FAULT-TOLERANT TIME-INVARIANT FEEDBACK CONTROL

Antonio Sala, Pedro Albertos

Department of Systems Engineering and Control
Universidad Politécnica de Valencia; Camino de Vera, 14;
46022 Valencia, Spain. Email:{asala,pedro}@isa.upv.es

Abstract: This paper discusses the problem of guaranteeing both a nominal performance and a degraded second-class control performance under actuators and/or sensors faults. LMI, BMI and other possible solutions are investigated in a time invariant feedback control framework. Also, the switching-on and shutting-down sequence of sensors and actuators in centralised control strategies may be critical to ensure continuous stability of the controlled plant. The advantage of measuring the actual input to the plant is also discussed. A bank angle control example illustrates the procedure. *Copyright* ©2005 *IFAC*

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1. INTRODUCTION

Multivariable systems allow for a variety of solutions in designing the control system. Usually, there is an interaction among the control actions leading to a redundancy in the control as well as in the information gathered from the process and, for simple requirements, different solutions are possible. This is clearly the case in pole assignment design for both state feedback control and state observer design: in a general case, the freedom to determine the control law or the observer gain can be used to get additional performances in the controlled plant and additional constraints or requirements, mainly oriented to the internal structure of the controlled plant or the robustness of the control.

In industrial control systems, the effect of the failure of one element in the control loop (sensor or actuator) should be analysed. Global *integrity* refers to the property of keeping acceptable plant behaviour under the failure of one of these components. The concept is becoming more and more relevant (Campo and Morari, 1994).

The general layout of a feedback controlled plant is depicted in fig. 1, where G represents the plant operator, S_i the i-sensor operator connecting the process variable y_i and the measurement m_i, C the control algorithm including the control law as well as measurements filtering, and H_j is the j-actuator operator converting the controller output v_j into the control variable u_j. For the sake of simplicity, the sensor and actuator operators may be considered as unitary, their properties being included in the plant operator G.

The integrity of the controlled plant w.r.t. the i-sensor requires the system stability under its failure, that is, if the attached connecting line is opened. In other words, the operator from m_i to y_i should be stable and the degrading of the controlled plant performances should be bounded. The integrity requirement can

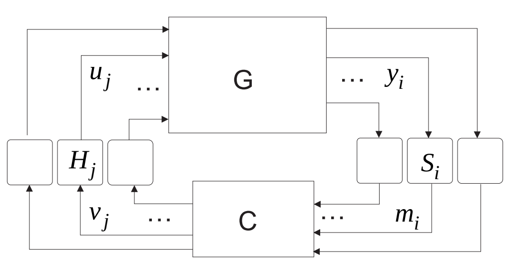

Fig. 1. Feedback control layout.

be extended to the failure of any subset of sensors or actuators. Necessary conditions for integrity under integral control can be obtained based on relative gain array considerations (Campo and Morari, 1994). An alternative approach to integrity-minded control design appears in (Zhao and Jiang, 1998), and the problem is closely related to the simultaneous stabilisation one (Blondel, 1994). Fault-tolerant control designs may include options for controller reconfiguration (Jiang, 1994), however, the implementation of fault detection and isolation algorithms and control reconfiguration in a short reaction time to avoid transient instability issues is not a trivial task. Some of the issues involved in controller commutation can be handled by hybrid control design techniques (Koutsoukos *et al.*, 2000; Hespanha *et al.*, 2003).

The option of an "autonomous feedback reconfiguration" has an advantage over supervision-based approaches because there is no need for a fault detection mechanism, plus switching, plus a sort of "probing" procedure to check if the defective actuator has been recovered from a transient fault. On the other hand, better performance can be reached by the nonlinear supervision approach as the controller can be fully redesigned for the faulty case.

This paper deals with the design of time invariant feedback control systems with integrity properties. That is, the set of controller requirements will include the desired performances under different operating modes but the controller will be fixed. The design with integrity is stated as a problem of multi-mode control design. Indeed, achieving control performance on a family of plants is the basic concept of robust control methodologies. In that area, the problem is a long standing one with lots of solutions proposed in literature. This paper discusses the practical insights that can be extracted from the requirement of *different* performance levels for different operation modes and the easy adaptation of well-established techniques to the problem under consideration. In some cases, the solution can be interpreted as involving an implicit reconfiguration in faulty conditions.

