DIRECT ROBUST ADAPTIVE NONLINEAR CONTROL with DERIVATIVES ESTIMATION

Efimov D.V. [†], **Tyukin I.Yu.** [‡]

[†] *Control of Complex Systems Laboratory*
Institute of Problem of Mechanical Engineering,
Bolshoi av., 61, V.O. , St-Petersburg, 199178 Russia
efde@mail.rcom.ru

[‡] *Laboratory for Perceptual Dynamics,*
RIKEN (Institute for Physical and Chemical Research) Brain Science Institute, 2-1,
Hirosawa, Wako-shi, Saitama, 351-0198, Japan,
tyukinivan@brain.riken.jp

Abstract: In this paper we propose a novel adaptive control algorithm which allows to handle the issues of detectability, robustness and transient performance of adaptive systems. The controller structurally is similar to that of Panteley et al (2002). It is designed, however, on the grounds of rather different philosophy – estimation of the derivatives of plant state. In addition to systems which can be rendered to be asymptotically stable by means of the state feedback, plants satisfying assumption of partial asymptotic stability are considered.

Keywords: adaptive control, robustness, persistent excitation.

1. INTRODUCTION

The solution of adaptive control problem of nonlinear dynamical systems with linear parameterization has a rather long history (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999; Ioannou and Sun, 1996; Krstić, *et al.*, 1995; Marino and Tomei, 1995). Direct adaptive control school is a direction in this field, where adjustment of regulator parameters is applied without previous identification of unknown coefficients of plant model. The main obstacle, which is met a designer of adaptation algorithms in such way, is robust properties of synthesized system. Using classical methods to solve mentioned problem (Fomin, *et al.*, 1981; Ioannou and Sun, 1996; Krstić, *et al.*, 1995) it is possible to ensure desired stability properties of the system for the case of external disturbance absence. But if some additional uncertainty is appeared, which can be modeled as external unmeasured disturbances, then in general adaptive system loses its overall stability property. To compensate disturbances influence and to save stability property for perturbed system many approaches were used. Some of them (Fomin, *et al.*, 1981; Krstić, *et al.*, 1995) results to steady-state error rise for vanishing disturbances, others disturbances model are required (Fradkov, *et al.*, 1999). Hybrid adaptive systems like (Hespanha, *et al.*, 2002; Morse, 1995; Efimov, 2003) provide the most convenient solution of this task, but it is a hard to check transient behavior quality in such systems. In fact, quality of transient processes is another one basic obstacle for wide practical application of adaptation algorithms. For implementation in real world applications identification ability of adaptation algorithm and desired upper bound on all variables are needed. The last obstacle, that should be pointed out here, is requirement for design purposes information about plant Lyapunov function for case without uncertainties. Such Lyapunov function should possesses negative definite time derivative, but in common case it is a hard task to find such function candidate for the system. In many applications information available for so-called storage functions, which have negative semidefinite time derivative. Construction of adaptation algorithms based on storage functions is important task for mechanical systems, where a total energy function can be viewed as a storage function.

Derivative estimation of components of plant state space vector is a variant of posed problem solution. In papers (Bartolini, *et al.*, 1999; Levant, 1998; Panteley, *et al.*, 2002; Tyukin, 2003; Tyukin, *et al.*, 2003) were shown that information about time derivative of state vector or specially chosen output function can help us to increase quality of transient behavior of adaptive system. From another side, information about time derivative of state space vector is equivalent to direct measurements of right hand side of system equations. The last ability can be viewed as parametric or signal uncertainty estimation, that can help us to provide desired robust properties of adaptive system. In this paper a solution overlapping all three mentioned above obstacles is presented, which is based on time derivative estimation of auxiliary output function. It is shown that proposed adaptation algorithm ensures robust stability property for overall system for any bounded external disturbances, additionally, zero steady-state error is guarantied for vanishing disturbances. An improved with respect to (Fomin, *et al.*, 1981) upper estimates for system solutions are presented. Statements and definitions included in Section 2. In Section 3 properties of proposed solution are substantiated. Auxiliary results are placed in Appendix.

2. STATEMENT of the PROBLEM

Let us consider model of uncertain system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\left(\omega(\mathbf{x},t)\boldsymbol{\theta} + \mathbf{u}\right) + \mathbf{B}(\mathbf{x})\mathbf{d}, \quad (1)$$

where $\mathbf{x} \in R^n$ is state space vector; $\mathbf{u} \in R^m$ is control input; $\mathbf{d} \in R^p$ is external disturbance, it is supposed, that $\mathbf{d} : R_{\geq 0} \to R^p$ is Lebesgue measurable and essentially bounded

$$\|\mathbf{d}\| = \text{ess sup}\left\{|\mathbf{d}(t)|,\, t \geq 0\right\} < +\infty$$

function of time, the class of such function we will denote as $\mathcal{M}^p$; $\boldsymbol{\theta} \in R^q$ is constant vector of unknown parameters. Function $\mathbf{f}$ and columns of $\mathbf{G}$ and $\mathbf{B}$ are locally Lipschitz continuous, $\mathbf{f}(0) = 0$; ω is also locally Lipschitz continuous function and uniformly bounded with respect to $t \geq 0$ for any fixed values of $\mathbf{x} \in R^n$. System has for any $\mathbf{x}_0 \in R^n$ and $\mathbf{u} \in \mathcal{M}^m$, $\mathbf{d} \in \mathcal{M}^p$ well defined at the least locally solution $\mathbf{x}(\mathbf{x}_0, \mathbf{u}, \mathbf{d}, t)$, $t \in [0, T_{\max})$, if $T_{\max} = +\infty$, then system is called forward complete.

