STATE DEPENDENT SWITCHING CONTROL FOR INVERTED PENDULUM SYSTEM

Satoko YAMAKAWA[1], Atsushi YAMADA[2] and Hideo FUJIMOTO[2]

[1] *Dept. of System Robotics, Toyo University, 2100 Kujirai, Kawagoe 350-8585, Japan*
yamakawa@eng.toyo.ac.jp
[2] *Dept. of Computer Science and Engineering, Nagoya Institute of Technology*
Gokiso-cho, Showa-ku, Nagoya 466-8555, Japan

Abstract: A switching control strategy for inverted pendulum systems is proposed based on the energy function and its transitions. We deal with a simplified second order model of cart-pendulum systems. Assume a virtual energy whose potential energy has an opposite sign. Paying attention to the motions of the pendulum, the energy changes are analyzed theoretically. According to these analyses, the conditions of a simple switching controller are derived and the stability of the switching controlled system is guaranteed. The role of the parameters in controllers is examined in numerical simulations.

Keywords: Switching control, Energy dependence, Inverted pendulum, Lyapunov function

1. INTRODUCTION

The control design based on the Lyapunov stability theory has often been used for stabilizing nonlinear systems. In these approaches, mechanical energy of systems is a useful candidate of Lyapunov functions because it helps us to intuitively recognize the physical characteristics of controlled behavior (e.g., Bloch, Chang, Leonard and Marsden(2001), Angeli (2001)). However, the design processes of continuous controllers often become comparatively complex because of the nonlinearity of the system, and obtained nonlinear controllers mostly become complex. On the contrary, by using the switching process in control, simple controllers for inverted pendulum systems, which are nonlinear systems, have been proposed (Åström and Furuta(2000), Shiriaev, Egeland and Ludvigsen(1998), Liberzon (2003)). These controllers swing the pendulum back and forward so that the mechanical energy of the pendulum increases. In many methods, two controllers are switched for the swinging-up phase and the stabilization phase. Therefore, another difficulty appears: when should the controllers be switched?

In this paper, a comparatively simple switching control law for inverted pendulum system is proposed. In the proposed method, the controllers are switched according to the angle of the pendulum. The pendulum goes to the lowest position due to gravity when control is not applied. Therefore, we assume a virtual energy function obtained reversing the sign of potential energy. This function becomes smallest when the pendulum remains stationary at the upright equilibrium. If there were controllers to always decrease this function, the states would converge to the upright equilibrium. However, there are not unbounded continuous control inputs which always decrease this virtual energy function. Consequently, the stability of the controlled system will be guaranteed by focusing on the typical modes of motion: rotation, swinging motion and convergence. The transitions of energy function are analyzed theoretically and the conditions of switching control law are clarified. In practically use, this switching control law can be used without consideration of transitions among these modes of motion and does not require the calculation of the energy function or another function for input switching. The simplest controllers are shown and the influences of the gain parameters on system behavior are investigated in numerical simulations.

2. DEFINITIONS AND THE MODEL OF SYSTEM

Consider the simplified inverted pendulum system described in Åström and Furuta(2000) and others. The equation of motion for this system is given as

$$J\frac{d^2\theta}{d\tau^2} - mgl\sin\theta + mlv\cos\theta = 0, \tag{1}$$

where θ is an angle between the pendulum and the vertical axis, g is the acceleration of gravity, m is the mass of the pendulum, l is the length of a rod from the pivot point to the center of mass, and J is the moment of inertia with respect to the pivot point. The external control force is horizontally applied at the pivot point. This force implies the acceleration of cart in the case of cart-pendulum systems. As described in Åström and Furuta(2000), the equation of motion (1) is normalized by introducing $t = \sqrt{mgl/J}\,\tau$, $u=v/g$.

$$\ddot{\theta} - \sin\theta + u\cos\theta = 0, \tag{2}$$

where $\ddot{\theta} = d^2\theta/dt^2$. The total mechanical energy of this system, which consists of kinetic energy and potential energy, is obtained as,

$$E(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + (\cos\theta - 1), \tag{3}$$

where $\dot{\theta} = d\theta/dt$. The purpose of this study is to stabilize the pendulum at the upright equilibrium. Therefore, the following definitions are used in this paper.