In the next section, the problem of state-feedback design with integrity regarding actuator faults is developed, with specifications set up as pole-region placement. An approach including the operating conditions, the faulty ones and the transient sequencing in the switching-on/shutting-down of the control, is developed by solving a set of Linear Matrix Inequalities (LMI) and, in some cases, bilinear ones (BMI). The control of the bank-angle of the well-known model of a jet aircraft, (Zhao and Jiang, 1998), illustrates the approach and some alternatives strategies are compared.

The design of state estimators with dynamic requirements expressed by the observer pole location, where sensor faults are foreseen, is a dual problem to be considered in Section 4. The separation principle can be claimed if the actual inputs are measured. Indeed, additional improvements in the global design can be achieved if these inputs are measured, as pointed out in Section 5 by considering redundancy in the actuators. Some conclusions and suggestions are outlined in the last section.

2. STATE FEEDBACK CONTROL

Usual robust control strategies aim for a prescribed level of performance for any plant in a user-defined set (robust performance). A variation, less common in bibliography, is trying to simultaneously ensure a certain level of performance for a plant G in a set $\mathscr{G}_1$, and another one for plants in a set $\mathscr{G}_2$. This design problem can be stated for several of these sets, as a particular case of multi-model control. This option can be investigated by using the freedom in the design of the state feedback control law for pole placement design requirements.

Given a linear time invariant plant $G(s)$, with realisation (A,B,C), the purpose is to design a fixed controller K ($u(s) = -K(s)y(s)$) such that, some basic requirements, S, are fulfilled. The freedom in selecting the controller parameters will be used to ensure, if it is possible, some reduced requirements, S_{MN} for the plant model $\mathscr{G}_{MN}$ under any subset of inputs and outputs, M and N, leading to the realisation matrices B_M, C_N, respectively, or to determine that such a controller is not feasible. The notation B_M will denote either a submatrix of B, when multiplied by K_M – a submatrix of K– or a matrix with the same dimensions as B setting the rest of the columns of B to zero, when multiplied by a full-size K.

In this section, the following simplifications are assumed: 1) $C = I$, the full state is measurable, 2) only changes in the input matrix B are considered, that is, the plant model is in one of $\mathscr{G}_M$, and 3) as a result of the control requirements, a constant feedback law is expected, $K(s) = K$.

Pole-region placement restrictions of a matrix A_{cl}, $\dot{x}(t) = A_{cl}x(t)$, can be cast as an LMI problem involving a positive definite matrix P in the case the target pole region is a convex subset of the complex plane in a given form (Boyd *et al.*, 1994). In the case of dominant pole restrictions, P corresponds to a quadratic Lyapunov function with a particular decay rate.

For instance, a decay rate faster than α is ensured if a solver is able to find a quadratic closed loop Lyapunov function $V(x) = x^T Px$ so that $\dot{V} \leq -2\alpha V$ This requires finding $P > 0$ such that

$$PA_{cl} + A'_{cl}P + 2\alpha P < 0 \tag{1}$$

In this setting, $\alpha = 0$ amounts to specifying only closed-loop stability.

However, in state feedback the closed-loop system matrix, $A_{cl} = A - BK$, is itself linear in the design parameter K, so the overall setup is cast as a BMI

(VanAntwerp and Braatz, 2000). The feedback gain $u = -Kx$ can be found by solving the following BMI:

$$P(A-BK)+(A-BK)'P+2\alpha P<0 \quad (2)$$

A standard change of variable (Boyd *et al.*, 1994)

$$Q=P^{-1}, \qquad KQ=Y \quad (3)$$

transforms (2) back into LMI form: a state feedback $u=-Kx$ exists so that the decay rate is at least α if an arbitrary $m\times n$ matrix Y and a positive definite $n\times n$ matrix Q exists so that the following LMI is satisfied:

$$QA^T+AQ-BY-Y^TB^T+2\alpha Q<0 \quad (4)$$

As above commented, this basic LMI can be modified to include other target pole regions and norm bounds ($\mathscr{H}_\infty$ or $\mathscr{H}_2$ performance)(Boyd *et al.*, 1994). These more complex cases will not be considered here for simplicity although the procedure would be completely analogous.