Assumption 1a. *There exists continuously differentiable function $V : R^n \to R_{\geq 0}$, such, that*

$$\alpha_1(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|),\ L_{\mathbf{f}}V(\mathbf{x}) \leq -\alpha(|\mathbf{x}|)$$

for some functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ for all $\mathbf{x} \in R^n$. □

Sign $|\cdot|$ denotes Euclidean norm of the vector. Expression $L_{\mathbf{f}}V(\mathbf{x})$ denotes scalar Lie derivative of function V with respect to vector field $\mathbf{f}$, i.e. $L_{\mathbf{f}}V(\mathbf{x}) = \partial V(\mathbf{x}) / \partial \mathbf{x}\, \mathbf{f}(\mathbf{x})$, term $L_{\mathbf{G}}V(\mathbf{x})$ further will be stated for covector

$$L_{\mathbf{G}}V(\mathbf{x}) = \partial V(\mathbf{x}) / \partial \mathbf{x}\, \mathbf{G}(\mathbf{x}).$$

It is said, that function $\rho : R_{\geq 0} \to R_{\geq 0}$ belongs to class $\mathcal{K}$, if it is strictly increasing and $\rho(0) = 0$; $\rho \in \mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and $\rho(s) \to \infty$ for $s \to \infty$.

Assumption 1b. *There exist continuously differentiable function $V : R^n \to R_{\geq 0}$ and continuous function $\boldsymbol{\psi} : R^n \to R^l$, such, that for all $\mathbf{x} \in R^n$*

$$\alpha_1(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|),\ L_{\mathbf{f}}V(\mathbf{x}) \leq -\varphi(|\boldsymbol{\psi}(\mathbf{x})|)$$

for some functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\varphi \in \mathcal{K}$. □

Both introduced assumptions fix stability properties of system (1) for vanishing disturbance $\mathbf{d}$ and control $\mathbf{u}$ under condition $\boldsymbol{\theta} = 0$, i.e. then control exactly compensates influence of parametric and signal uncertainties. In Assumption 1a the case of global asymptotic stability of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is considered, in Assumption 1b the case of a kind "total energy" function is discussed with negative semidefinite time derivative $\dot{V}$. To prove asymptotic convergence of vector $\mathbf{x}$ to zero a detectability notion should be introduced. System (1) is detectable with respect to continuous output function $\boldsymbol{\psi} : R^n \to R^l$, if condition $\boldsymbol{\psi}(t) \equiv 0$ for all $t \geq 0$ implies, that $\lim_{t \to +\infty} \mathbf{x}(t) = 0$. Thus, if system (1) is detectable with respect to output $\boldsymbol{\psi}$, then Assumption 1b is equivalent to Assumption 1a, both establish global asymptotic stability property for autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Let us introduce auxiliary output function $\mathbf{y} = \mathbf{h}(\mathbf{x})$ with continuously differentiable $\mathbf{h} : R^n \to R^q$, which importance will be explained later. The next assumption introduces technical growth condition for function α.

Assumption 2. *There exist functions $\lambda \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and constant $X_\gamma \geq 0$ such, that*

$$\begin{aligned} &L_{\mathbf{B}}V(\mathbf{x})\mathbf{d} + |L_{\mathbf{B}}\mathbf{h}(\mathbf{x})|^2 |\mathbf{d}|^2 - \alpha(|\mathbf{x}|) \leq \\ &\leq -\lambda(|\mathbf{x}|) + \sigma(|\mathbf{d}|) \end{aligned}, \quad |\mathbf{x}| \geq X_\gamma,$$

where $L_{\mathbf{B}}\mathbf{h}(\mathbf{x}) = \partial \mathbf{h}(\mathbf{x}) / \partial \mathbf{x}\, \mathbf{B}(\mathbf{x})$. □

This assumption will be used only to base robust properties of overall adaptive system. And that is more, Assumption 2 always can be satisfied by appropriate choice of intermediate control law.

Before we proceed, persistent excitation (PE) and positive in average (PA) properties should be mentioned. Definitions of these properties are placed in Appendix (Definitions A1 and A2). In fact, PE condition is widely used in theory of adaptive control (Fomin, *et al.*, 1981; Loria, *et al.*, 2002) to base identification ability property of adaptation algorithm. Positive in average property was introduced in (Efimov and Fradkov, 2003) and in Lemma A2 equivalence conditions between PE and PA properties are established. But according to Definitions A1 and A2, PA property can be more simpler verified. In our work PA property also will be used to prove robust stability of proposed adaptive system taking in account results of Lemma A1.

