ROBUST GLOBAL STABILIZING BOUNDED CONTROL OF A PVTOL AIRCRAFT WITH LATERAL COUPLING

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Abstract: This note provides a detailed stability analysis of a global stabilizing control algorithm for a PVTOL aircraft with lateral coupling ($\varepsilon \neq 0$) and bounded inputs. Such control approach was originally proposed considering $\varepsilon = 0$. The analysis furnished here proves the robustness of the original scheme with respect to the existence of lateral coupling. The presented methodology is based on the use of embedded saturation functions and a result of global asymptotic stabilization for cascade systems.

Keywords: nonlinear control, aircraft control, bounded inputs, saturations

1. INTRODUCTION

The recent literature shows that the planar vertical take-off and landing (PVTOL) aircraft always produces a great interest in the control community. Indeed, its mathematical model represents a challenge in nonlinear control design. The PVTOL aircraft system is also extensively used to develop and/or approximate models of flying vehicles. This can be confirmed through numerous works that have been recently contributed on Unmanned Autonomous Vehicles (UAV).

The dynamical model of the PVTOL aircraft proposed in (Hauser, et al., 1992) is given by

$$\ddot{x} = -u_1 \sin\theta + \varepsilon u_2 \cos\theta \tag{1a}$$
$$\ddot{y} = u_1 \cos\theta + \varepsilon u_2 \sin\theta - 1 \tag{1b}$$
$$\ddot{\theta} = u_2 \tag{1c}$$

where x, y, and θ refer to the center of mass position and the roll angle of the aircraft with the horizon. The variable u_1 and u_2 are respectively the thrust and the angular acceleration inputs. The constant "-1" is the normalized gravitational acceleration and ε is a coefficient which characterizes the coupling between the rolling moment u_2 and the lateral acceleration of the aircraft.

Numerous authors have proposed control methodologies for the stabilization or the trajectory tracking of the PVTOL aircraft system. A few of them are (Hauser, et al., 1992; Lin, et al., 1999; Olfati-Saber, 2002; Teel, 1996; Zavala-Río, et al., 2003). In these works, the controllers have been designed either neglecting the coupling between the rolling moment and the lateral acceleration or considering the exact knowledge of this term. In the first case, ε is regarded as so small that $\varepsilon = 0$ is supposed in (1) (see for instance (Hauser, et al., 1992, §2.4)). In the second situation, the authors mostly use a globally invertible nonlinear coordinate transformation such that in the new state representation such coupling effect does not explicitly appear (Olfati-Saber, 2002). The control designs and the stability analyses have thus been developed for the transformed system without considering the coupling.

From all the works previously cited, only (Zavala-Río, et al., 2003) has developed a global stabilizing scheme considering bounded inputs. Moreover, the control algorithm proposed in such work takes into account the positive nature of the thrust.

However, robustness of the previously proposed algorithms have been scarcely addressed. As far as the authors are aware, only Lin, et al. (1999) have developed a robust control setting for the PVTOL aircraft with respect to uncertainty of the coupling parameter. A nominal value of ε is however required. Their algorithm is based on an optimal control solution. Furthermore, Teel (1996) proposed a control law depending on the exact value of ε and showed through numerical simulations the robustness of his approach when initial conditions are close enough to the origin. Due to its dependence on the physical parameters of the aircraft, the supposition that ε is exactly known can be defended (see (Olfati-Saber, 2002)). Nevertheless, its exact value can be difficult to measure or estimate in real experiments.

In the present paper, the crucial contribution is to demonstrate that using the control methodology previously presented in (Zavala-Río, et al., 2003), where $\varepsilon = 0$ was supposed, global stabilization is achieved despite the presence of lateral coupling. This corroborates the robustness of such a control approach. The algorithm is based on the use of the embedded saturation function methodology proposed by Teel (1996). The closed-loop stability analysis leans on the result stated by Sontag (1989) for cascade systems relying on a *converging input bounded state* (CIBS) property.

The paper is organized as follows. Section 2 recalls the approach presented in (Zavala-Río, et al., 2003). Section 3 details the closed-loop stability analysis including the lateral coupling. Some experimental results are provided in Section 4. Finally, conclusions are given in Section 5.

2. GLOBAL STABILIZATION ALGORITHM

Before recalling the control law, the reader is invited to consult (Zavala-Río, et al., 2003) for a detailed description of the conceptual setting underlying the proposed approach. In such reference, we have considered the PVTOL aircraft dynamics with $\varepsilon = 0$, i.e

$$\ddot{x} = -u_1 \sin\theta \ , \quad \ddot{y} = u_1 \cos\theta - 1 \ , \quad \ddot{\theta} = u_2 \quad (2)$$

We recall the control objective, stated as the global asymptotic stabilization of the system towards $(x, \dot{x}, y, \dot{y}, \theta, \dot{\theta}) = (0, 0, 0, 0, 0, 0)$ considering bounded inputs, i.e. $0 \le u_1 \le U_1$ and $|u_2| \le U_2$ for some constants $U_1 > 1$ and $U_2 > 0$.