***Definition* 1** The upright equilibriums are the states such that

$$\dot{\theta} = 0, \quad \cos\theta = 1. \tag{4}$$

***Definition* 2** The pendent equilibriums are the states such that

$$\dot{\theta} = 0, \quad \cos\theta = -1. \tag{5}$$

***Definition* 3** The inverted pendulum system (2) is asymptotically stable when almost all states except the pendent equilibriums converge to any one of the upright equilibriums, i.e.,

$$\lim_{t\to\infty}\dot{\theta} = 0, \quad \lim_{t\to\infty}\cos\theta = 1 \tag{6}$$

are satisfied.

3. PROPOSED SWITCHING CONTROL LAW

When applying inputs decreasing the energy of system (2), the pendulum is stabilized at the pendent equilibrium, where the potential energy becomes the minimum. Paying attention to this fact, we assume a virtual energy which is obtained if gravity worked in the opposite direction. Moreover, the pendulum will be stabilized at the upright equilibrium if control inputs are applied to decrease this virtual energy. Thus, in this paper, we suppose the following function E_d which is obtained by reversing the sign of potential energy in equation (3).

$$E_d(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \cos\theta \tag{7}$$

The time-derivative of E_d is obtained as

$$\begin{aligned}\dot{E}_d &= \dot{\theta}(\ddot{\theta} + \sin\theta)\\ &= \dot{\theta}(2\sin\theta - u\cos\theta).\end{aligned} \tag{8}$$

The derivative $\dot{E}_d$ is controlled by the control input u on the right-hand side of (8). However, $\dot{E}_d$ cannot be controlled when $\dot{\theta}$ =0 or $\cos\theta$=0, because the coefficient of u becomes 0.

The upright equilibriums defined as *Definition* 1 are not only one. That is, all states satisfying $\theta=\pm 2n\pi$, n=0, 1, 2, … are candidates for the upright equilibriums. Therefore, in this study, we confine control inputs to the class satisfying

$$u(\theta) = u(\theta + 2\pi). \tag{9}$$

The feedback controllers satisfying (9) perform the same role in the neighborhoods of any upright equilibrium. From this fact, the controlled system has a possibility of stabilizing the pendulum at an appropriate upright equilibrium among $\theta=\pm 2n\pi$, n=0, 1, 2,… according to the initial states. As a class of control inputs satisfying (9), we choose the inputs as

$$u = 2F(\theta)\sin\theta, \tag{10}$$

where $F(\theta)$ is a bounded function and $F(\theta) = F(\theta+2\pi)$. Applying (10) to (8), we obtain

$$\dot{E}_d = 2\dot{\theta}\sin\theta(1 - F(\theta)\cos\theta). \tag{11}$$

In order to satisfy $\dot{E}_d \le 0$, we have to determine the function $F(\theta)$ so that $(1-F(\theta)\cos\theta)$ has the opposite sign of $\dot{\theta}\sin\theta$. When $\dot{\theta}\sin\theta < 0$, it is appropriate to select the function $F(\theta)$ such that

$$F(\theta)\cos\theta \le 1, \ \text{for } \dot{\theta}\sin\theta < 0. \tag{12}$$

On the other hand, when $\dot{\theta}\sin\theta \ge 0$, the conditions for $\dot{E}_d \le 0$ is rewritten as

$$F(\theta)\cos\theta \ge 1, \ \text{for } \dot{\theta}\sin\theta \ge 0. \tag{13}$$

There are no bounded functions $F(\theta)$ satisfying (13) at the points where $\cos\theta$=0. Therefore, we propose the controllers satisfying conditions (12) and (13) in as wide a range as possible. Moreover, outside of this range, the controller is switched to 0 taking into consideration the conditions for finite inputs. To summarize, a proposed switching law of the function

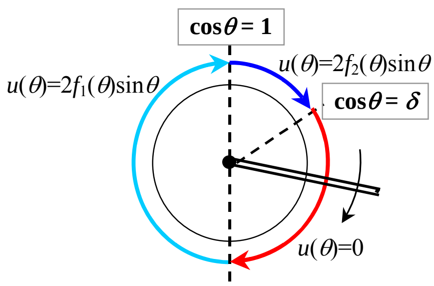

Fig. 1 Input switching in clockwise rotation

$F(\theta)$ is described as follows.

