LINEAR OPTIMIZATION OF PARAMETERS IN DYNAMIC THRESHOLD GENERATORS

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Abstract: A model-based fault detection algorithm for linear systems with uncertain parameters is treated. An error system, bilinear in the uncertainties, generates the residual. The residual is compared to a threshold, which is generated by a linear system with the unknown uncertainty upper bounds as parameters. These unknown uncertainty upper bounds can be substituted by design parameters and this article suggests an algorithm to choose design parameter values such that the threshold is larger than the residual when no fault is present. This parameter design algorithm is applied to a sensor fault detection algorithm for a jet engine.

Keywords: Fault detection, Threshold selection, Parameter optimization, Linear programming, Uncertain linear systems

1. INTRODUCTION

A fault detection algorithm contains essentially two parts, *residual generator* and *residual evaluator*. One way to design the residual generator is to use dynamic process models. Methods which have been used during the years are *e.g.* state observers, (Frank 1997), parity equations (Chow and Willsky 1984) and on-line identification algorithms (Gustafsson 2000). To determine if a fault is present, the residual evaluator compares the residual, or a function of the residual, to a threshold. The character of the threshold will depend on the assumptions on the disturbances and uncertainties. The case of stochastic disturbances are treated in (Gustafsson 2000) while frequency domain uncertainty is considered by (Emami-Naeini *et al.* 1988) and (Frank and Ding 1994). Time-domain uncertainty description are utilized in *e.g.* (Zhang *et al.* 2002), (Ding *et al.* 2003), (Bask and Johansson 2004) and (Johansson and Bask 2005). In the latter, the threshold is generated by a linear system with the uncertainty upper bounds, which are unknown, as parameters. The upper bounds are substituted by a vector of parameters which, so far, has been tuned manually in (Bask and Johansson 2004) and (Johansson and Bask 2005). The main purpose of this study is thus to provide an algorithm to choose these parameters in an automatic way.

The class of systems considered in this paper are linear with uncertain parameters and can be described by the following system of bilinear differential equations

$$\begin{aligned} \dot{x} &= Ax + N(\pi \otimes x) + Bu + E\pi \\ y &= Cx + Du + F\pi \end{aligned} \tag{1}$$

where $x(t) \in \mathrm{R}^n$, $u(t) \in \mathrm{R}^m$ and $y(t) \in \mathrm{R}^l$ are the state, known input and output vector, respectively. The matrices A, N and C and the matrix valued functions $B(t), D(t), E(t)$ and $F(t)$ are of appropriate dimensions. Furthermore, it is assumed that $\mathrm{rank}(C) = l < n$. Disturbances and parametric uncertainties are represented in $\pi(t) \in \mathrm{R}^p$ which is bounded as $|\pi(t)| \leq \Pi$. With this assumption it is easy to show that the right hand side of (1) satisfies a Lipschitz condition and thus (1) has a solution.

If a Luenberger observer is applied to (1) then the error system will also have the form of (1) but no known input signal will be present, *i.e.* $B = D = 0$. In this case, x represents the estimation error and $y \triangleq r$ is the residual.

A process where fault detection algorithms may be a significant advantage is the jet engine in a single engine aircraft where faults can have catastrophic consequences. A fault detection algorithm with a dynamic detection threshold for a sensor in a turbofan engine is presented in (Johansson and Norlander 2003). There, constant parameter uncertainties are assumed and the uncertainty bound design parameters are tuned manually. In (Johansson and Bask 2005) the approach from (Johansson and Norlander 2003) is generalized to allow time-varying parameter uncertainties but the bounds are still tuned manually. In this paper, an automatic method to determine the threshold design parameters, substituting the upper bounds, is derived. The design method is successfully tested on data from a turbofan engine.

2. PRELIMINARIES

An inequality between two matrices $X, Y \in \mathrm{R}^{n\times m}$ is to be interpreted as element-wise. The notation $|\cdot|$ means matrix modulus, *i.e.* element-wise absolute value. The following inequalities for matrix operations are trivial but included in order to increase readability of the proofs in the sequel.

Property 1. Let A, B, and C be matrices of compatible dimension.