Multi-model system. Let us now consider the faulty conditions. The same problem can be solved for each operating condition, B_M, leading to different matrices P_M and control laws K_M

$$P_M(A-B_MK_M)+(A-B_MK_M)'P_M+2\alpha P_M<0 \quad (5)$$

Thus, the control implementation would require the detection of the faulty condition as well as the switching of the controller parameters. The above expression amounts to solving separately (via LMI) each of the possible operating regimes. To avoid the need of fault detection and control switching, a common K can be sought in the above setup:

$$P_M(A-B_MK)+(A-B_MK)'P_M+2\alpha P_M<0 \quad (6)$$

However, the change of variable (3) no longer applies and the problem is in this case a full BMI. An efficient computation method (polynomial time) for solving general BMIs is not available. The simplest one is the *P-K* iteration (iterated solving of LMI conditions on one variable considering the other as fixed); however, that algorithm only converges locally, and it may even not do so. Alternative algorithms are discussed in, for instance, (VanAntwerp and Braatz, 2000; Zheng *et al.*, 2002) and references therein.

Different specifications (α_M) may be considered for different actuator faults. For instance, in a two actuator case, three sets of BMIs may be specified: the full B and the nominal α, B_1, (fault of actuator 2) stating a performance level of α_1 and similarly B_2 (fault of actuator 1) stating a performance level α_2.

Random faults. Stability under random changes of operating regime is *not* guaranteed (Hespanha *et al.*, 2003) by the previous approach unless each regime is active for a time significantly longer than the closed-loop settling time. A conservative solution, by looking for a common $P=P_M$ for any actuator subset M and, as before, a common K, is possible. In this way, due to the common Lyapunov function, specifications are guaranteed for arbitrary actuators coming in and out of service at any instant. Nevertheless, faults in industrial practice usually occur for times longer than the loops time constant so the above requirement might be overkill in many situations.

To solve combined requirements with different specifications as an LMI, the same Q and Y should be enforced for all models. The result is a set of LMIs in the form:

$$\begin{aligned} QA^T+AQ-B_MY-Y^TB_M^T+2\alpha_MQ<0 \\ QA^T+AQ-BY-Y^TB^T+2\alpha Q<0 \end{aligned} \quad (7)$$

where M represents a set of LMIs for each considered fault condition and the non-subscripted matrices denoting the nominal (all actuators working) situation. So, a common Lyapunov function is sought, and the approach is conservative (a solution to the slowly-switching integrity problem may exist even if the solver finds the LMIs unfeasible).

Sequential strategies. As previously mentioned, a practical problem in setting up a control strategy is the switching on, from manual to automatic, of the different actuators in the control system. Even if the global LMI/BMI approach fails a straightforward sequential strategy can ensure stability for some of the partial actuator configurations.

Let us assume that the actuators are arranged in B in the order they are going to be switched on. Then, design a controller to place the poles of $A-B_1K_1$ at p_1. Afterwards, a controller is designed to place the poles of $(A-B_1K_1)-B_2K_2$ at a desired location p_2. Successively, the process goes on until $A-\sum B_iK_i$ has its poles placed at the final target p_m. Assuming all controllability requirements are verified, the solution is unique and the design parameters are the sequence of intermediate and final pole positions. In this way, stability and a particular performance level can be guaranteed if actuators are put in and out of service in the particular sequence determined by the ordering of the columns of B. However, the rest of the configurations must be explicitly checked.

Servo systems. In the previous designs, the target has been specified by the closed-loop dynamic behaviour (closed-loop poles). If the controlled system is stable, all the (incremental) variables go to zero in the steady-state condition. However, loss of availability of one particular actuator may be understood as a change in B plus a step input disturbance on that channel (as it will be probably stuck in a constant value different from the nominal operating point). To cope with this issue, classical solutions based on integral actions can be applied.

3. EXAMPLE: BANK ANGLE JET CONTROL

In (Zhao and Jiang, 1998), a “reliable state feedback” control for a 4th order jet bank-angle dynamic model is discussed, under the same actuator failure setting as treated in this paper. The plant has two actuators (aileron and rudder), and its model is:

$$y(s) = [G_1(s)G_2(s)]u(s) \quad (8)$$

$$G_1(s) = \frac{1.1476s^2 - 2.0036s - 13.726}{\kappa(s)} \quad (9)$$

$$G_2(s) = \frac{10.729s^2 + 2.3169s + 10.237}{\kappa(s)} \quad (10)$$

$$\kappa(s) = s^4 + 0.6358s^3 + 0.9389s^2 + 0.5116s + 0.0037 \quad (11)$$

Based on a canonical state space representation, (A, B, C), a state feedback control, K, with the requirement of decay rate of $-.1$ is sought. The Matlab command "K=place(A,B,p)", with poles at $p = \{-0.1, -0.1, -0.15, -0.15\}$, yields the (non-unique) possible solution:

```
K= -0.1926    1.0275    0.0453   -0.1147
    0.0206    1.3814   -0.2925   -0.1142
```