$$F(\theta)=\begin{cases} f_1(\theta) & if\ \ \dot{\theta}\sin\theta<0 \\ f_2(\theta) & if\ \ \dot{\theta}\sin\theta\geq 0,\ \cos\theta>\delta \\ 0 & if\ \ \dot{\theta}\sin\theta\geq 0,\ \cos\theta\leq\delta \end{cases} \quad (14)$$

where $0<\delta<1$, $f_1(\theta)\cos\theta\leq 1$ for $^\forall\theta$ and $f_2(\theta)\cos\theta\geq 1$ for $\cos\theta>\delta$. Furthermore, the points where $f_1(\theta)\cos\theta=1$ or $f_2(\theta)\cos\theta=1$ are at most finite points in the range of $-\pi<\theta<\pi$.

Since $\dot{E}_d$ has a positive sign when $F(\theta)=0$, the global stability cannot be discussed by using E_d as a candidate of the Lyapunov function. In the next section, stability is discussed by focusing on the characteristic motion modes of pendulum systems and investigating the change of E_d.

4. PARAMETER CONDITIONS FOR STABILITY

Motions of system (2) from arbitrary states $\boldsymbol{\theta}=(\theta, \dot{\theta})$ are divided into two motions: the pendulum makes a full rotation through $\theta\pm 2n\pi$, and the pendulum stops before making a full rotation. The latter is a swinging motion such that the motion direction is reversed due to gravity whenever the pendulum stops. Since system (2) has no friction, the total mechanical energy (3) is kept when $u=0$. In this case, motion of the system depends only on initial states and the pendulum repeats eternally either rotations or swinging motions, or remains stationary at the equilibrium points. In this section, transitioning the motion modes of the pendulum as shown in Fig. 2, the conditions for stabilization are derived using a switching control law proposed in the previous section.

Using switching control law (10)(14), an invariant set (Bacciotti and Rosier(2001)) appears in the neighborhood of each upright equilibrium.

***Lemma* 1** Applying switching control law (10)(14) to system (2), $\mathrm{O}=\{\boldsymbol{\theta}|E_d<-\delta\}$ is an invariant set. □

Proof Let $\boldsymbol{\theta}\in\mathrm{O}$ at t=0. Then, $\cos\theta>\delta$ because the states satisfy $E_d<-\delta$. In such case, control input function $F(\theta)$ is $f_1(\theta)$ or $f_2(\theta)$. Since the input functions $f_1(\theta)$ and $f_2(\theta)$ satisfy conditions (12) and (13) and $\dot{E}_d\leq 0$, it is evidently satisfied that $E_d<-\delta$, i.e., $\boldsymbol{\theta}\in\mathrm{O}$, for $t\geq 0$. ■

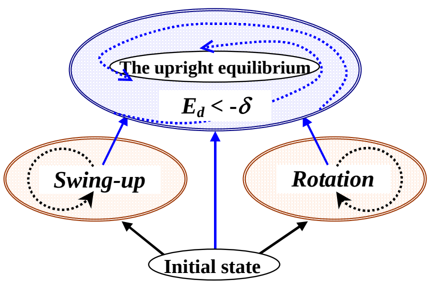

Fig. 2 Illustration of motion modes and state transitions by proposed method

From the LaSalle's invariance principle, the following lemma is obtained.

***Lemma* 2** Applying switching control law (10)(14) to system (2), arbitrary states in the invariant set O converge to an upright equilibrium if $f_1(\theta)\cos\theta>1/2$ for $\boldsymbol{\theta}\in\mathrm{O}$. □