(a) If $A \geq 0$ and $B \geq C$, then $AB \geq AC$ and $BA \geq CA$.
(b) $|A + B| \leq |A| + |B|$
(c) $|AC| \leq |A||C|$

Throughout the article the notation 1_n represents a column vector of ones of dimension n and I_n is the identity matrix of size n. The pseudoinverse of a matrix $A \in \mathrm{R}^{n\times m}$ is denoted A^+. If rank$(A) = n < m$ then $AA^+ = I$ since in this case $A^+ = A^T(AA^T)^{-1}$.

Some properties regarding the Kronecker product $\otimes$ will be required further on in the article.

Property 2. Let $A \in \mathrm{R}^{n\times m}, B \in \mathrm{R}^{p\times q}, C \in \mathrm{R}^{m\times r}$ and $D \in \mathrm{R}^{q\times s}$ for arbitrary natural numbers m, n, p, q, r, s. Then

(a) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$
(b) $A \otimes (B \pm C) = A \otimes B \pm A \otimes C$
(c) $(x \otimes A) = (x \otimes I_m)A = (I_p \otimes A)(x \otimes I_m)$

Proof. Part a and b can be found in (Lütkepohl 1996), and c is straightforward to prove using part a. □

All signals are assumed to be causal. A star between functions denotes convolution, *i.e.*

$$(F * G)(t) \triangleq \int_{0^-}^{t} F(t - \tau)G(\tau)d\tau$$

where $F(t) \in \mathrm{R}^{n\times q}$ and $G(t) \in \mathrm{R}^{q\times m}$ are matrix-valued signals. Instead of writing the convolution with a star it can be expressed as a linear operator written with the symbol of the weighting function in bold-face font, *i.e.* $\mathbf{F}G \triangleq F*G$. Some inequalities involving the convolution are derived in (Johansson and Bask 2005) and is also stated in Lemma 1 below.

Lemma 1. Let $F(t) \in \mathrm{R}^{n\times m}$ and $H_1(t), H_2(t) \in \mathrm{R}^{m\times q}$ then

a) $F \geq 0$ and $H_1 \geq H_2$ then $F * H_1 \geq F * H_2$
b) $|F * H_1| \leq |F| * |H_1|$

Proof. See (Johansson and Bask 2005). □

The modulus and inequalities of functions are intended to be point-wise. Assume that $F(t), G(t) \in \mathrm{R}^{n\times q}$ then the modulus $|F|$ is defined by, $|F|(t) \triangleq |F(t)|$ for all $t \geq 0$, and the inequality, $F \leq G$ means that $F(t) \leq G(t)$ for all $t \geq 0$.

3. INEQUALITIES FOR LINEAR SYSTEMS WITH PARAMETER UNCERTAINTY

An inequality for the modulus of the state of a linear system with a general input signal $g(t)$ and uncertain time-varying parameters π was given in (Johansson and Bask 2005) as

Theorem 1. Consider the bilinear differential equation

$$\begin{aligned} \dot{x} &= Ax + N(\pi \otimes x) + g \\ x(0) &= x_0 \end{aligned} \tag{2}$$

where $A \in \mathrm{R}^{n\times n}$, $N \in \mathrm{R}^{n\times np}$, $\pi(t) \in \mathrm{R}^p$ and $g(t) \in \mathrm{R}^p$. Assume that $|\pi(t)| \leq \Pi$ for all $t \geq 0$ and let $G(t) \triangleq e^{At}$. Let $H(t) \in \mathrm{R}^{n\times n}$ be a function that satisfies $H \geq |GN|(\Pi \otimes I_n)$. If $\|\mathbf{H}\| < 1$ for some induced operator norm $\|\cdot\|$ then $(I - \mathbf{H})^{-1}$ is a bounded operator and

$$|x| \leq (I - \mathbf{H})^{-1}(|\mathbf{G}g + Gx_0|) \tag{3}$$

Proof. See (Johansson and Bask 2005). □

In the special case (1) of (2) the input, g, is composed of both known, u and unknown, π input signals linearly.

Corollary 1. Consider the uncertain system (1) and define $G(t)$ and $H(t)$ as in Theorem 1. Let $\Gamma(t) \geq |G(t)|$ for all $t \geq 0$, then

$$\begin{aligned} |y| \leq& |C|(I - \mathbf{H})^{-1}(|\mathbf{G}Bu| + \mathbf{\Gamma}|E|\Pi + |Gx_0|) \\ &+ |Du| + |F|\Pi \end{aligned} \tag{4}$$

Proof. The modulus of the output is

$$\begin{aligned}|y| =&|Cx+Du+F\pi| \\ \le&|C|(I-\mathbf{H})^{-1}(|\mathbf{G}g+Gx_0|)+|Du|+|F||\pi| \\ \le&|C|(I-\mathbf{H})^{-1}(|\mathbf{G}Bu|+\mathbf{\Gamma}|E||\pi|+|Gx_0|) \\ &+|Du|+|F||\pi|\end{aligned}$$

where the first inequality comes from Property 1a-1c and Theorem 1. The second inequality results from using $g = Bu + E\pi$ in combination with Property 1b, Lemma 1, and the assumption $|G| \le \Gamma$. Utilizing the assumption $|\pi| \le \Pi$ completes the proof. □

4. THRESHOLD PARAMETER DESIGN

The residual evaluation algorithm is composed of a threshold σ which is compared to the modulus of the residual $|r|$ and an alarm is raised at time t if $|r|(t) \ge \sigma(t)$. The threshold is generated by a linear dynamical system and depends on the upper bounds on the uncertainties, Π which are unknown. So far the thresholds have been designed by manually tuning a set of parameters, π^*, see (Johansson and Bask 2005), (Bask and Johansson 2004), such that the threshold is larger than the residual for a set of test data. In this section, an automatic way to determine the parameters π^* is suggested.

Assume that the fault detection residual can be described as the output of an error system of the form (1) with $B = D = 0$. Then an upper bound for the modulus of the residual is given by Corollary 1, *i.e.* $|r| \le |C|(I-\mathbf{H})^{-1}(\mathbf{\Gamma}|E|\Pi + |Gx_0|) + |F|\Pi$. This expression would be an ideal threshold but unfortunately Π is unknown. This problem is solved by substituting Π with the design parameters π^* in the threshold. Another problem is finding $\mathbf{H}$ to satisfy $H \ge |GN|(\pi^* \otimes I_n)$ which may be solved by finding a realizable

$$\Gamma(t) \triangleq C_\Gamma e^{A_\Gamma t} B_\Gamma \ge |G(t)| \quad (5)$$

and choosing

$$\begin{aligned}H^*(t) \triangleq&\Gamma(t)|N|(\pi^*\otimes I_n) \\ \ge&|G(t)||N|(\pi^*\otimes I_n) \\ \ge&|G(t)N|(\pi^*\otimes I_n)\end{aligned} \quad (6)$$

The resulting threshold is

$$\sigma =|C|(I-\mathbf{H}^*)^{-1}(\mathbf{\Gamma}|E|\pi^* + |Gx_0|) + |F|\pi^* \quad (7)$$

and the design parameter π^* is chosen such that $\sigma \ge |r|$ when no fault is present. Later on, it will be convenient to express the error system in transformed coordinates which is enabled by Lemma 2.

Lemma 2. Assume the bilinear system is described by (1) and let

$$z = T_x x + T_\pi \pi \quad (8)$$

then, assuming that the uncertainties are small so that $\pi \otimes \pi$ may be neglected, the dynamics of z can be expressed as

$$\begin{aligned}\dot{z}=&A_T z + N_T(\pi\otimes z) + B_T u + E_T \pi_T \\ y=&C_T z + D_T u + F_T \pi_T\end{aligned} \quad (9)$$

where

$$\begin{array}{ll}A_T=T_x A T_x^{-1} & E_T=\left[\, T_x E - T_x A T_x^{-1} T_\pi \;\; T_\pi \,\right] \\ B_T=T_x B & N_T=T_x N(I_p\otimes T_x^{-1}) \\ C_T=C T_x^{-1} & F_T=\left[\, F - C T_x^{-1} T_\pi \;\; 0 \,\right] \\ D_T=D & \pi_T=\left[\, \pi \;\; \dot{\pi} \,\right]^T\end{array} \quad (10)$$

Proof.

$$\begin{aligned}&\dot{z} = T_x\dot{x} + T_\pi\dot{\pi} \\ =&T_x(Ax+N(\pi\otimes x)+Bu+E\pi)+T_\pi\dot{\pi} \\ =&T_xAT_x^{-1}z + T_xAT_x^{-1}T_\pi\pi + T_xN(\pi\otimes(T_x^{-1}z)) \\ &-T_xN(\pi\otimes(T_x^{-1}T_\pi\pi)) + T_xBu + T_xE\pi + T_\pi\dot{\pi} \\ =&T_xAT_x^{-1}z + T_xN(I_p\otimes T_x^{-1})(\pi\otimes z) \\ &-T_xN(I_p\otimes T_x^{-1}T_\pi)(\pi\otimes\pi) + T_xBu \\ &+(T_xE - T_xAT_x^{-1}T_\pi)\pi + T_\pi\dot{\pi}\end{aligned}$$

where the second, third and fourth equality comes from (1), (8) and Property 2a, respectively. The output can be described as

$$\begin{aligned}y=&Cx+Du+F\pi \\ =&CT_x^{-1}(z-T_\pi\pi)+Du+F\pi \\ =&CT_x^{-1}z+Du+\left[\, F-CT_x^{-1}T_\pi) \;\; 0\,\right]\pi_T \\ =&C_Tz+D_Tu+F_T\pi_T\end{aligned}$$

where (8) was used. Utilizing the assumption that $\pi \otimes \pi$ can be neglected completes the proof. □

The design parameters, π^*, shall be determined such that $\sigma \ge |r|$. The purpose of the following theorem is to recast the problem of finding π^* into satisfying a linear inequality condition.

Theorem 2. Let the residual r be generated by the error system (1) with $B = D = 0$, $r = y$ and assume that $|N|(I_p \otimes |C|^+|F|) = 0$. Define σ, Γ and H^* as in (7), (5) and (6), respectively, with $G(t) \triangleq e^{At}$ and let

$$\begin{aligned}\Omega\triangleq&\mathbf{\Gamma}|E| + |C|^+|F| + \mathbf{\Gamma}(|N|(I_p\otimes|C|^+|r|)) \\ \beta\triangleq&|C|^+|r| - |Gx_0|\end{aligned}$$

Choose π^* such that

$$\Omega\pi^* - \beta \ge 0$$

then $\sigma \ge |r|$.

Proof. Define the function $\zeta \triangleq \Omega\pi^* - \beta \ge 0$. The last term of $\Omega\pi^*$ can be written as

$$\begin{aligned}\mathbf{\Gamma}(|N|(I_p\otimes|C|^+|r|))\pi^*=&\Gamma * |N|(I_p\otimes|C|^+|r|)\pi^* \\ =&\Gamma * |N|(\pi^*\otimes I_n)|C|^+|r| \\ =&\mathbf{H}^*|C|^+|r|\end{aligned}$$

where Property 2c and (6) was used. Furthermore,

$$\begin{aligned}\mathbf{H}^*|C|^+|F| &= \Gamma * |N|(\pi^* \otimes I_n)|C|^+|F| \\ &= \Gamma * |N|(I_p \otimes |C|^+|F|)(\pi^* \otimes I_p) = 0\end{aligned} \tag{11}$$

where Property 2c was used in the second equality and the last equality follows from the assumption that $|N|(I_p \otimes |C|^+|F|) = 0$. Thus

$$\begin{aligned}0 =& \Omega\pi^* - \beta - \zeta \\ =& \mathbf{\Gamma}|E|\pi^* + |C|^+|F|\pi^* + \mathbf{H}^*|C|^+|r| \\ & -\mathbf{H}^*|C|^+|F|\pi^* + \mathbf{H}^*|C|^+|F|\pi^* \\ & -|C|^+|r| + |Gx_0| - \zeta\end{aligned}$$

Some rearrangement yields

$$\begin{aligned}&(I - \mathbf{H}^*)|C|^+|r| = \mathbf{\Gamma}|E|\pi^* + |Gx_0| - \zeta \\ &+(I - \mathbf{H}^*)|C|^+|F|\pi^* + \mathbf{H}^*|C|^+|F|\pi^*\end{aligned}$$

Applying the operator $(I - \mathbf{H}^*)^{-1}$ results in

$$\begin{aligned}&|C|^+|r| = (I - \mathbf{H}^*)^{-1}(\mathbf{H}^*|C|^+|F|\pi^* - \zeta) \\ &+(I - \mathbf{H}^*)^{-1}(\mathbf{\Gamma}|E|\pi^* + |Gx_0|) + |C|^+|F|\pi^*\end{aligned}$$

Using (11) and multiplying with $|C|$ form the left and using the knowledge that $|C||C|^+ = I$ gives

$$\begin{aligned}|r| =& |C|(I - \mathbf{H}^*)^{-1}(\mathbf{\Gamma}|E|\pi^* + |Gx_0|) + |F|\pi^* \\ & -|C|(I - \mathbf{H}^*)^{-1}\zeta \\ =& \sigma - |C|((I - \mathbf{H}^*)^{-1} - I)\zeta - \zeta\end{aligned}$$

Finally, since $\zeta \geq 0$ and the impulse response of $(I - \mathbf{H}^*)^{-1} - I$ is positive (see Lemma 3 in (Johansson and Bask 2005)) it is concluded that $\sigma \geq |r|$. □

Remark 1. In Theorem 2 it was assumed that $|N|(I_p \otimes |C|^+|F|) = 0$. Since, by Property 2a, $|N|(I_p \otimes |C|^+|F|) = |N|(I_p \otimes |C|^+)(I_p \otimes |F|)$ this is true if either of the following is true.

$$\begin{aligned}&|N|(I_p \otimes |C|^+) = 0 \\ &|F| = 0\end{aligned} \tag{12}$$

The latter can always be accomplished by the transformation described in Lemma 2 so that the new matrix F becomes $F - CT_x^{-1}T_\pi = 0$. To achieve this, choose *e.g.* $T_\pi = \begin{bmatrix} F^T & 0 \end{bmatrix}^T$ and $T_x^{-1} = \begin{bmatrix} C^+ & \Lambda \end{bmatrix}$ where Λ is chosen such that T_x is invertible which is possible since $C^+ \in \mathrm{R}^{n \times l}$ and $\mathrm{rank}(C^+) = l$.

4.1 Optimizing the threshold

When the vector π^* is to be determined, using Theorem 2, measurements without faults are required. These measurements will be discrete in time and consequently the conditions in Theorem 2 may be checked at the time instances kh where h is the sampling interval and k is the sample number. Since it is also desirable that $(I - \mathbf{H}^*)^{-1}$ is stable, the set of admissible parameters is defined as $\mathrm{D} = \{\pi^* \geq 0 | \Sigma\pi^* \geq \varsigma, (I - \mathbf{H}^*)^{-1} \text{ is stable}\}$ where

$$\begin{aligned}\Sigma =& \begin{bmatrix} \Omega(h)^T & \Omega(2h)^T & \cdots & \Omega(Mh)^T \end{bmatrix}^T \\ \varsigma =& \begin{bmatrix} \beta(h)^T & \beta(2h)^T & \cdots & \beta(Mh)^T \end{bmatrix}^T\end{aligned}$$

and M is the total number of time instances. It is desirable to write the criterion that $\pi^* \in \mathrm{D}$ as simple as possible. Therefore, the condition that $(I - \mathbf{H}^*)^{-1}$ is stable, will be approximated by a linear inequality in π^*, see Lemma 3.

Lemma 3. Let $H(t) = Ce^{At}B \geq 0$. Then $(I - \mathbf{H})^{-1}$ is stable if

$$-CA^{-1}B1_n - 1_n \leq 0 \tag{13}$$

Proof. The small gain theorem states that if $\|\mathbf{H}\| < 1$ then $(I - \mathbf{H})^{-1}$ is stable. The norm $\|\mathbf{H}\|$ is any induced operator norm, *e.g.* the ∞-norm defined by $\|\mathbf{H}\|_\infty = \max_i \|H_i\|_1$ where $H_i(t)$ is the *i*:th row of the impulse response of $\mathbf{H}$ and $\|\cdot\|_1$ is the 1-norm for signals. Thus

$$\begin{aligned}\|H_i\|_1 =& \int_0^\infty |H_i(\tau)|1_n d\tau = \int_0^\infty e_i^T|H(\tau)|1_n d\tau \\ =& e_i^T \int_0^\infty |H(\tau)|d\tau 1_n = e_i^T \int_0^\infty H(\tau)d\tau 1_n \\ =& -e_i^T CA^{-1}B1_n\end{aligned}$$

where e_i is column i of an identity matrix. The last equality is easy to show by taking the Laplace transform of the impulse response and using the final value theorem. Thus $\|\mathbf{H}\|_\infty < 1$ if and only if $-e_i^T CA^{-1}B1_n < 1 \ \forall i$ or equivalently $-CA^{-1}B1_n < 1_n$ □

By using Lemma 3, a stability criterion for $(I - \mathbf{H}^*)^{-1}$ can be defined as $-C_\Gamma A_\Gamma^{-1}B_\Gamma|N|(\pi^* \otimes I_n)1_n < 1_n$ which using Property 2c may be written as $-C_\Gamma A_\Gamma^{-1}B_\Gamma|N|(I_p \otimes 1_n)\pi^* < 1_n$.

In conclusion, the criterions on π^* can be expressed as linear inequality conditions. An optimal choice of π^* can be found by minimizing $\Omega\pi^* - \varsigma$ with respect to some norm. When the 1-norm is used then this is equivalent to minimizing $\sum_{i=1}^{Mn} \Sigma_i\pi^*$ where Σ_i is the *i*:th row of Σ. The linear optimization problem to find π^* can thus be stated as

$$\min_{\left\{\begin{smallmatrix} \Sigma^*\pi^* \leq \varsigma^* \\ \pi^* \geq 0 \end{smallmatrix}\right.} \Psi\pi^* \tag{14}$$

where

$$\begin{aligned}\Psi =& 1_{Mn}^T\Sigma, \quad \varsigma^* = \begin{bmatrix} \varsigma^T & 1_n^T \end{bmatrix}^T \\ \Sigma^* =& \begin{bmatrix} \Sigma^T & (C_\Gamma A_\Gamma^{-1}B_\Gamma|N|(I_p \otimes 1_n))^T \end{bmatrix}^T\end{aligned}$$

Equation (14) can be solved by linear programming and the global minimum can be found.

5. APPLICATION TO JET ENGINE FAULT DETECTION

5.1 Jet engine process model

The temperature at the compressor inlet (T25), in a turbofan engine, is modelled by the following linear time-varying, first order differential equation,

$$\begin{aligned} \dot{x}(t) &= a(t)x(t) + b(t) \\ y(t) &= x(t) + \pi_y(t) \end{aligned} \tag{15}$$

where $x(t)$ and $y(t)$ are the T25 temperature and the measurement of T25, respectively. The signal π_y is measurement noise while $a(t)$ and $b(t)$ are nonlinear functions of other measurements in the jet engine, but the functions in detail are omitted here. The functions a and b are uncertain and it is thus assumed that $a = \hat{a} + \pi_a$ and $b = \hat{b} + \pi_b$ where hat signifies the known nominal value. The uncertainties can be collected into one vector as $\pi = \begin{bmatrix} \pi_a & \pi_b & \pi_y \end{bmatrix}$

5.2 Residual generation

As residual generator a linear observer with integral action and time-varying feedback is chosen. The observer is

$$\begin{aligned} \dot{\hat{x}} &= \hat{a}\hat{x} + \hat{b} + \iota + K_x(y - \hat{y}) \\ \dot{\iota} &= L_\iota(y - \hat{y}) \\ \hat{y} &= \hat{x} \end{aligned} \tag{16}$$

By choosing $K_x(t) \triangleq \hat{a}(t) + L_x$, the estimation error dynamics becomes time-invariant. The initial conditions, $\hat{x}(0)$ and $\iota(0)$ are chosen to be zero. Due to the integral action the residual, $r \triangleq y - \hat{y}$, will converge to zero even in the presence of the uncertainties π. The integral action introduces high-pass filtering of the residual, which is desirable since increased sensor noise is to be detected by the fault detection algorithm.

The dynamics of the estimation error, $\tilde{x} = x - \hat{x}$ and the integral state can be written as

$$\begin{aligned} \dot{\tilde{x}} &= -L_x\tilde{x} + \pi_a x - \iota - K_x\pi_y + \pi_b \\ \dot{\iota} &= L_\iota(y - \hat{y}) = L_\iota\tilde{x} + L_\iota\pi_y \end{aligned} \tag{17}$$

This system will not fulfill $|N|(I_p \otimes |C|^+|F|) = 0$, see Remark 1, so it needs to be transformed which can be accomplished using Lemma 2. In (Johansson and Norlander 2003) another parametrization was used and will also be chosen here. A new state variable is introduced as $\xi = \pi_a x - \iota$ for which the dynamics are

$$\begin{aligned} \dot{\xi} &= \pi_a((\hat{a} + \pi_a)x + \hat{b} + \pi_b) + \dot{\pi}_a x - L_\iota(y - \hat{y}) \\ &= \pi_a(\hat{a}y + \hat{b} + \pi_b + \iota) + \pi_a\xi - \pi_a\hat{a}\pi_y \\ &\quad + \dot{\pi}_a(y - \pi_y) - L_\iota\tilde{x} - L_\iota\pi_y \end{aligned} \tag{18}$$

where ι was added and subtracted in the first parentheses to get the equality. Products between uncertainties, $\pi_a a\pi_y$, $\pi_a\pi_b$ and $\dot{\pi}_a\pi_y$ are assumed to be small and are therefore neglected. The transformed system together with the residual, $r = y - \hat{y}$, can thus be written as

$$\begin{aligned} \dot{\xi} &= -L_\iota\tilde{x} + \pi_a\xi + H_\xi\pi_T \\ \dot{\tilde{x}} &= -L_x\tilde{x} + \xi + H_x\pi_T \\ r &= \tilde{x} + H_r\pi_T \end{aligned} \tag{19}$$

where

$$\begin{aligned} H_\xi &= \begin{bmatrix} \hat{a}y + \hat{b} + \iota & 0 & -L_\iota & y \end{bmatrix} \\ H_x &= \begin{bmatrix} 0 & 1 & -K_x & 0 \end{bmatrix} \\ H_r &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \\ \pi_T &= \begin{bmatrix} \pi_a & \pi_b & \pi_y & \dot{\pi}_a \end{bmatrix} \end{aligned}$$

By defining the new state vector $z = \begin{bmatrix} \xi & \tilde{x} \end{bmatrix}^T$ the error system can be written in the form (1) with

$$\begin{aligned} A &\triangleq \begin{bmatrix} 0 & -L_\iota \\ 1 & -L_x \end{bmatrix} \; N \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \; B \triangleq 0 \\ E &\triangleq \begin{bmatrix} H_\xi \\ H_x \end{bmatrix} \; C \triangleq \begin{bmatrix} 0 & 1 \end{bmatrix} \; D \triangleq 0 \; F \triangleq H_r \end{aligned} \tag{20}$$

It is then straightforward to show that $|N|(I_p \otimes |C|^+) = 0$ which, according to Remark 1 implies that $|N|(I_p \otimes |C|^+|F|) = 0$, which is a prerequisite for Theorem 2.

5.3 Uncertainty level derivation

The threshold for the detection algorithm is (7). The observer is assumed to have converged before using the detection algorithm and therefore the initial value x_0 is assumed to be zero. The threshold can thus be written as

$$\sigma = C(I - \mathbf{H}^*)^{-1}\mathbf{\Gamma}|E|\pi^* + F\pi^*$$

where $H^*(t) = \Gamma(t)|N|(\pi^* \otimes I_2)$. In the threshold, an upper bound $\Gamma(t)$ for the modulus of the impulse response matrix $G(t) = e^{At}$ is required. It is straightforward to show that a suitable Γ, expressed in the Laplace domain, is

$$(\mathcal{L}\Gamma)(s) = \frac{1}{s^2 + L_x s + L_\iota}\begin{bmatrix} s + L_x & L_\iota \\ 1 & \frac{sL_x/2 + L_\iota}{\sqrt{L_x^2/4 - L_\iota}} \end{bmatrix}$$

Furthermore, a vector of parameters, π^*, needs to be obtained which can be done by using (14). The functions β and Ω in Theorem 2 are

$$\begin{aligned} \Omega &= \Gamma E + |C|^+F + \Gamma|N|(I_4 \otimes |C|^+|r|) \\ \beta &= |C|^+|r| \end{aligned}$$

where $|C|^+ = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

5.4 Experimental result

The fault detection algorithm has been tested on data collected from an turbofan engine. The observer parameters are

$$L_x = 5, \; L_\iota = 2$$

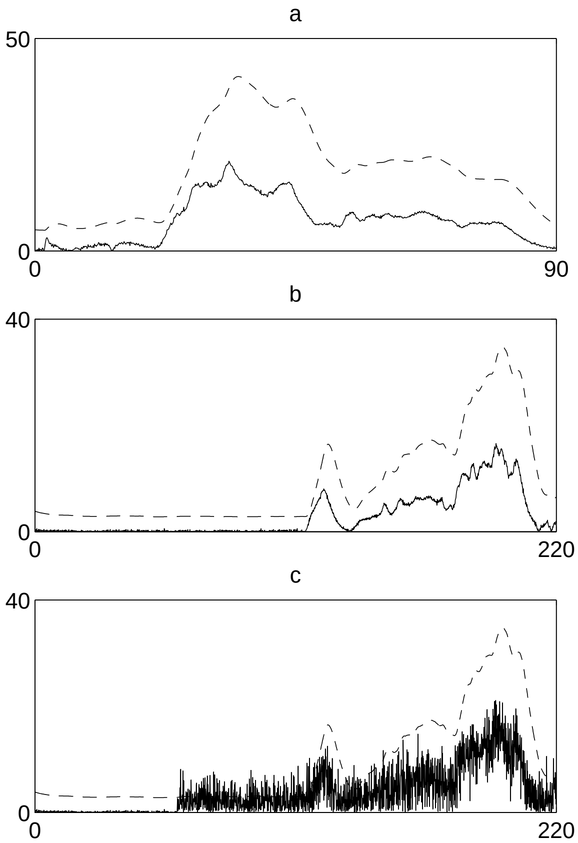

Fig. 1. A part of the identification data set is shown in figure a. Figure b shows the validation data set without any fault and c the validation data set with a white noise disturbance added to the measured T25 signal from $t = 60$s.

which were tuned manually in order to obtain an observer residual resembling white noise.

The total identification data set, from which π^* is obtained using (14), is 450 seconds and a part of the this data set is shown in Figure 1a. A validation data set where no fault in the temperature sensor is present is shown in Figure 1b. The third figure, Figure 1c, shows the residual and the threshold when a fault is simulated by adding a disturbance to the measured signal, $y(t)$, from time $t = 60$. The disturbance is a white noise signal with standard deviation 0.85. The residual exceeds the threshold and an alarm is raised at 1.1 second after the fault has occurred.

6. CONCLUSIONS AND FUTURE WORK

A model-based fault detection algorithm for linear systems with parameter uncertainty is presented. The residual is the output of a bilinear system with the uncertainties as input. The modulus of the residual is compared to a threshold which is generated by a linear system with upper bounds on the uncertainties as parameters. These bounds are in general unknown and are therefore replaced by a vector of parameters that has, so far, been tuned manually. In this paper, an automatic way to determine them by linear optimization is presented.

A fault detection algorithm to identify faults in a temperature sensor in the turbofan engine is developed using the presented algorithm and tested successfully with measured data.

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