Control goals, which should be reached by appropriate design of control $\mathbf{u}(\mathbf{x})$ (it is supposed, that vec-

tor $\mathbf{x}$ is available for measurements) and adaptation algorithm, that helps to handle parametric uncertainty problem, can be formulated as follows:
1. Global Lyapunov stability of the system solution and asymptotic convergence of $\mathbf{x}(t)$ to zero for vanishing disturbance $\mathbf{d}(t)\equiv 0$, $t\geq 0$;
2. Global boundedness of solutions for any $\mathbf{d}\in\mathcal{M}^p$.

3. ROBUST ADAPTIVE CONTROL

During this work we will use the following conventional "certainty equivalence" control law

$$\mathbf{u}=-\omega(\mathbf{x},t)\hat{\boldsymbol{\theta}}, \quad (2)$$

where $\hat{\boldsymbol{\theta}}\in R^q$ is an estimate of unknown parameters vector $\boldsymbol{\theta}$. When matching condition $\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}(t)$, $t\geq 0$ is satisfied, system (1), (2) with Assumptions 1 and 2 becomes robustly stable and all control goals are reached. Thus, further we should design an algorithm for $\hat{\boldsymbol{\theta}}$ adjusting, which help us to ensure this matching condition satisfying.

Here Speed Gradient (SG) approach (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999) will be utilized to synthesize adaptation algorithm for (2). In SG approach adaptation algorithm has the following general form:

$$\dot{\hat{\boldsymbol{\theta}}}=-\gamma\nabla_{\hat{\boldsymbol{\theta}}}\dot{Q}\left(\mathbf{x},\hat{\boldsymbol{\theta}},t\right),$$

where $\gamma>0$ is a design parameter; $Q:R^n\times R\to R_{\geq 0}$ is continuously differentiable auxiliary goal functional, which time derivative with respect to system (1), (2) explicitly depends on vector $\hat{\boldsymbol{\theta}}$. In our work we will use goal functional

$$Q(\mathbf{x},t)=V(\mathbf{x})+0.5\int_0^t\left|\dot{\mathbf{y}}(\tau)-L_{\mathbf{f}}\mathbf{h}\left(\mathbf{x}(\tau)\right)\right|^2 d\tau .$$

It is casual due to time derivative is included under sign of integral. Its time derivative is as follows:

$$\dot{Q}(t)=\dot{V}(t)+0.5\left|\dot{\mathbf{y}}(t)-L_{\mathbf{f}}\mathbf{h}\left(\mathbf{x}(t)\right)\right|^2 .$$

Therefore, adaptation algorithm takes form:

$$\begin{aligned}\dot{\tilde{\boldsymbol{\theta}}}=&-\gamma\,\omega(\mathbf{x},t)^T L_{\mathbf{G}}V(\mathbf{x})^T-\\&-\gamma\,\omega(\mathbf{x},t)^T L_{\mathbf{G}}\mathbf{h}(\mathbf{x})^T\left(\dot{\mathbf{y}}(t)-L_{\mathbf{f}}\mathbf{h}(\mathbf{x})\right),\end{aligned} \quad (3)$$

where $\tilde{\boldsymbol{\theta}}=\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}$ is parameter estimation error. Algorithm (3) explicitly depends on time derivative of function $\mathbf{y}$, which is not measured due to statement of the task. So an estimator of $\dot{\mathbf{y}}$ should be developed, this problem will be considered later in the section. Here let us suppose, that an estimation signal $\mathbf{w}(t)$ of $\dot{\mathbf{y}}(t)$ is available with some error $\mathbf{e}(t)$:

$$\mathbf{w}(t)=\dot{\mathbf{y}}(t)+\mathbf{e}(t),\ t\geq 0 .$$

In this case algorithm (3) should be rewritten in realizable form:

$$\begin{aligned}\dot{\tilde{\boldsymbol{\theta}}}=&-\gamma\,\omega(\mathbf{x},t)^T L_{\mathbf{G}}V(\mathbf{x})^T-\\&-\gamma\,\omega(\mathbf{x},t)^T L_{\mathbf{G}}\mathbf{h}(\mathbf{x})^T\left(\mathbf{w}(t)-L_{\mathbf{f}}\mathbf{h}(\mathbf{x})\right).\end{aligned} \quad (4)$$