Proof From LaSalle's invariance principle, the states in $\boldsymbol{\theta}\in\mathrm{O}$ converge into the largest invariant set contained in $\mathrm{N}=\{\boldsymbol{\theta}\in\mathrm{O}\,|\,\dot{E}_d=0\}$, because $\dot{E}_d\leq 0$ (Imura (2000)). Therefore, we shall show that the largest invariant set consists of only the upright equilibriums. From equation (11), N is easily rewritten as

$$\mathrm{N}=\{\boldsymbol{\theta}\in\mathrm{O}\,|\,\dot{\theta}=0\}\cup\{\boldsymbol{\theta}\in\mathrm{O}\,|\,\sin\theta=0\}\cup \{\boldsymbol{\theta}\in\mathrm{O}\,|\,F(\theta)\cos\theta=1\}. \quad (15)$$

Under the conditions that the points where $F(\theta)\cos\theta=1$ are at most finite points, the states $\boldsymbol{\theta}$ in the set $\{\boldsymbol{\theta}|\ F(\theta)\cos\theta=1\}$ are discrete points and are not continuous points. Thus, the states in invariant sets contained in the set N surely satisfy $\dot{\theta}\equiv 0$. It is required for $\dot{\theta}\equiv 0$ to satisfy $\ddot{\theta}=0$. Namely, from equation (2), it is necessary that the right-hand side of the following equation becomes 0.

$$\ddot{\theta}=\sin\theta(1-2F(\theta)\cos\theta) \quad (16)$$

Since this condition is satisfied when $\sin\theta=0$, the set $\{\boldsymbol{\theta}|\ \sin\theta=0,\ \dot{\theta}=0\}$ is the invariant set. On the other hand, the set $\{\boldsymbol{\theta}|\ F(\theta)\cos\theta=1,\ \dot{\theta}=0\}$ is not the invariant set, because $\ddot{\theta}\neq 0$.

Furthermore, when using the switching control, there is a possibility that the states remain on the switching surfaces by switching indefinitely between different control modes. This phenomenon is called *livelock* (van der Shaft and Schumacher(2000)). If the states satisfy $\ddot{\theta}\dot{\theta}\leq 0$ after the input function is switched from $f_2(\theta)$ to $f_1(\theta)$, then the pendulum cannot escape from the switching point which is not the upright equilibrium and *livelock* occurs. From equation (2), $\ddot{\theta}\dot{\theta}$ is obtained as follows.

$$\dot{\theta}\ddot{\theta}=\dot{\theta}\sin\theta(1-2f_i(\theta)\cos\theta),\ \ i=1,2 \quad (17)$$

Using $f_1(\theta)$, $\ddot{\theta}\dot{\theta}$ is positive when $f_1(\theta)\cos\theta>1/2$. Therefore, the states escape from the switching points if $f_1(\theta)\cos\theta>1/2$. Furthermore, *livelock* never occurs on the other switching surfaces except the upright position. ■

From the above-mentioned, the following theorem is obtained in the neighborhood of the upright equilibrium.

***Theorem* 1** Assume that $f_1(\theta)\cos\theta>1/2$ for $\boldsymbol{\theta}\in\mathrm{O}$. Then, the states of system (2) with (10)(14) converge to the upright equilibrium if $E_d<-\delta$ is satisfied once.□

Consider the case where the states are outside of the invariant set O described in *Lemma* 1. In this case, the pendulum rotates or swings. We consider the conditions for decreasing energy E_d. From the form of the time-derivative $\dot{E}_d$ given as (11), the change of energy E_{12} when the angle of the pendulum changes from θ_1 to θ_2 is calculated regardless of the angular velocity.

$$\begin{aligned} E_{12} &= \int_{\theta_1}^{\theta_2} \dot{E}_d dt \\ &= \int_{\theta_1}^{\theta_2} 2\sin\theta(1-F(\theta)\cos\theta)\, d\theta \\ &= 2\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta - \int_{\theta_1}^{\theta_2} F(\theta)\sin 2\theta\, d\theta \end{aligned} \quad (18)$$

First, the energy change through a full rotation E_a as shown in Fig. 1 is calculated. Since the energy change depends only on the angle, assume that θ_1=0, and $\theta_2=2\pi$ without loss of generality. The energy change E_a is obtained as

$$\begin{aligned} E_a &= 2\int_0^{2\pi} \sin\theta d\theta - \left(\int_0^{\cos^{-1}\delta} f_2(\theta)\sin 2\theta d\theta \right. \\ &\qquad \left. + \int_{\cos^{-1}\delta}^{\pi} 0\, d\theta + \int_{\pi}^{2\pi} f_1(\theta)\sin 2\theta d\theta \right). \\ &= -\left(\int_0^{\cos^{-1}\delta} u_2(\theta)\sin 2\theta d\theta + \int_{\pi}^{2\pi} f_1(\theta)\sin 2\theta d\theta \right) \end{aligned} \quad (19)$$

Consequently, the following conditions are derived.