Note that the idea underlying the control algorithm proposed in this article and that in (Olfati-Saber, 2002) are similar except that the latter considers unbounded inputs. Moreover, the present approach is robust in the sense that it does not depend on the exact value of ε, but on a partial knowledge of it (i.e. the global stabilization objective is achieved provided that ε is small enough).

The approach is based on *linear saturation functions*, as defined in (Teel, 1992), and a special type of them named *2-level linear saturation functions*, whose definitions are recalled here.

Definition 1. Given positive constants L and M, with $L \le M$, a function $\sigma : \mathbb{R} \to \mathbb{R}$ is said to be a ***linear saturation*** for (L, M) if it is a continuous, nondecreasing function satisfying

(a) $\sigma(s) = s$ when $|s| \le L$
(b) $|\sigma(s)| \le M$ for all $s \in \mathbb{R}$

Definition 2. Given positive constants L^+, M^+, N^+, L^-, M^-, and N^-, with $L^\pm \le \min\{M^\pm, N^\pm\}$, a function $\sigma : \mathbb{R} \to \mathbb{R}$ is called a ***2-level linear saturation*** for $(L^+, M^+, N^+, L^-, M^-, N^-)$ if it is a continuous, nondecreasing function satisfying

(a) $\sigma(s) = s$ for all $s \in [-L^-, L^+]$
(b) $-M^- < \sigma(s) < M^+$ for all $s \in (-N^-, N^+)$
(c) $\sigma(s) = -M^-$ for all $s \le -N^-$
(d) $\sigma(s) = M^+$ for all $s \ge N^+$

Notice that a 2-level linear saturation for $(L^+$, M^+, N^+, L^-, M^-, $N^-)$ is a linear saturation for $(\min\{L^-, L^+\}, \max\{M^-, M^+\})$.

We recall the proposed control algorithm (from (Zavala-Río, et al., 2003)): the thrust input u_1 and the rolling moment u_2 are expressed by

$$u_1 = \sqrt{r_1^2 + (1 + r_2)^2} \quad (3)$$

$$\begin{aligned} u_2 = \sigma_{41}(\ddot{\theta}_d) - \sigma_{32}(\dot{\theta} - \sigma_{42}(\dot{\theta}_d) \\ + \sigma_{31}(\dot{\theta} - \sigma_{43}(\dot{\theta}_d) + \theta - \theta_d)) \end{aligned} \quad (4)$$

where r_1, r_2 and θ_d are defined as follows

$$r_1 = -k\sigma_{12}(\dot{x} + \sigma_{11}(kx + \dot{x})) \quad (5)$$

$$r_2 = -\sigma_{22}(\dot{y} + \sigma_{21}(y + \dot{y})) \quad (6)$$

$$\theta_d = \arctan(-r_1, 1 + r_2) \quad (7)$$

with k in (5) a constant satisfying

$$0 < k < 1 \quad (8a)$$

the functions $\sigma_{ij}(\cdot)$ in (5) and (6) are twice differentiable 2-level linear saturations for given (L^+_{ij}, M^+_{ij}, N^+_{ij}, L^-_{ij}, M^-_{ij}, N^-_{ij}) such that

$$B_{u_1} \triangleq \sqrt{(kM_{12})^2 + (1+M^-_{22})^2} < U_1 \tag{8b}$$
$$M^+_{22} < 1 \tag{8c}$$
$$M_{i1} < \frac{L_{i2}}{2}, \forall i = 1,2 \tag{8d}$$

with $M_{ij} \triangleq \max\{M^-_{ij}, M^+_{ij}\}$ and $L_{ij} \triangleq \min\{L^-_{ij}, L^+_{ij}\}$, $i = 1,2$, $j = 1,2$, and the functions $\sigma_{mn}(\cdot)$ in (4) are linear saturations for given (L_{mn}, M_{mn}) such that

$$M_{41} + M_{32} < U_2 \tag{9a}$$
$$M_{41} + 2M_{42} + 2M_{31} < L_{32} \tag{9b}$$
$$M_{41} + M_{42} + 2M_{43} + 2B_{\theta_d} < L_{31} \tag{9c}$$

with $B_{\theta_d} \triangleq \arctan\left(\frac{kM_{12}}{1-M^+_{22}}\right)$. Further, defining $\bar{r}_1 \triangleq -\sigma_{12}(\dot{x} + \sigma_{11}(kx + \dot{x})) = \frac{r_1}{k}$, the first and second time-derivatives of θ_d, used in (4), are given by

$$\dot{\theta}_d = k\left(\frac{\bar{r}_1\dot{r}_2 - (1+r_2)\dot{\bar{r}}_1}{u_1^2}\right) \tag{10}$$
$$\ddot{\theta}_d = k\left(\frac{\bar{r}_1\ddot{r}_2 - (1+r_2)\ddot{\bar{r}}_1}{u_1^2}\right) - \frac{2\dot{u}_1\dot{\theta}_d}{u_1} \tag{11}$$