To emphasize main advance of algorithm (4) let us incorporate into consideration the following equality

$$\dot{\mathbf{y}}-L_{\mathbf{f}}\mathbf{h}(\mathbf{x})=L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\omega(\mathbf{x},t)\tilde{\boldsymbol{\theta}}+L_{\mathbf{B}}\mathbf{h}(\mathbf{x})\mathbf{d},$$

substituting it in (4) we receive:

$$\begin{aligned}\dot{\tilde{\boldsymbol{\theta}}}=&-\gamma\,\omega(\mathbf{x},t)^T\begin{bmatrix}L_{\mathbf{G}}V(\mathbf{x})^T+\\+L_{\mathbf{G}}\mathbf{h}(\mathbf{x})^T\left(L_{\mathbf{B}}\mathbf{h}(\mathbf{x})\mathbf{d}+\mathbf{e}(t)\right)\end{bmatrix}-\\&-\gamma\,\omega(\mathbf{x},t)^T L_{\mathbf{G}}\mathbf{h}(\mathbf{x})^T L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\omega(\mathbf{x},t)\tilde{\boldsymbol{\theta}} .\end{aligned} \quad (5)$$

Thus, according to (5) algorithm (4) has a negative parametric feedback with matrix functional gain $\gamma\Omega(t)^T\Omega(t)$, where $\Omega(t)=L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\omega(\mathbf{x},t)$. If the smallest eigenvalue of $\Omega^T\Omega$ possesses PA property (for example, this is the case if system has well defined relative degree $\{1,...,1\}$ from input $\tilde{\boldsymbol{\theta}}$ to output $\mathbf{y}$), then as it follows from Lemma A1, parameter error $\tilde{\boldsymbol{\theta}}$ stays bounded for any essentially bounded variables $\mathbf{x}$, $\mathbf{d}$ and $\mathbf{e}$ (differential equation (5) repeats form of equation (A1)). Therefore, the first obstacle mentioned in introduction is handled. Further, this negative feedback with gain Ω allows to increase identification ability of adaptation algorithm (4), that help us to handle the second obstacle mentioned in introduction and improve the quality of transient processes in the system.

From other point of view, if we would compare algorithm (5) for $q=1$ and $y(\mathbf{x})=V(\mathbf{x})$ with classical one (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999) obtained with SG approach for auxiliary goal functional $\breve{Q}(\mathbf{x},t)=V(\mathbf{x})$, that is

$$\dot{\tilde{\boldsymbol{\theta}}}=-\breve{\gamma}\,\omega(\mathbf{x},t)^T L_{\mathbf{G}}V(\mathbf{x})^T,\ \breve{\gamma}>0, \quad (6)$$

then it is possible to conclude, that algorithm (5) without negative parametric feedback coincides with (6) under substitution

$$\breve{\gamma}=\gamma\left(1+e(t)+L_{\mathbf{B}}V(\mathbf{x})\mathbf{d}\right).$$

The advances of algorithm (5) were mentioned before, but here we recover that, comparing with classical algorithm (6), proposed solution has sign varying design parameter $\breve{\gamma}$.

Before we proceed let us introduce the main ideas of signal $\mathbf{w}$ construction. As it possible to conclude that presence of time derivative in adaptation algorithm equations allows to introduce additional (with respect to classical solution) negative parametric feedback. The feedback gain Ω improves quality of transient processes and adds robust stability properties. Exact form and kind of dependence on system state vector of function Ω is not important, this function Ω should only satisfy PA property. Therefore it is possible to generalize equation of adaptation algorithm (4) or (5) as follows:

$$\dot{\tilde{\boldsymbol{\theta}}}=-\gamma\omega(\mathbf{x},t)^T L_{\mathbf{G}}V(\mathbf{x})^T-\gamma\Omega(t)^T\left(\Omega(t)\tilde{\boldsymbol{\theta}}-\mathbf{e}(t)\right), \quad (7)$$

where $\Omega:R_{\geq 0}\to R^{q\times q}$ is some regressor continuous matrix function and $\mathbf{e}:R_{\geq 0}\to R^q$ is a "derivative" estimation error function. To apply results of Lemma A1 to system (7) this error should satisfy the following series of properties

$$\mathbf{d}(t)\equiv 0,\ t\geq 0\ \Rightarrow\ \int_t^{+\infty}\left|\mathbf{e}(\tau)\right|^2 d\tau<+\infty,\ t\geq 0; \quad (8)$$

$$\mathbf{d}\in\mathcal{M}^p\ \Rightarrow\ \mathbf{e}\in\mathcal{M}^q . \quad (9)$$

Therefore, the last properties define requirements which should be possessed by derivative estimator:

$$\dot{\mathbf{z}} = -r\left(\mathbf{h}(\mathbf{x}) + \mathbf{z}\right) - L_{\mathbf{f}}\mathbf{h}(\mathbf{x}) + L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\omega(\mathbf{x},t)\hat{\boldsymbol{\theta}}\,; \quad (10)$$

$$\dot{\Omega} = -r\Omega - L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\omega(\mathbf{x},t)\,; \quad (11)$$

$$\mathbf{e} = \Omega^T\boldsymbol{\theta} + \mathbf{h}(\mathbf{x}) + \mathbf{z}\,, \quad (12)$$

where $r > 1$ is a constant, $\mathbf{z} \in R^q$ is an auxiliary variable. The closely connected regulator was proposed in (Panteley, *et al.*, 2002) for case $\mathbf{h}(\mathbf{x}) = \mathbf{x}$. Let us calculate dynamics of error $\mathbf{e}$ basing on overall system equations (1), (2), (7) , (10) – (12):

$$\dot{\mathbf{e}} = \dot{\Omega}\boldsymbol{\theta} + \dot{\mathbf{y}} + \dot{\mathbf{z}} = -r\,\mathbf{e} + L_{\mathbf{B}}\mathbf{h}(\mathbf{x})\mathbf{d}\,. \quad (13)$$