***Lemma* 3** Using switching control law (10)(14), system (2) is asymptotically stable only if

$$\int_0^{\cos^{-1}\delta} f_2(\theta)\sin 2\theta d\theta + \int_{\pi}^{2\pi} f_1(\theta)\sin 2\theta d\theta > 0 \quad (20)$$

is satisfied. □

Proof Assume E_a≥0 and consider the initial states such that $\cos\theta=\delta$, $\dot{\theta}\sin\theta \geq 0$ and E_d>1. Then, if the pendulum makes a full rotation without satisfying $\dot{\theta}$ =0, $|\dot{\theta}|$ does not become smaller than the initial value because E_a≥0. Furthermore, from condition (9), the pendulum repeats the rotations and $|\dot{\theta}|$ never decreases. Therefore, to satisfy $\dot{\theta}$ =0 before a full rotation is required so that the states converge to the upright equilibriums. However, using switching control law (10)(14), the energy increases from this initial state to the lowest position, and then decreases as shown in Fig. 3. Under the assumption of E_a≥0, the energy during a full rotation satisfies E_d >1 and $\dot{\theta}$ never becomes 0. Namely, the pendulum cannot stabilize at the upright position from the initial state if E_a≥0. Therefore, condition (20) implying E_a<0 is required. ■

Next, swinging motions are considered. Assume that the angular velocity $\dot{\theta}$ becomes 0 when $\theta=\theta_1$ and $\theta=\theta_2$. When $\boldsymbol{\theta} \notin$O at $\theta=\theta_1$, $\cos\theta_1<\delta$. Thus, the input function $F(\theta)$ is switched to 0 at $\theta=\theta_1$. The pendulum goes downward and the control function is switched to $f_1(\theta)$ at the lowest point. Noticing that the energy is -cosθ when $\dot{\theta}$ =0, the following equation concerned with the energy change from θ_1 to θ_2 is obtained.

$$-\cos\theta_2 = \int_{\theta_1}^{\theta_2} \dot{E}_d dt + (-\cos\theta_1)$$

$$-\cos\theta_2 = -\cos\theta_1 + \int_{\pi}^{\theta_2} f_1(\theta)\sin 2\theta d\theta \quad (21)$$

This equation indicates that θ_2 is closer to the upright position than θ_1 if the energy change ΔE

$$\Delta E = \int_{\pi}^{\theta_2} f_1(\theta)\sin 2\theta d\theta \quad (22)$$

is negative. The following lemma concerned with ΔE is obtained.

***Lemma* 4** Using switching control law (10)(14), system(2) is asymptotically stable only if

$$\int_{\pi}^{\theta_2} f_1(\theta)\sin 2\theta d\theta < 0 \quad (23)$$

is satisfied for all θ_2 satisfying (21) for -1<cosθ_1<δ.□

Proof Assume ΔE≥0 when -1<cosθ_1≤ε, ε<δ. Then, the pendulum never reaches an upright position from the initial state such that -1<cosθ≤ε, $\dot{\theta}$ =0. Therefore, ΔE<0 is required for asymptotic stabilization for -1< cosθ_1≤ε. Furthermore, assume that the pendulum reaches an upright position when ΔE<0 for -1< cosθ_1≤ε and ΔE≥0 for ε<cosθ_1<ε'. Then, there is γ_1≥0 so that the pendulum swings up from cos$\theta_1=\varepsilon-\gamma_1$, $\dot{\theta}$ =0 to cos$\theta_2=\varepsilon'$ without stopping. Under this condition, consider the initial state such that cos$\theta_1=\varepsilon+\gamma_2<\varepsilon'$, $\dot{\theta}$ =0. Then, in the range where cosθ changes from $\varepsilon-\gamma_1$ to ε' through –1 as shown in Fig. 4, the energy is the larger of $2(\gamma_1+\gamma_2)$ than the energy of the motion from cos$\theta_1=\varepsilon-\gamma_1$. Therefore, the