Let us denote by B_1 and B_2 the matrix obtained by setting to zero the columns 2 and 1 of B, respectively (*i.e.*, denoting the situation with only actuator 1 working and only actuator 2, respectively). The integrity of the proposed solution is not satisfactory: if the second actuator fails, the system $(A - B_1K)$ becomes unstable. This also points out the relevance of the first actuator (Albertos, 2004) by using this solution.

Our purpose is then to design a feedback control law such that, with the same relative degree of stability (α at least -0.1), the controlled plant remains stable if the control loop is opened at any of the actuators. Moreover, we are also interested in checking the maximum achievable performance (decay rate) with both actuators working while preserving integrity.

LMI solution. The following LMI problem is posed according to specifications with the 4th order system in the variables $Q > 0$, Y, aiming for a decay rate in the interval $[\alpha_1, \alpha_2]$:

$$QA^T + AQ - B_1Y - Y^TB_1^T < 0 \quad (12)$$

$$QA^T + AQ - B_2Y - Y^TB_2^T < 0 \quad (13)$$

$$QA^T + AQ - BY - Y^TB^T < -\alpha_1 * 2Q \quad (14)$$

$$QA^T + AQ - BY - Y^TB^T > -\alpha_2 * 2Q \quad (15)$$

With $\alpha_1 = 0.1$, $\alpha_2 = 1.2$ the problem renders feasible. The state feedback gain found by Matlab $K_{fb} = YQ^{-1}$ and the closed-loop poles are:

```
K= -0.8194   -0.8179    0.3084    0.0868
    2.4920    0.9824   -1.0740   -0.0926
eig(A-B*Kfb)
  -0.4971 + 0.8387i     -0.5626
  -0.4971 - 0.8387i     -0.2252
eig(A-B1*Kfb)
  -0.1030     -0.4099 + 1.1755i
  -0.5626     -0.4099 - 1.1755i
 eig(A-B2*Kfb)
  -0.1121 + 0.8334i     -0.5626
  -0.1121 - 0.8334i     -0.1456
```

In the above case, a new LMI constraint to minimise a bound on the singular value norm of the feedback gain matrix was also enforced, and decreased until no feasible solution existed. The gain guarantees the decay rate of $-.1$ even if actuators are failing with an arbitrary time pattern, improving the time-invariant pole-based result of (Zhao and Jiang, 1998). Also, Zhao's results required a system augmentation to 6th order.

Plant augmentation can be used to enforce integral action, by adding an integrator at the output. On the following, the resulting 5th-order realisation will be used.

In this case, in order to improve the closed loop performances under normal operating conditions, but with guaranteed integrity (decay rate 0.02), a generalised eigenvalue problem is posed on the augmented model: maximising α_1 subject to $Q > 0$ and conditions (12)–(15), with fixed $\alpha_2 = 1.2$. The resulting optimisation determines that an achievable nominal decay rate of 0.155 can be obtained, remaining stable if any of the actuators is disconnected. The closed-loop poles are $(-0.201, -0.196, -0.256)$ in the nominal, B_1 and B_2 cases respectively. Note that the time-invariant pole figures (for the "permanent" fault case) are better than the achieved value of α. This arises from the conservativeness of the shared Lyapunov function approach.

BMI solution. The nonexistence of a better LMI solution does not preclude finding it by another methodology. The alternating approach (*P*-*K* iteration with progressively increasing decay rate specifications at small steps) may provide a local solution to the full BMI problem (with a different P sought for each restriction, i.e., slow switching). Indeed, starting from the previously obtained gain, the BMI approach finds better solutions. After each successful iteration, the nominal required decay rate is increased by a factor of 1.0025 and the faulty ones by 1.003. The result is:

```
K=-4.602 -16.033 -12.529  -1.699  -1.665
  -1.216   0.778   3.986   5.476   3.094
```

with dominant poles (-1.15,-0.834,-0.109) for each of the three considered cases. The nested LMI conditions kept feasible until the nominal decay limit was 0.637 and the fault one 0.109. So in this case, the iterations seem to obtain reasonable solutions sensibly improving those from the LMI approach.