while those of $\bar{r}_1$ and r_2, and $\dot{u}_1$ by

$$\begin{aligned}
\dot{\bar{r}}_1 &= -\sigma'_{12}(s_{12})\dot{s}_{12} \\
\dot{r}_2 &= -\sigma'_{22}(s_{22})\dot{s}_{22} \\
\ddot{\bar{r}}_1 &= -\sigma''_{12}(s_{12})\dot{s}^2_{12} - \sigma'_{12}(s_{12})\ddot{s}_{12} \\
\ddot{r}_2 &= -\sigma''_{22}(s_{22})\dot{s}^2_{22} - \sigma'_{22}(s_{22})\ddot{s}_{22} \\
\dot{u}_1 &= \frac{k^2\bar{r}_1\dot{\bar{r}}_1 + (1+r_2)\dot{r}_2}{u_1}
\end{aligned} \tag{12}$$

with $\sigma'_{ij}(s_{ij}) = \frac{d\sigma_{ij}}{ds_{ij}}$, $\sigma''_{ij}(s_{ij}) = \frac{d^2\sigma_{ij}}{ds^2_{ij}}$,

$$\begin{aligned}
s_{12} &= \dot{x} + \sigma_{11}(s_{11}) \\
s_{22} &= \dot{y} + \sigma_{21}(s_{21}) \\
\dot{s}_{12} &= a_x + \sigma'_{11}(s_{11})\dot{s}_{11} \\
\dot{s}_{22} &= a_y + \sigma'_{21}(s_{21})\dot{s}_{21} \\
\ddot{s}_{12} &= \dot{a}_x + \sigma''_{11}(s_{11})\dot{s}^2_{11} + \sigma'_{11}(s_{11})\ddot{s}_{11} \\
\ddot{s}_{22} &= \dot{a}_y + \sigma''_{21}(s_{21})\dot{s}^2_{21} + \sigma'_{21}(s_{21})\ddot{s}_{21}
\end{aligned}$$

$$\begin{aligned}
s_{11} &= kx + \dot{x} & & \\
s_{21} &= y + \dot{y} & & \\
\dot{s}_{11} &= k\dot{x} + a_x \quad , & a_x &= -u_1 \sin\theta \\
\dot{s}_{21} &= \dot{y} + a_y \quad , & a_y &= u_1\cos\theta - 1 \\
\ddot{s}_{11} &= ka_x + \dot{a}_x \quad , & \dot{a}_x &= -u_1\dot{\theta}\cos\theta - \dot{u}_1\sin\theta \\
\ddot{s}_{21} &= a_y + \dot{a}_y \quad , & \dot{a}_y &= -u_1\dot{\theta}\sin\theta + \dot{u}_1\cos\theta
\end{aligned}$$

where the accelerations ($\ddot{x}$ and $\ddot{y}$) have been replaced by their expressions in (2) every time they appeared in the derivation procedure (recall that $\varepsilon = 0$ was originally supposed). Subsequently, $\|\cdot\|$ will represent the standard Euclidean vector norm i.e. $\|\xi\| = \left[\sum_{i=1}^n \xi_i^2\right]^{1/2}$, $\forall \xi \in I\!R^n$.

3. ROBUST GLOBAL STABILIZATION ALGORITHM

Theorem 1. Consider the PVTOL aircraft dynamics (1) with input saturation bounds $U_1 > 1$ and $U_2 > 0$. Let the input thrust u_1 be defined as in (3),(5),(6), with constant k and parameters (L^+_{ij}, M^+_{ij}, N^+_{ij}, L^-_{ij}, M^-_{ij}, N^-_{ij}) of the twice differentiable 2-level linear saturation functions $\sigma_{ij}(\cdot)$ in (5) and (6) satisfying inequalities (8), and the input rolling moment u_2 as in (4),(7), with parameters (L_{mn}, M_{mn}) of the linear saturation functions $\sigma_{mn}(\cdot)$ in (4) satisfying inequalities (9). Then, provided that k and ε are sufficiently small,

(i) global asymptotic stabilization of the closed-loop system (1),(3)–(7) towards (x, $\dot{x}$, y, $\dot{y}$, θ, $\dot{\theta}$) = (0,0,0,0,0,0) is achieved, with
(ii) $0 < 1 - M^+_{22} \leq u_1(t) \leq B_{u_1} < U_1$ and $|u_2(t)| \leq M_{41} + M_{32} < U_2$, $\forall t \geq 0$.