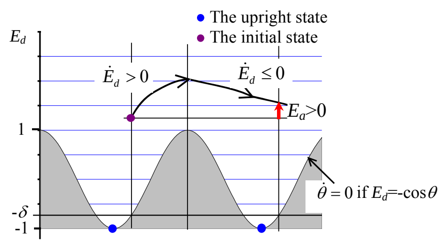

Fig. 3 Energy transition in the proof of *Lemma* 3

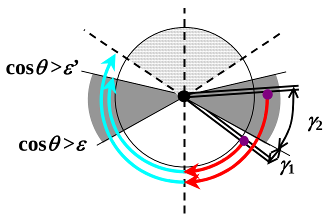

Fig. 4 Illustration of pendulum motion in the proof of *Lemma* 4

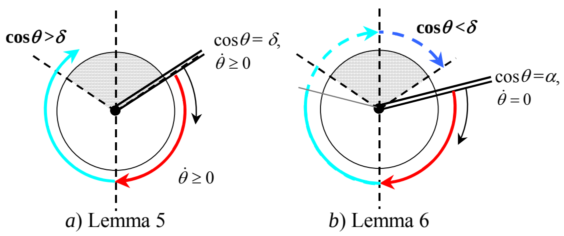

Fig. 5 Illustrations of pendulum motion in the proofs of *Lemmas* 5 and 6

assumption is contradicted because the pendulum swings up to $\cos\theta=\varepsilon'$. Namely, it is required that $\Delta E<0$ for all θ_1 such that $-1<\cos\theta_1<\delta$. ■

The following lemmas concerned with transitions between rotations and swinging motions are obtained.

***Lemma* 5** Assume that *Lemma* 4 is satisfied. Then, it is not satisfied that $\boldsymbol{\theta} \notin \mathrm{O}$, $\dot{\theta}=0$ after a full rotation of the pendulum. □

Proof In a full rotation motion, the input is switched to 0 when the pendulum goes downward through the point where $\cos\theta=\delta$. Then, the pendulum evidently reaches to the lowest position without stopping. Therefore, the pendulum swings up to a higher position than the position where $\cos\theta=\delta$ because *Lemma* 4 is satisfied. Namely, the pendulum never stops in the range where $\cos\theta\le\delta$, after a full rotation as shown in Fig. 5a). ■

***Lemma* 6** Assume that *Lemma* 3 is satisfied. Then, the pendulum never makes a full rotation after $\boldsymbol{\theta} \notin \mathrm{O}$, $\dot{\theta}=0$. □

Proof Assume that $\dot{\theta}$ becomes 0 with the point where $\cos\theta=a$, $\delta>a$. Then, the input is switched to 0 and the pendulum goes downward. Assume again that the pendulum reaches the position where $\cos\theta=\delta$ via the lowest and the upright positions as shown in Fig. 5b). Then, the energy at the position where $\cos\theta=\delta$ is obtained as follows, taking into account $E_a<0$ from *Lemma* 3 and $\delta>a$.

$$\begin{aligned} E_d &= -a+(E_a-2\int_{\cos^{-1}\delta}^{\cos^{-1}a}\sin\theta d\theta) \\ &= -a+(E_a+2(a-\delta)) \\ &= -\delta+(E_a+(a-\delta)) \\ &< -\delta \end{aligned} \tag{24}$$

This inequality implies that the energy becomes smaller than the allowable minimum value $-\delta$ at this angle. Namely, $\dot{\theta}$ becomes 0 before the pendulum reaches this angle. ■

Under the conditions described in the above lemmas, the states of the system transition as shown in Fig. 2. Therefore, the following theorem is obtained.