Therefore, property (8) is satisfied for such estimator, property (9) also can be deduced if boundedness of state space vector can be substantiated for $\mathbf{d} \in \mathcal{M}^p$. Denote $a(t) = \min_t \{\lambda_1(t),...,\lambda_q(t),\lambda_{\max}\}$, where $\lambda_i(t)$, $i = \overline{1,q}$ are eigenvalues of matrix $\Omega(t)^T\,\Omega(t)$, $\lambda_{\max} \geq \max\{\lambda_1(0),...,\lambda_q(0)\} > 0$, it is clear that in this case signal $a(t)$ is always bounded and admits inequality:

$$\Omega(t)^T\,\Omega(t) \geq a(t)\mathbf{I}_q\,,$$

where $\mathbf{I}_q$ is identity matrix of dimension $(q \times q)$. And now let us consider separately cases of disturbance $\mathbf{d}$ absence and presence.

T h e o r e m 1 . *Let* $\mathbf{d}(t) \equiv 0$, $t \geq 0$ *and Assumptions 1a hold. Then system (1), (2), (7), (10) – (12) is globally stable with respect to* $\mathbf{X} = \left(\mathbf{x},\hat{\boldsymbol{\theta}},\mathbf{z},\Omega,\mathbf{e}\right)$, $\lim_{t\to+\infty} \mathbf{x}(t) = 0$ *and the following estimate holds*:

$$R\left(\mathbf{x}(t),\tilde{\boldsymbol{\theta}}(t)\right) \leq R\left(\mathbf{x}(0),\tilde{\boldsymbol{\theta}}(0)\right) - \int_0^t \left[\alpha\left(|\mathbf{x}(\tau)|\right) + 0.5\,a(\tau)\left|\tilde{\boldsymbol{\theta}}(\tau)\right|^2 - \left|\mathbf{e}(\tau)\right|^2\right] d\tau\,, \quad (14)$$

where $R(\mathbf{a},\mathbf{b}) = V(\mathbf{a}) + 0.5\gamma^{-1}|\mathbf{b}|^2$. *If also signal* $a(t)$ *is* (μ,Δ)-PA *for some* $\mu > 0$, $\Delta > 0$, *then* $\lim_{t\to+\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$.

P r o o f . Let us analyze Lyapunov-like function:

$$U(\mathbf{x},\hat{\boldsymbol{\theta}},t) = R(\mathbf{x},\tilde{\boldsymbol{\theta}}) + 0.5\int_t^{+\infty} \left|\mathbf{e}(\tau)\right|^2 d\tau\,. \quad (15)$$

According to differential equation (13) integral of $\mathbf{e}$ is bounded in this case and it can be used as a part of Lyapunov-like function. Additionally, the presence of this integral component is the reason why function U is called Lyapunov type. Time derivative of U with respect to system equations admits inequality:

$$\dot{U} \leq -\alpha\left(|\mathbf{x}|\right) - 0.5\,a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2.$$

From condition $\dot{U} \leq 0$ stability property for $(\mathbf{x},\hat{\boldsymbol{\theta}})$ follows. Further $(\mathbf{z},\Omega)$ are bounded due to they are solutions of asymptotically stable linear systems with bounded inputs. Integrating of above inequality on time interval $[0,t)$ yields

$$V\left(\mathbf{x}(t)\right) + 0.5\gamma^{-1}\left|\tilde{\boldsymbol{\theta}}(t)\right|^2 + 0.5\int_t^{+\infty}\left|\mathbf{e}(\tau)\right|^2 d\tau - V\left(\mathbf{x}(0)\right) - 0.5\gamma^{-1}\left|\tilde{\boldsymbol{\theta}}(0)\right|^2 - 0.5\int_0^{+\infty}\left|\mathbf{e}(\tau)\right|^2 d\tau \leq$$

$$\leq -\int_0^t \left[\alpha\left(|\mathbf{x}(\tau)|\right) + 0.5\,a(\tau)\left|\tilde{\boldsymbol{\theta}}(\tau)\right|^2\right] d\tau.$$

From which inequality (14) can be obtained. From LaSalle invariance principle also follows, that $\mathbf{x}(t)$ asymptotically converges to zero. If $a(t)$ is (μ,Δ)-PA, then applying result of Lemma A1 to system (7) it is possible to prove desired property $\hat{\boldsymbol{\theta}}(t) \to \boldsymbol{\theta}$. ■

R e m a r k 1 . According to inequality (14) system has stability property not uniformly with respect to initial conditions $\mathbf{X}(0)$. Indeed, presence in right hand side of (14) or (15) a time-variant term

$$E(t) = \int_0^t \left|\mathbf{e}(\tau)\right|^2 d\tau < +\infty$$

introduces additional dependence of trajectories upper bound on properties of derivative estimation error $\mathbf{e}$. Form another side, comparing this inequality with the same derived for conventional algorithm (6) (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999):

$$R\left(\mathbf{x}(t),\tilde{\boldsymbol{\theta}}(t)\right) \leq R\left(\mathbf{x}(0),\tilde{\boldsymbol{\theta}}(0)\right) - \int_0^t \alpha\left(|\mathbf{x}(\tau)|\right) d\tau\,,$$

it is possible to conclude, that upper estimate (14) has additional negative term

$$-0.5\int_0^t a(\tau)\left|\tilde{\boldsymbol{\theta}}(\tau)\right|^2 d\tau\,,$$

which seriously improves quality of transient processes in the system, especially for small amplitudes of signal $E(t)$ (Tyukin, 2003). □

T h e o r e m 2 . *Let* $\mathbf{d}(t) \equiv 0$, $t \geq 0$ *and Assumptions 1b hold. Then system (1), (2), (7), (10) – (12) is globally stable with respect to* $\mathbf{X}$ *and globally asymptotically stable with respect to output* $\boldsymbol{\psi}$:

1. *If signal* $a(t)$ *is* (μ,Δ)-PA *for some* $\mu > 0$, $\Delta > 0$ *and system (1) is detectable with respect to output* $\boldsymbol{\psi}$, *then* $\lim_{t\to+\infty} \mathbf{x}(t) = 0$ *and* $\lim_{t\to+\infty} \hat{\boldsymbol{\theta}}(t) = \boldsymbol{\theta}$;
2. *If* $q = 1$, $y(\mathbf{x}) = V(\mathbf{x})$ *and signal* $a(t)$ *is not* (μ,Δ)-PA *for any* $\mu > 0$, $\Delta > 0$ *and system (1) is detectable with respect to output* $(\boldsymbol{\psi}, L_{\mathbf{G}}V(\mathbf{x})\omega(\mathbf{x},t))$, *then* $\lim_{t\to+\infty} \mathbf{x}(t) = 0$.

Also the following estimate holds:

$$R\left(\mathbf{x}(t),\tilde{\boldsymbol{\theta}}(t)\right) \leq R\left(\mathbf{x}(0),\tilde{\boldsymbol{\theta}}(0)\right) - \int_0^t \left[\varphi\left(|\boldsymbol{\psi}(\tau)|\right) + 0.5\,a(\tau)\left|\tilde{\boldsymbol{\theta}}(\tau)\right|^2 - \left|\mathbf{e}(\tau)\right|^2\right] d\tau \,. \quad (16)$$

P r o o f . Let us consider time derivative of function (15) in this case:

$$\dot{U} \leq -\varphi\left(|\boldsymbol{\psi}(\mathbf{x})|\right) - 0.5\,a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2.$$

As before $\dot{U} \leq 0$ and system is stable (not uniformly with respect to initial conditions). Integrating of the inequality again yields estimate (16). Due to system solution $\mathbf{X}$ is bounded, then it has nonempty compact invariant set of ω-limit points, which according to LaSalle invariance principle is contained in

$$\mathcal{Q}=\left\{\mathbf{X}:\boldsymbol{\psi}(\mathbf{x})=0\cap\sqrt{a(t)}\left|\tilde{\boldsymbol{\theta}}\right|=0\right\},\ t\geq 0.$$

Thus, system is globally asymptotically stable with respect to output $\boldsymbol{\psi}$ (Rumyantsev and Oziraner, 1987). Let us consider the first part of conditions imposed in the Theorem. From detectability property of system (1) with respect to output $\boldsymbol{\psi}$ it is possible to conclude, that on trajectories in set $\mathcal{Q}$ property $\lim_{t\to+\infty}\mathbf{x}(t)=0$ holds. If $a(t)$ is (μ,Δ)-PA, then applying result of Lemma A1 to system (7) it is possible to prove desired convergence property $\lim_{t\to+\infty}\hat{\boldsymbol{\theta}}(t)=\boldsymbol{\theta}$. According to second part of conditions of the Theorem, for any $\mu>0$ and $\Delta>0$ there exists a $T>0$ such, that inequality

$$\int_T^{T+\Delta}a(\tau)\,d\tau\leq\mu\,\Delta$$

holds, in other words it means, that there exists a $T>0$, such, that average value of function $a(t)$ is less than any $\mu>0$. So, taking in mind, that function $a(t)$ takes only nonnegative values and bounded, there exists a $T>0$, such, that $a(t)\leq\mu$ for all $t\geq T$ and any $\mu>0$. Therefore function $a(t)$ vanishes into set of ω-limit points, but for $q=1$ function $a(t)=\left|\Omega(t)\right|^2$ and convergence to zero of signal $a(t)$ is equivalent to asymptotic convergence to zero of vector Ω. Taking in mind (11) it is possible to conclude, that input $L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\omega(\mathbf{x},t)$ of this subsystem also converges to zero in set of ω-limit points of the system. But for $y(\mathbf{x})=V(\mathbf{x})$ it ensures, that set of ω-limit points is located into set

$$\breve{\mathcal{Q}}=\left\{\mathbf{x},\hat{\boldsymbol{\theta}}:\boldsymbol{\psi}(\mathbf{x})=0\cap\left|L_{\mathbf{G}}V(\mathbf{x})\omega(\mathbf{x},t)\right|=0\right\},$$

where system is detectable and relation $\lim_{t\to+\infty}\mathbf{x}(t)=0$ holds. ■

R e m a r k 2 . As it was mentioned, PA property is satisfied for $a(t)$ if system has well defined relative degree $\{1,...,1\}$ from output $\mathbf{y}$ to input $\tilde{\boldsymbol{\theta}}$. And that is more, this property holds if smallest eigenvalue of matrix $\Omega(t)^T\,\Omega(t)$ is bigger than zero in average. It is worth to indicate, that the case of PA property of signal $a(t)$ is not interesting from the practical point of view, due to in general it is too hard to establish conditions of PA property realization in a system with vanishing state $\mathbf{x}$ and disturbances $\mathbf{d}$. More suitable for practical implementation is the second part of conditions proposed in Theorem 2. In this case detectability assumption should be satisfied for extended output $(\boldsymbol{\psi},L_{\mathbf{G}}V(\mathbf{x})\omega(\mathbf{x},t))$, that simplifies stability investigation. □

But requirement of PA property for signal $a(t)$ becomes rather mild if we assume presence of non vanishing disturbance $\mathbf{d}$ (in this case state $\mathbf{x}$ also loses its convergence to zero property generally). Before we proceed let us introduce several auxiliary functions from class $\mathcal{K}$:

$$\delta(s)=\max_{|\mathbf{x}|\leq s}\left|L_{\mathbf{B}}V(\mathbf{x})\right|,\ \nu(s)=\max_{|\mathbf{x}|\leq s}\left|L_{\mathbf{B}}\mathbf{h}(\mathbf{x})\right|^2,$$

$$\tilde{\lambda}(s)=\min\{\lambda(s),\alpha(s)\},$$

$$\tilde{\sigma}(s)=\max\left\{\sigma(s),\delta(X_\gamma)s+\nu(X_\gamma)s^2\right\}.$$

Such functions δ and ν exist due to continuity property of functions to be majorized.

T h e o r e m 3 . *Let* $\mathbf{d}\in\mathcal{M}^p$; $\tilde{\lambda}(|\mathbf{x}|)\geq\chi V(\mathbf{x})$ $\chi>0$; *signal* $a(t)$ *is* (μ,Δ)-PA *for some* $\mu>0$, $\Delta>0$ *and Assumptions 1a and 2 hold. Then solution of system (1), (2), (7), (10) – (12) is globally bounded.*

P r o o f . Let us analyze Lyapunov function:

$$U(\mathbf{x},\hat{\boldsymbol{\theta}},t)=R(\mathbf{x},\tilde{\boldsymbol{\theta}})+0.5\mathbf{e}^T\mathbf{e}.$$

Time derivative of U with respect to system equations (1), (2), (7), (10) – (13) takes form:

$$\dot{U}\leq-\alpha(|\mathbf{x}|)-0.5a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2-(r-1)|\mathbf{e}|^2+$$
$$+L_{\mathbf{B}}V(\mathbf{x})\mathbf{d}+\left|L_{\mathbf{B}}\mathbf{h}(\mathbf{x})\right|^2|\mathbf{d}|^2.$$

Taking in mind Assumption 2, let us consider separately two cases $|\mathbf{x}|\geq X_\gamma$ and $|\mathbf{x}|<X_\gamma$:

$$|\mathbf{x}|\geq X_\gamma\ \Rightarrow\ \begin{aligned}&\dot{U}\leq-\lambda(|\mathbf{x}|)-0.5a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2-\\&-(r-1)|\mathbf{e}|^2+\sigma(|\mathbf{d}|);\end{aligned}$$

$$|\mathbf{x}|<X_\gamma\ \Rightarrow\ \begin{aligned}&\dot{U}\leq-\alpha(|\mathbf{x}|)-0.5a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2-\\&-(r-1)|\mathbf{e}|^2+\delta(X_\gamma)|\mathbf{d}|+\nu(X_\gamma)|\mathbf{d}|^2.\end{aligned}$$

Thus, the following inequality holds independently on vector $\mathbf{x}$ amplitude:

$$\begin{aligned}\dot{U}&\leq-\tilde{\lambda}(|\mathbf{x}|)-0.5a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2-(r-1)|\mathbf{e}|^2+\tilde{\sigma}(|\mathbf{d}|)\leq\\&\leq-\chi V(\mathbf{x})-0.5a(t)\left|\tilde{\boldsymbol{\theta}}\right|^2-(r-1)|\mathbf{e}|^2+\tilde{\sigma}(|\mathbf{d}|).\end{aligned}\tag{17}$$

Due to signal $a(t)$ is bounded it is possible to apply Lemma A3 to inequality (17) and obtain estimate:

$$\dot{U}\leq-k(t)U+\tilde{\sigma}(|\mathbf{d}|),$$

where $k(t)=\min\left\{1,\min(\chi,2r-2)\gamma^{-1}\|a\|^{-1}\right\}\gamma a(t)$. From which basing on result of Lemma A1 boundedness of variables $\mathbf{x}$, $\tilde{\boldsymbol{\theta}}$ and $\mathbf{e}$ follows. From (11) variable Ω is also bounded, variable $\mathbf{z}$ is an additive item of $\mathbf{e}$ and all other items are bounded. ■

4. CONCLUSION

In this paper an extension of proposed in (Panteley, *et al.*, 2002) adaptive controller is presented. The robust properties are substantiated and estimates on transient processes quality are given. The dimension of proposed adaptive controller is q^2+2q, that is smaller than $nq+n+q$ in (Panteley, *et al.*, 2002).

5. ACKNOWLEDGEMENT

This work is partly supported by Russian Science Support Foundation and by Program of Presidium of Russian Academy of Science № 19.

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APPENDIX

D e f i n i t i o n A 1 (Fomin, *et al.*, 1981). *It is said, that essentially bounded matrix function* $\mathbf{R}(t)$, $t \ge 0$ *with dimension* $l_1 \times l_2$ *admits* (L, ϑ)*–persistent excitation (*PE*) condition, if there exist strictly positive constants* L *and* ϑ *such, that for any* $t \ge 0$

$$\int_t^{t+L} \mathbf{R}(s)\mathbf{R}(s)^T \, ds \ge \vartheta \mathbf{I}_{l_1},$$

where $\mathbf{I}_{l_1}$ *is identity matrix of dimension* $l_1 \times l_1$. □

The following property was proposed in (Efimov and Fradkov, 2003).

D e f i n i t i o n A 2 . *Function* $a : R_{\ge 0} \to R$ *is called* (μ, Δ)*–positive in average (PA), if for any* $t \ge 0$ *and any* $\delta \ge \Delta$, $\mu > 0$,

$$\int_t^{t+\delta} a(\tau) d\tau \ge \mu\delta .$$ □

Importance of PA property is explained in the following lemma. Proofs are omitted due to space limitation.

L e m m a A 1 . *Let us consider time-varying linear dynamical system*

$$\dot{p} = -a(t)p + b(t), \; t_0 \ge 0, \tag{A1}$$

where $p \in R$ *and functions* $a : R_+ \to R$, $b : R_+ \to R$ *are Lebesgue measurable,* b *is essentially bounded, function* a *is* (μ, Δ)*–PA for some* $\mu > 0$, $\Delta > 0$ *and essentially bounded from below, i.e. there exists* $A \in R_+$*, such, that:*

$$ess\inf\{a(t), \; t \ge t_0\} \ge -A .$$

Then solution of system (A1) is defined for all $t \ge t_0$ *and it admits estimate*

$$|p(t)| \le \begin{cases} |p(t_0)| e^{A\Delta} e^{-\mu(t-t_0)+(A+\mu)\Delta} + \\ + \|b\| \max\{A^{-1}e^{At_0}, \mu^{-1}e^{-\mu t_0}\}, \; A \ne 0; \\ |p(t_0)| e^{-\mu(t-t_0)+\mu\Delta} + \\ + \|b\| \max\{\Delta, \mu^{-1}e^{-\mu t_0}\}, \qquad A = 0. \end{cases}$$ ■

L e m m a A 2 . *Suppose, that function* $a(t)$, $t \ge 0$ *admits* (L, ϑ)*–PE condition, then function* $a^2(t)$ *is* $(0.5\vartheta/L, L)$*– PA. Conversely, if function* $a^2(t)$, $t \ge 0$ *is* (ϑ, L)*–PA, then function* $a(t)$ *also possesses* $(L, L\vartheta)$*–PE condition too.* ■

L e m m a A 3 . *Let* $V_1 : R_{\ge 0} \to R_{\ge 0}$, $V_2 : R_{\ge 0} \to R_{\ge 0}$ *and* $k_1 : R_{\ge 0} \to R_{\ge 0}$ *be Lebesgue measurable and essentially bounded functions,* $k_2 \in R_{\ge 0}$; k_1 *is* (μ, Δ)*-PA. Then inequality*

$$k_1(t)V_1(t) + k_2 V_2(t) \ge k_3(t)\big(V_1(t) + V_2(t)\big), \; t \ge 0$$

holds for $(\mu\mu^*, \Delta)$ *-PA function*

$$k_3(t) = \mu^* k_1(t), \; \mu^* = \min\left\{1, k_2 \|k_1\|^{-1}\right\}.$$ ■