Proof. Property (ii) of the statement is a direct consequence of the definitions of u_1, u_2, r_1, and r_2. Its proof is consequently straightforward. The proof of property (i) is divided in four parts. The first part shows that θ_d, $\dot{\theta}_d$, and $\ddot{\theta}_d$ are (uniformly or ultimately) bounded signals whose (uniform or ultimate) bounds are directly influenced by the parameter k. This is essential within the closed-loop stability analysis which is developed in the remaining stages of the proof. The second part shows that for any initial condition vector $\zeta(0) \in I\!R^6$, with $\zeta \triangleq (x, \dot{x}, y, \dot{y}, \theta, \dot{\theta})^T$, (provided that k is sufficiently small) there exists a finite time $t' \geq 0$ after which the system trajectories evolve within a positively invariant set (containing the origin of $I\!R^6$) where every linear saturation function $\sigma_{mn}(\cdot)$ in (4) is equal to its argument. As a consequence, by defining $z = (z_1, z_2, z_3, z_4)^T \triangleq (x, \dot{x}, y, \dot{y})^T$ and $e = (e_1, e_2)^T \triangleq (\theta - \theta_d, \dot{\theta} - \dot{\theta}_d)^T$, the closed-loop dynamics get (from t' on) a state-space representation of the form

$$\dot{z} = f(z, e) \tag{13a}$$
$$\dot{e} = g(e) \tag{13b}$$

with $f(0_4, 0_2) = 0_4$ and $g(0_2) = 0_2$, 0_n denoting the origin of $I\!R^n$, where $e = 0_2$ is a globally asymptotically stable (GAS) equilibrium state of subsystem (13b). From the expressions in (3)–(12), it is not hard to see that $(z^T, e^T)^T = 0_6 \iff \zeta = 0_6$. The third part of the proof shows that (provided that k and ε are sufficiently small) the trajectories of subsystem (13a) exist and are bounded for any bounded e converging to 0_2 asymptotically in time. Consequently, according to the result stated in (Sontag, 1989), system (13) has $(z^T, e^T)^T = 0_6$ as a GAS equilibrium state if $z = 0_4$ is itself a GAS equilibrium state of

$$\dot{z} = f(z, 0_2) \tag{14}$$

Such a stability property of system (14) is shown to be satisfied in the fourth part of the proof.

1st and 2nd parts. These are thoroughly developed within the proof of Theorem 1 in (Zavala-Río, et al., 2003). Due to space limitations, the reader is invited to consult such reference. For the development of the subsequent stages, two facts shown therein shall be retained:

F1. for all $t \geq t'$, every linear saturation function $\sigma_{mn}(\cdot)$ in (4) is equal to its argument;
F2. $|\ddot{\theta}_d(t)| \leq kE_2$, $\forall t \geq t'$, for some initial-condition-independent constant $E_2 > 0$. [1]

3rd part. As a consequence of fact F1, under the state-space representation adopted above, u_2 in (4) becomes (from t' on)

$$u_2 = \ddot{\theta}_d - 2e_2 - e_1 \tag{15}$$

while subsystem (13b) takes the form

$$\dot{e} = Ae \tag{16}$$

with $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$. Since A is Hurwitz, $e = 0_2$ is a GAS equilibrium state of subsystem (13b). Thus, $e(t)$ is indeed bounded and converges to 0_2, i.e. $\exists B_e = B_e(\|e(t')\|)$ such that $\|e(t)\| \leq B_e$, $\forall t \geq t'$, and $\|e(t)\| \to 0$ as $t \to \infty$. Furthermore, subsystem (13a) can be expressed as

$$\dot{z}_1 = z_2 \tag{17}$$
$$\dot{z}_2 = -k\sigma_{12}(z_2 + \sigma_{11}(kz_1 + z_2)) + R_1(z, e) \tag{18}$$
$$\dot{z}_3 = z_4 \tag{19}$$
$$\dot{z}_4 = -\sigma_{22}(z_4 + \sigma_{21}(z_3 + z_4)) + R_2(z, e) \tag{20}$$

where $R_1(z, e) = -u_1[\sin(e_1 + \theta_d) - \sin\theta_d] + \varepsilon u_2 \cos(e_1 + \theta_d)$ and $R_2(z, e) = u_1[\cos(e_1 + \theta_d) - \cos\theta_d] + \varepsilon u_2 \sin(e_1 + \theta_d)$. Let us note that from (15), fact F2, and the facts that $|\sin(e_1 + \theta_d) - \sin\theta_d| \leq |e_1|$, $|\cos(e_1 + \theta_d) - \cos\theta_d| \leq |e_1|$, $|e_1| \leq \|e\|$, and $|2e_2 + e_1| = |(1, 2)e| \leq \|(1, 2)^T\| \, \|e\| = \sqrt{5}\|e\|$, we have

$$|R_i(z, e)| \leq \varepsilon kE_2 + B'\|e\| \tag{21}$$

$i = 1, 2$, where $B' \triangleq B_{u_1} + \sqrt{5}\varepsilon$ (see (8b)). Let us further note that due to the smoothness properties of every term in system (16)–(20), global existence and uniqueness of the system state trajectories follow if they are proved to be bounded (see for instance (Khalil, 2002, Thrm. 3.3)). Boundedness of $e(t)$ has already been shown. That of $z(t)$ is proved next.

Let us first analyze the vertical motion dynamics, i.e. equations (19) and (20). Notice from (20) and (21) that

$$|\dot{z}_4(t)| \leq M_{22} + \varepsilon kE_2 + B'B_e \tag{22}$$

$\forall t \geq t'$ (where the fact that $\|e(t)\| \leq B_e$, $\forall t \geq t'$, for some B_e, has been considered). Inequality (22) shows that the absolute value of the vertical acceleration is bounded by a (positive) constant. Therefore, $z_3(t)$ and $z_4(t)$ exist and are bounded at any finite time. Now, since $\|e(t)\| \to 0$ as $t \to \infty$, for any positive δ, there exists a time $t_1 \geq t'$ such that $\|e(t)\| \leq \delta$, $\forall t \geq t_1$. Take

$$\delta < \frac{\gamma - \varepsilon kE_2}{B'} \tag{23}$$

where

$$\gamma \triangleq \min\{L_{21}, L_{22} - 2M_{21}, kL_{11}, k(L_{12} - 2M_{11})\} \tag{24}$$

and suppose that k and ε satisfy

$$\varepsilon kE_2 < \gamma \tag{25}$$

Let us define the function $V_1 = z_4^2$. Its derivative along the system trajectories is given by

$$\begin{aligned} \dot{V}_1 &= 2z_4\dot{z}_4 \\ &= 2z_4 \left[-\sigma_{22}(z_4 + \sigma_{21}(z_3 + z_4)) + R_2(z, e)\right] \end{aligned} \tag{26}$$

Notice from (21) that, for all $t \geq t_1$, $|R_2(z, e)| \leq \varepsilon kE_2 + B'\delta$. Suppose for the moment that $z_4 > M_{21} + \varepsilon kE_2 + B'\delta > 0$. Under such an assumption, we have $z_4 + \sigma_{21}(\cdot) > \varepsilon kE_2 + B'\delta > 0$. Then, according to Definition 2, either $\sigma_{22}(\cdot) \in (0, L_{22}^+]$ implying (from (20)) $\dot{z}_4 = -z_4 - \sigma_{21}(\cdot) + R_2(z, e) < M_{21} + \varepsilon kE_2 + B'\delta - z_4 < 0$, or $\sigma_{22}(\cdot) \in (L_{22}^+, M_{22}^+]$ entailing $\dot{z}_4 = -\sigma_{22}(\cdot) + R_2(z, e) < \varepsilon kE_2 + B'\delta - L_{22}^+ < 0$, since, from (23) and (24), $\varepsilon kE_2 + B'\delta < \gamma \leq L_{21} \leq M_{21} < \frac{L_{22}}{2} < L_{22}^+$. Hence,

$$z_4 > M_{21} + \varepsilon kE_2 + B'\delta > 0 \implies \dot{z}_4 < 0 \tag{27}$$

Similarly, if $z_4 < -M_{21} - \varepsilon kE_2 - B'\delta < 0$, which implies $z_4 + \sigma_{21}(\cdot) < -\varepsilon kE_2 - B'\delta < 0$, then either $\sigma_{22}(\cdot) \in [-L_{22}^-, 0)$ entailing $\dot{z}_4 = -z_4 - \sigma_{21}(\cdot) + R_2(z, e) > -M_{21} - \varepsilon kE_2 - B'\delta - z_4 > 0$, or $\sigma_{22}(\cdot) \in [-M_{22}^-, -L_{22}^-)$ implying $\dot{z}_4 = -\sigma_{22}(\cdot) + R_2(z, e) > L_{22}^- - \varepsilon kE_2 - B'\delta > 0$, since, from (23) and (24), $\varepsilon kE_2 + B'\delta < \gamma \leq L_{21} \leq M_{21} < \frac{L_{22}}{2} < L_{22}^-$. Thus,

$$z_4 < -M_{21} - \varepsilon kE_2 - B'\delta < 0 \implies \dot{z}_4 > 0 \tag{28}$$

Therefore, from (27) and (28), we see that $|z_4| > M_{21} + \varepsilon kE_2 + B'\delta \implies \text{sign}(z_4) \neq \text{sign}(\dot{z}_4) \iff \dot{V}_1 < 0$. This proves that, for any $z(t_1) \in I\!R^4$, there is a time $t_2 \geq t_1$ such that $|z_4(t)| \leq M_{21} + \varepsilon kE_2 + B'\delta$, $\forall t \geq t_2$. Then, for all $t \geq t_2$, we have $|z_4 + \sigma_{21}(\cdot)| \leq |z_4| + M_{21} \leq 2M_{21} + \varepsilon kE_2 + B'\delta < L_{22}$, since, from (23) and (24), $\varepsilon kE_2 + B'\delta < \gamma \leq L_{22} - 2M_{21}$. Consequently (according to property (a) of Definition 2) (20) becomes $\dot{z}_4 = -z_4 - \sigma_{21}(z_3 + z_4) + R_2(z, e)$ (from t_2 on). Let us now define $q_1 \triangleq z_3 + z_4$ and the function $V_2 \triangleq q_1^2$. The derivative of V_2 along the system trajectories is given by $\dot{V}_2 = 2q_1\dot{q}_1 = 2q_1\left[-\sigma_{21}(q_1) + R_2(z, e)\right]$. Following a similar reasoning that the one developed above for the analysis of (26), one sees that $|q_1| > \varepsilon kE_2 +$

[1] A *worst-case* estimation of such a constant, E_2, is given in the proof of Theorem 1 in (Zavala-Río, et al., 2003).

$B'\delta \implies \mathrm{sign}(q_1) \neq \mathrm{sign}(\dot{q}_1) \iff \dot{V}_2 < 0$. Hence, for any $z(t_1) \in I\!R^4$, there is a time $t_3 \geq t_2$ such that $|q_1(t)| \leq \varepsilon k E_2 + B'\delta$, $\forall t \geq t_3$. Therefore,

$$(z_3(t), z_4(t)) \in \mathcal{S}_1 \triangleq \big\{(z_3, z_4) \in I\!R^2 \mid |z_4| \leq M_{21} + \varepsilon k E_2 + B'\delta, \ |z_3 + z_4| \leq \varepsilon k E_2 + B'\delta\big\} \quad (29)$$

$\forall t \geq t_3$. Notice that $\mathcal{S}_1$ is a compact subset of $I\!R^2$. So far, existence and boundedness of $z_3(t)$ and $z_4(t)$ for all $t \geq t'$ are concluded.

Let us now analyze the horizontal motion dynamics, i.e. equations (17) and (18). Notice from (18) and (21) that $|\dot{z}_2(t)| \leq kM_{12} + \varepsilon k E_2 + B'B_e$, $\forall t \geq t'$, showing that $z_1(t)$ and $z_2(t)$ exist and are bounded at any finite time. Let us define the function $V_3 = z_2^2$. Its derivative along the system trajectories is $\dot{V}_3 = 2z_2\dot{z}_2 = 2z_2\left[-k\sigma_{12}(z_2 + \sigma_{11}(kz_1 + z_2)) + R_1(z, e)\right]$. Following a similar procedure that the one developed above for the analysis of (26), one sees that $|z_2| > M_{11} + \varepsilon E_2 + \frac{B'\delta}{k} \implies \mathrm{sign}(z_2) \neq \mathrm{sign}(\dot{z}_2) \iff \dot{V}_3 < 0$. This proves that, for any $z(t_1) \in I\!R^4$, there is a time $t_4 \geq t_1$ such that $|z_2(t)| \leq M_{11} + \varepsilon E_2 + \frac{B'\delta}{k}$, $\forall t \geq t_4$. Then, for all $t \geq t_4$, we have $|z_2 + \sigma_{11}(\cdot)| \leq |z_2| + M_{11} \leq 2M_{11} + \varepsilon E_2 + \frac{B'\delta}{k} < L_{12}$, since, from (23) and (24), $\varepsilon k E_2 + B'\delta < \gamma \leq k(L_{12} - 2M_{11})$. Consequently (according to property (a) of Definition 2) (20) becomes $\dot{z}_2 = -kz_2 - k\sigma_{11}(kz_1 + z_2) + R_1(z, e)$ (from t_4 on). Let us now define $q_2 \triangleq kz_1 + z_2$ and the function $V_4 \triangleq q_2^2$. The derivative of V_4 along the system trajectories is $\dot{V}_4 = 2q_2\dot{q}_2 = 2q_2\left[-k\sigma_{11}(q_2) + R_1(z, e)\right]$. Following a similar reasoning that the one developed above for the analysis of (26), one sees that $|q_2| > \varepsilon E_2 + \frac{B'\delta}{k} \implies \mathrm{sign}(q_2) \neq \mathrm{sign}(\dot{q}_2) \iff \dot{V}_4 < 0$. Hence, for any $z(t_1) \in I\!R^4$, there is a time $t_5 \geq t_4$ such that $|q_2(t)| \leq \varepsilon E_2 + \frac{B'\delta}{k}$, $\forall t \geq t_5$. Therefore,

$$(z_1(t), z_2(t)) \in \mathcal{S}_2 \triangleq \left\{(z_1, z_2) \in I\!R^2 \mid |z_2| \leq M_{11} + \varepsilon E_2 + \frac{B'\delta}{k}, \ |z_1 + z_2| \leq \varepsilon E_2 + \frac{B'\delta}{k}\right\} \quad (30)$$

$\forall t \geq t_5$. Note that $\mathcal{S}_2$ is a compact subset of $I\!R^2$. Therefore, existence and boundedness of $z_1(t)$ and $z_2(t)$ for all $t \geq t'$ are concluded. Finally, from (29) and (30) we see that $z(t) \in \mathcal{S}_{12} \triangleq \mathcal{S}_1 \times \mathcal{S}_2$, $\forall t \geq t'' \triangleq \max\{t_3, t_5\}$, $\mathcal{S}_{12}$ being a compact subset of $I\!R^4$. Existence and boundedness of $z(t)$ for all $t \geq t'$ are therefore concluded.

4th part. As a consequence of fact F1, system (14) takes the form

$$\dot{z}_1 = z_2 \quad (31)$$
$$\dot{z}_2 = -k\sigma_{12}(z_2 + \sigma_{11}(kz_1 + z_2)) + \varepsilon\ddot{\theta}_d \cos\theta_d \quad (32)$$
$$\dot{z}_3 = z_4 \quad (33)$$
$$\dot{z}_4 = -\sigma_{22}(z_4 + \sigma_{21}(z_3 + z_4)) + \varepsilon\ddot{\theta}_d \sin\theta_d \quad (34)$$

A careful reading of the 3rd part of the proof shows that if $e_1 = e_2 = \delta = B_e = 0$ is taken, the analysis holds with $t_1 = t'$. Then $z(t) \in \mathcal{S} \triangleq \big\{z \in I\!R^4 \mid |z_2| \leq M_{11} + \varepsilon E_2, \ |z_1 + z_2| \leq \varepsilon E_2, \ |z_4| \leq M_{21} + \varepsilon k E_2, \ |z_3 + z_4| \leq \varepsilon k E_2\big\}$, $\forall t \geq t''$, for some finite time $t'' \geq t'$, $\mathcal{S}$ being a compact subset of $I\!R^4$ containing the origin. Consequently, for all $t \geq t''$, we have $|z_4 + \sigma_{21}(\cdot)| \leq |z_4| + M_{21} \leq 2M_{21} + \varepsilon k E_2 < 2M_{21} + \gamma \leq L_{22}$, $|z_3 + z_4| \leq \varepsilon k E_2 < \gamma \leq L_{21}$, $|z_2 + \sigma_{11}(\cdot)| \leq |z_2| + M_{11} \leq 2M_{11} + \varepsilon E_2 < 2M_{11} + \frac{\gamma}{k} \leq L_{12}$, and $|z_1 + z_2| \leq \varepsilon E_2 < \frac{\gamma}{k} \leq L_{11}$ (see (25) and (24)). Hence, the 2-level linear saturation functions $\sigma_{ij}(\cdot)$ in equations (32) and (34) are equal to their argument. Consequently, for all $t \geq t''$, system (31)–(34) becomes

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -k^2 z_1 - 2kz_2 + \varepsilon\ddot{\theta}_d \cos\theta_d \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -z_3 - 2z_4 + \varepsilon\ddot{\theta}_d \sin\theta_d \end{aligned} \quad (35)$$

On the other hand, $\sigma'_{ij}(\cdot) = 1$ and $\sigma''_{ij}(\cdot) = 0$ for every $\sigma'_{ij}(\cdot)$ and $\sigma''_{ij}(\cdot)$ in equations (12). Therefore

$$\begin{aligned} \ddot{r}_1 &= 2k(\dot{\theta}_d u_1 \cos\theta_d + \dot{u}_1 \sin\theta_d) + k^2 u_1 \sin\theta_d \\ \ddot{r}_2 &= 2(\dot{\theta}_d u_1 \sin\theta_d - \dot{u}_1 \cos\theta_d) - (u_1 \cos\theta_d - 1) \end{aligned} \quad (36)$$

(recall that $e = 0_2$, hence $\theta = \theta_d$ and $\dot{\theta} = \dot{\theta}_d$, is being considered). Since $u_1 \sin\theta_d = -r_1$ and $u_1 \cos\theta_d = 1 + r_2$, and from the expressions of $\dot{\theta}_d$ in (10) and $\dot{u}_1$ in (12), the equations in (36) are actually equivalent to

$$\begin{aligned} \ddot{r}_1 &= -2k\dot{r}_1 - k^2 r_1 \\ \ddot{r}_2 &= -2\dot{r}_2 - r_2 \end{aligned} \quad (37)$$

Let $\rho \triangleq (r_1, \dot{r}_1, r_2, \dot{r}_2)^T$. Observe from (11) and (37) that $\ddot{\theta}_d$ is a function of ρ, $\ddot{\theta}_d = \ddot{\theta}_d(\rho)$, with $\ddot{\theta}_d(0_4) = 0$. From this and equations (37), one sees that system (35) may be represented as

$$\dot{z} = f_1(z, \rho) = A'z + \varepsilon R(\rho) \quad (38a)$$
$$\dot{\rho} = g_1(\rho) \quad = A'\rho \quad (38b)$$

with $A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k^2 & -2k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$ and $R(\rho) = \left(0\ ,\ \ddot{\theta}_d \cos\theta_d\ ,\ 0\ ,\ \ddot{\theta}_d \sin\theta_d\right)^T$. Notice that $R(0_4) = 0_4$ (since $\ddot{\theta}_d(0_4) = 0$), and consequently $f_1(0_4, 0_4) = g_1(0_4) = 0_4$. Since A' is a Hurwitz matrix, $\rho = 0_4$ is a GAS equilibrium state of subsystem (38b). Then, $\rho(t)$ is bounded and converges to 0_4, i.e. $\exists B_\rho = B_\rho(\|\rho(t'')\|)$ such that $\|\rho(t)\| \leq B_\rho$,

$\forall t \geq t''$, and $\|\rho(t)\| \to 0$ as $t \to \infty$. Now, from (11) and (37), one can verify that $\|R(\rho)\| = |\ddot{\theta}_d(\rho)| \leq r(\|\rho\|)\|\rho\|$ for some function $r(\|\rho\|) > 0$. Then $\|R(\rho)\| \leq r(B_\rho)B_\rho$ and $\|R(\rho(t))\| \to 0$ as $t \to \infty$. Since the states of GAS linear time invariant systems with bounded inputs exist and are bounded globally in time (see for instance (Khalil, 2002, §4.9)), system (38) has $(z^T, \rho^T)^T = 0_8$ as a GAS equilibrium state if $z = 0_4$ is itself a GAS equilibrium state of $\dot{z} = A'z$, according to (Sontag, 1989). Therefore, since A' is a Hurwitz matrix, the proof follows. □

4. EXPERIMENTAL RESULTS

Numerical results with several values of $\varepsilon \neq 0$ are shown in (Zavala-Río, et al., 2003, §4). Here, we present some preliminary experimental results obtained when the control strategy proposed above is applied to a real prototype: the four-rotor Draganflyer III helicopter. In this device, the front and rear motors rotate counter-clockwise while the other two rotate clockwise. When the yaw and roll angles are set to zero, this helicopter reduces to a PVTOL system. We have used a Futaba Skysport 4 radio for transmitting the control signals; these are referred as the *throttle* (u_1) and the *pitch* (u_2) control inputs. They are constrained in the radio to satisfy $0.66\,\mathrm{V} < u_1 < 4.70\,\mathrm{V}$ and $1.23\,\mathrm{V} < u_2 < 4.16\,\mathrm{V}$. In order to measure the configuration (x, y, θ) of the mini helicopter, we have used a 3D tracker system (POLHEMUS). The computation of the control inputs requires the knowledge of various angular and linear velocities. We have obtained the angular velocity by means of a gyro Murata ENV-05F-03. Linear velocities were approximated as $\dot{q} = \frac{q_t - q_{t-T}}{T}$ where T is the sampling period ($T = 0.05\,\mathrm{sec}$, in our experiment). The initial conditions and desired configuration were $(x_0, y_0, \theta_0) = (0, 30\,\mathrm{cm}, 0.1\,\mathrm{rad})$ and $(x_d, y_d, \theta_d) = (0, 50\,\mathrm{cm}, 0)$. In order to ease the displacement of the helicopter altitude, small step inputs were gradually added to y_d around the reference value (50 cm) between 10 sec and 80 sec. In Fig. 1, we can see that the altitude y follows the reference. Concerning the position x, we observe a small deviation (2 cm) due to, among others, uncertainties and cable connections between the PC and the mini helicopter. The angle θ converges to zero and the control inputs are bounded. In all the figures, we note that the signals are corrupted by noise due to mechanical gears of motors and propellers. Furthermore, uncertainties in the responses are also caused by the difficulty to adjust gains and couplings existing in the four-rotor helicopter, which have not been taken into account in the analysis. However, the preliminary experimental results presented here show that the control strategy works on a real experiment.

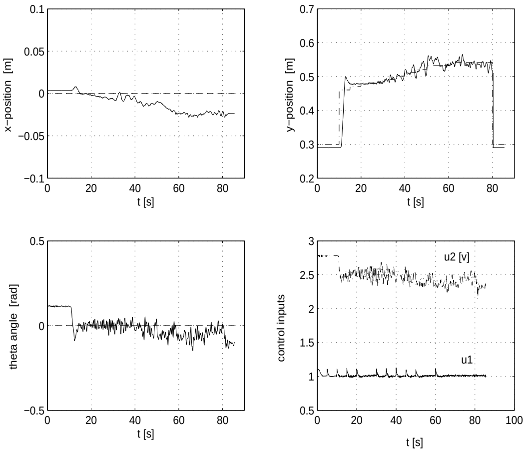

Fig. 1. System states and control inputs (— real data, $\cdot - \cdot$ reference)

5. CONCLUSIONS

In this work, robustness of a global stabilizing control for the PVTOL aircraft with bounded inputs has been addressed. The control approach, which takes into account the positive nature of the thrust, had been recently published considering $\varepsilon = 0$. Here, it has been proved that such algorithm achieves the global stabilization objective even with (small enough) $\varepsilon \neq 0$. The analysis developed is based on the use of embedded saturations and a result of global asymptotic stabilization for cascade systems.

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