***Theorem* 2** Assume that $f_1(\theta)$, $f_2(\theta)$ and δ satisfy the conditions in *Theorem* 1 and *Lemmas* 3, 4. Then, applying switching control law (10)(14), controlled system(2) is asymptotically stable. □

Proof When *Lemmas* 3 and 4 are satisfied, transitions between swinging and rotation motion modes never occur. Therefore, the energy E_d decreases by repeating one of these motions and then, E_d surely becomes smaller than $-\delta$. From *Theorem* 1, the states converge to the upright equilibriums. ■

Remark The pendulum actually rotates decreasing its kinetic energy when an initial angular velocity is comparatively large. On the other hand, the pendulum swings up by increasing its potential energy when the initial angular velocity is small. In addition, in this case, the pendulum is attracted to the neighborhood of the upright equilibrium without redundantly increasing the energy.

5. SIMPLE CONTROLLERS

5.1 The simplest class of switching controllers

Simple controllers satisfying the above-mentioned conditions are shown. At first, let the functions $f_1(\theta)$ and $f_2(\theta)$ be constant gains k_1 and k_2, respectively. Since the infimum of $k_1\cos\theta$ in the range of $\boldsymbol{\theta}\in\mathrm{O}$ is $k_1\delta$, the conditions in *Lemma* 2 are calculated as,

$$k_1\delta>1/2. \tag{25}$$

On the other hand, the conditions in *Lemmas* 4 is calculated as

$$k_1(1-\cos^2\theta_2)<0. \tag{26}$$

Condition (26) indicates $k_1<0$, because $(1-\cos^2\theta_2)$ is always nonnegative. This condition conflicts with condition (25). Namely, there is no parameter k_1 simultaneously satisfying all conditions in *Theorem* 2, in the case of $f_i(\theta)=k_i$, i=1,2.

Next, let functions $f_1(\theta)$ and $f_2(\theta)$ be trigonometric functions. Assume $k_i\sin(\theta)$, i=1, 2. Then, there is no k_2 satisfying

$$k_2\sin\theta\cos\theta\ge1 \tag{27}$$

when θ=0. Thus, the conditions for (14) are not satisfied.

Therefore, we assume the functions $k_i\cos(\theta)$, i=1, 2 as $F(\theta)$. The conditions are represented as follows.

$$k_2\delta^2\ge1,\ 1\ge k_1 \tag{28}$$

$$k_1>1/(2\delta^2) \tag{29}$$

$$k_2(1-\delta^3)-2k_1>0 \tag{30}$$

$$-k_1(1+\cos^3\theta_2)<0 \tag{31}$$

Reducing these equations, the conditions for stabilization are simplified as,

$$1\ge k_1>\frac{1}{2\delta^2},\ k_2\ge\frac{1}{\delta^2}\ \text{ and }\ k_2>\frac{2}{1-\delta^3}k_1. \tag{32}$$

We can easily find the parameters k_1, k_2 satisfying (32). From the first inequality in (32), the conditions of a switching surface are obtained as $\delta>1/\sqrt{2}$.

The maximum value of input u is $u_{\max}=2k_2\delta\sqrt{1-\delta^2}$ when $\cos\theta=\delta$. Under condition (32) of parameters k_1, k_2, the maximum input $u_{\max}$ has lower bound as,

$$u_{\max}>2\frac{\sqrt{1-\delta^2}}{\delta(1-\delta^3)}. \tag{33}$$

The parameter minimizing this lower bound of $u_{\max}$ is obtained as δ=0.7625 from a derivation of the right-hand side of (33). The lower bound of $u_{\max}$ is 3.048 when k_1=0.860 and k_2=3.090.

5.2 Simulation results

Simulation results in the case of $\delta=\sqrt{3}/2$ are shown in Figs. 6 and 7. Condition (32) indicates that $1 \geq k_1 > 2/3$ and $k_2 \geq 5.706k_1$ when $\delta=\sqrt{3}/2$. k_2 is chosen as 6.0. The differences of behaviors from the initial values θ=0.99π[rad] and $\dot{\theta}$ =0.0 [rad/s] are compared in changing k_1 in 0.70, 0.80 and 0.90. In this case, the pendulum swings up early on as shown in Fig. 6a). In Fig. 7a), the points where the energy trajectories contact with the curve E_d=-cosθ indicate the points where the kinetic energy becomes 0 and the pendulum motion stops. The energy in these points decreases in proportion to k_1. As k_1 is larger, angular velocity becomes larger early on and the pendulum swings up to higher positions. When the pendulum does not swing up to a sufficiently high position, for example at t=3.0[s], the input is switched to 0. The pendulum goes through the lowest position and swings up in contrary direction. On the other hand, after t =6[s], the pendulum goes over the upright position and the input function is switched to $f_2(\theta)$. After the pendulum motion is stopped by a large $f_2(\theta)$, the input function is switched to $f_1(\theta)$ again. Repeating these switches in the set O, the states converge to the upright equilibriums. As for the range of $\boldsymbol{\theta} \in$O, the larger parameters k_1, k_2 are, the larger the absolute values of $\dot{E}_d$ becomes. Therefore, the states converge fast after E_d<-δ is satisfied.

The pendulum rotates as shown in Fig. 6b) when the initial states are θ=π/3[rad] and $\dot{\theta}$ = 2.5[rad/s]. In this simulation, k_1 is 0.7 and k_2 is changed in 4.1, 4.3 and 4.5. From *Lemma* 3, as k_2 becomes larger, the energy change $|E_a|$ during a full rotation becomes larger and the states reach into the set O during a few rotations as shown in Fig. 7b). Furthermore, it is verified that the input becomes large if k_2 is large.

6. CONCLUSIONS

A switching control strategy for inverted pendulum systems was proposed based on the energy function and its transitions. If gravity worked in the reverse direction, the pendulum would remain at the upright equilibrium. Thus, we assumed a virtual energy whose potential has an opposite sign. Then, a switching control law was designed so that this virtual energy decreases in as wide a range as possible. Paying attention to motions of the pendulum, the energy changes were analyzed theoretically. The conditions of simple switching controller were derived and the stability of the switching controlled system was guaranteed. In simple examples, the influences of gain parameters in the controllers on the motion of the system were shown.

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REFERENCES

Angeli,D. (2001). Almost global stabilization of the inverted pendulum via continuous state feedback, *Automatica*, *37*, 1103-1108.

Åström,K.J., & Furuta,K. (2000). Swinging up a pendulum by energy control, *Automatica*, *36*, 287-295.

Bacciotti,A., & Rosier,L. (2001). Liapunov functions and stability in control theory (LNCIS267), London:Springer

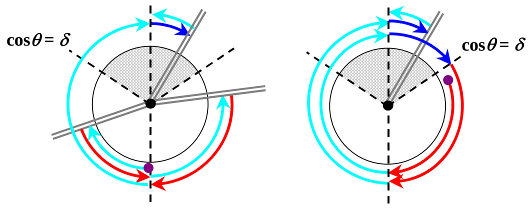
Bloch,A.M., Chang,D.E., Leonard, N.E., & Marsden,J.E. (2001). Controlled Lagrangians and the Stabilization of Mechanical Systems II: Potential Shaping, *IEEE Trans on Automat. Contr.*, *46*(*10*), 1556-1571.

Imura,J. (2000). Stability theory in systems and control, Tokyo:Colona publishing co., ltd..

Liberzon,D.(2003). Switching in systems and control, Boston:Birhäuser.

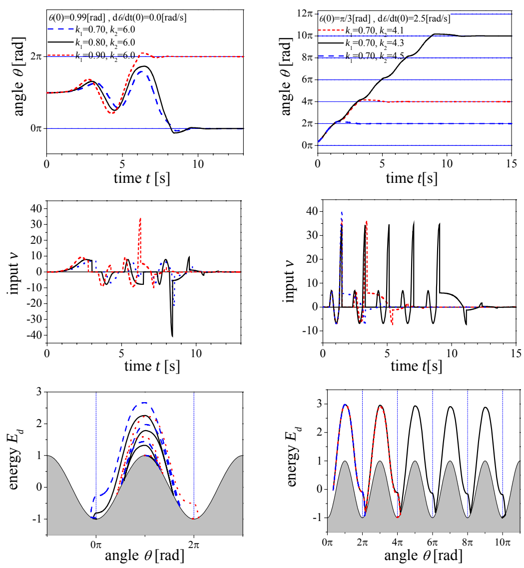
Shiriaev,A., Egeland,O., & