ -1.1741 & -1.1741 \\ 1.3395 & 0.5370 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -0.1630 & 0.6472 & 0.5461 & -0.6639 \\ 0.7545 & 0.0260 & 0.5640 & 0.0405 \end{bmatrix}$$

$$D_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The $\mathcal{H}_2$ cost computed with the branch-and-bound strategy for the presented controller is 1.7312 and the $\mathcal{H}_\infty$ is 2.9912. In (Veillette *et al.*, 1992), aiming only the $\mathcal{H}_\infty$ cost optimization, the best synthesis achieves the worst case value of 3.38 for the $\mathcal{H}_\infty$ cost.

Example 3

Consider the same system of Example 2. Now, the objective is to design a decentralized reliable control to deal with three different scenarios: the nominal plant ($b_1 = 1, b_2 = 1$), outage of the first actuator ($b_1 = 0, b_2 = 1$), and outage of the second actuator ($b_1 = 1, b_2 = 0$). Again, the system is modelled by a polytope of matrices with three vertices in the 2-D space corresponding to these scenarios. The decentralized control consists of two transfer functions of second order: $G_{c1}(s) = U_1(s)/Y_1(s)$ and $G_{c2}(s) = U_2(s)/Y_2(s)$. The proposed synthesis approach generates the following controller matrices in three external iterations:

$$A_c = \begin{bmatrix} -1.9080 & 0.4299 & 0 & 0 \\ 0.4726 & -2.0023 & 0 & 0 \\ 0 & 0 & -3.7075 & 0.4929 \\ 0 & 0 & 0.1813 & -3.7962 \end{bmatrix}$$

$$B_c = \begin{bmatrix} -1.0083 & 0 \\ 1.4678 & 0 \\ 0 & 1.3632 \\ 0 & -1.8784 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1.1183 & -1.2692 & 0 & 0 \\ 0 & 0 & -1.8162 & 1.4413 \end{bmatrix}$$

$$D_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

that corresponds to the following transfer functions

$$G_{c1}(s) = \frac{-2.9905(s + 1.505)}{(s + 2.408)(s + 1.502)}$$

$$G_{c2}(s) = \frac{-5.1831(s + 3.357)}{(s + 4.054)(s + 3.45)}$$

The $\mathcal{H}_2$ cost computed with the branch-and-bound strategy for the presented controller is 3.91 and the $\mathcal{H}_\infty$ cost is 6.50. In (Veillette *et al.*, 1992), aiming only the $\mathcal{H}_\infty$ cost optimization, the best synthesis achieve the worst case value of 7.03 for the $\mathcal{H}_\infty$ cost. And as reported in (de Oliveira *et al.*, 2000), aiming only the $\mathcal{H}_2$ cost optimization for the nominal plant, the best synthesis achieves the $\mathcal{H}_2$ guaranteed cost of $\sqrt{169.92} \approx 13$.

5. CONCLUSIONS

The proposed strategy was shown to be a valid approach for the robust $\mathcal{H}_2$ / $\mathcal{H}_\infty$ dynamic output-feedback design with regional pole placement applied to systems with polytope uncertainty. The proposed approach can also handle structure-constrained control problems, e.g., decentralized or reduced-order control synthesis. Although the proposed synthesis step is based on a non-convex optimization, the analysis step is performed with convex algorithms that lead to exact results, valid in the whole uncertainty set. The exact worst-case points are found, to be dealt with by the next step of synthesis. The resulting controller is, therefore, robust, with low conservativity.

The problem formulation allows dealing with other kinds of objectives and constraints without substantial modification. It can also be directly applied to the discrete-time case. The proposed procedure can also be a good alternative to solve control problems when there is no LMI-based synthesis procedure, or when they do not provide acceptable controllers.

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