The step responses under a step input disturbance in both channels (nominal performance) are depicted in figure 2. Being at an equilibrium state, the loss of actuator 2 can be understood as a step disturbance at that input channel. The referred step response (cutting the feedback path before actuator 2 so only actuator 1 remains in operation) appears in Figure 3. The converse situation when only actuator 2 is available is plotted in figure 4. Note that closed-loop stability under arbitrary patterns of actuator availability cannot be guaranteed.

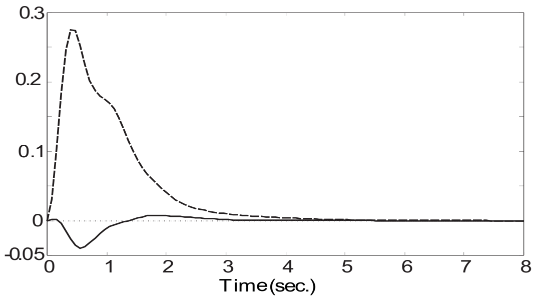

Fig. 2. Nominal Performance.

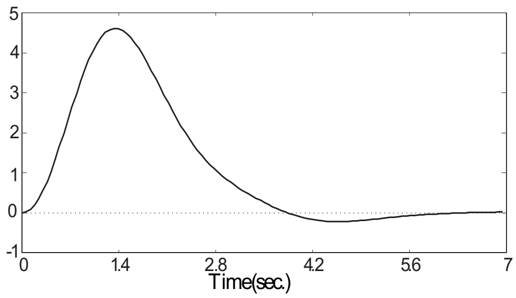

Fig. 3. Transient if actuator 1 is working and 2 fails.

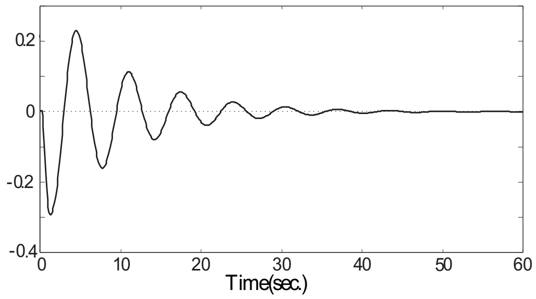

Fig. 4. Transient if actuator 2 is working and 1 fails.

4. OBSERVER DESIGN

The pole assignment approach to design a state observer such as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \qquad (16)$$

is dual to the state feedback control design described in Section 2. A BMI setting similar to (2) can be defined as

$$P(A-LC) + (A-LC)'P + 2\alpha P < 0 \qquad (17)$$

and, given A and C, find the observer gain L to get a decay rate α of the observed state error.

Sensors faults. This solution does not guarantee any integrity w.r.t. the failure of one sensor. Following the same approach previously described and allowing a slower convergence of the observed state, an LMI setting similar to (7) can be used,

$$PA + A^T P - YC_N - C_N^T Y^T + 2\alpha_M P < 0 \qquad (18)$$
$$PA + A^T P - YC - C^T Y^T + 2\alpha P < 0$$

leading to an observer gain $L = P^{-1}Y$. Nevertheless, the estimation error $x - \hat{x}$ can be shown to behave according to the equation:

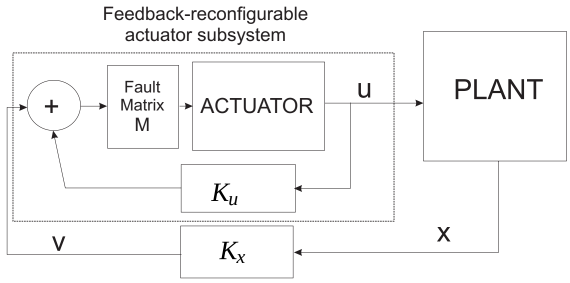

Fig. 5. Feedback of the actuator output.

$$\dot{e} = (A - LNC)e + L(I - N)C\hat{x} \qquad (19)$$

Thus, this error only converges to zero in the nominal case ($N = I$). Otherwise, supervision is needed so that the term that multiplies $\hat{x}$ at the right hand side goes to zero.

If sensors and actuators failures are considered altogether, the global control schema would be:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(m(t) - NC\hat{x}(t)) \qquad (20)$$
$$v(t) = K\hat{x}(t) \quad u(t) = Mv(t) \quad m(t) = NCx(t) \qquad (21)$$

The usual separation principle approach (controller, observer) does not hold in this framework and the design of K and L must be done simultaneously. Again, a supervisory level would be necessary. This implies the controller being time-varying. Thus, as commented in the introduction, other procedures could lead to better results.

5. ACTUATORS OUTPUT MEASUREMENT

In the case of actuators failure, if measurement of input variables is available, the performance and fault-tolerance properties of the state feedback approach can be extended, due to two reasons: the consideration of actuator dynamics may be significant when extending the performance specifications, and more information is available for reconfiguration, so that an actuator subsystem reconfigured by feedback can be designed, as in Figure 5, with the fault matrix M. There is no additional theory, apart from extending the state vector to incorporate actuator dynamics. However, some interesting options appear, as shown in the following illustrative example.

Example. Let us consider the academic simple system $1/(s+1)^2$, represented in canonical controllable form. Approximately, the K achieving pole-placement at -7, is

```
K =   48.1400   12.0200
```

whose norm is 50. Deemed acceptable, the objective is inserting a redundant actuator trying to improve performance when both are connected, and keeping the poles when only one is working at close to the one-actuator design.

Assume a double identical actuator $B = [0\ 0; 1\ 1]$. To achieve the same type of nominal response, a K

roughly (duplicated) half of these would be a reasonable starting point, $K' = [K/2; K/2]$. Its norm is 35 and, of course, the nominal behaviour is identical to the one-actuator setting. The faulty behaviour, however, has -4 as dominant pole.

Keeping the norm bound in K, a BMI approach will be pursued, increasing by small steps the integrity decay rate limit until BMI iterations fail to find a feasible solution. Unfortunately, the final gain is almost identical to K' indicating that nothing can be done, apparently. The alternative, other than allowing a higher gain, is to measure the input.

Let us explore the input measurement enhancement, with 1st-order actuators having gain 1 and pole at -40. The state-space matrices are suitably augmented, and the starting point for new BMI iterations is K' (extended by zeroes). In this case, due to the actuator dynamics, the closed-loop dominant poles are at -5.5 and -4.4 in the nominal and faulty situations: note the degradation when including actuator dynamics into the previous results.

BMI iterations are carried out progressively increasing performance specifications until both the nominal and faulty performances have a guaranteed decay greater than -6.9. The result places dominant poles at -6.99 and -7.02 in the nominal and faulty cases respectively with gain:

```
K= 16.299  3.8941  -0.5199   0.3053
   16.299  3.8941   0.3053  -0.5199
```

To interpret the result, let us separate the analysis of the state feedback and feedback-actuator subsystems (Figure 5): Closing the actuator subsystem and carrying out some elementary operations, the control action in the nominal case can be written as:

$$v = -16.299x_1 - 3.8941x_2; \quad u_1 = u_2 = \frac{40}{s + 31.42} v$$

so that, at low frequency, it is approximately:

$$u_1 = u_2 = -20.75x_1 - 4.96x_2$$

i.e., a solution behaving similarly to `KN` above has been found (better, in fact, as it takes actuator dynamics into account). In the faulty case, closing the loop with one of the actuators outputting zero, the result is $v = -16.299x_1 - 3.8941x_2$, $u_1 = \frac{40}{s+19.21}v$, $u_2 = 0$. At low frequency, it approximates:

$$u_1 = -33.95x_1 - 8.11x_2$$

so the system has automatically "reconfigured" and increased its gain to keep performance. Of course, an "optimal" fault detector would have increased the gain 100%, but the found solution lies pretty close and keeps the same decay rate in nominal and faulty cases, being that the primary design objective.

6. CONCLUSION

This paper has discussed the practical importance of integrity requirements, posing the problem in the particular case of state feedback under actuator faults. The suitability of BMI, LMI and sequential pole-placement strategies has been discussed, and a jet control example has been presented. In a similar way, dual state estimation problems can be studied and the treatment can be easily extended to the sensors' fault condition. The results seem promising.

The main advantage of this approach with respect to the classical hybrid control solutions is that the control feedback law is time invariant. Thus, detection and isolation of faults in the devices (sensors and actuators) is not needed and commutation of controllers is not required. Integral action is needed to ensure setpoints are maintained.

Nevertheless, the stability of some of the solutions rely on different Lyapunov functions for each operating condition. Thus, the BMI results are not valid for arbitrary times of in and out of service events. This issue does not arise if only LMI formulations are used.

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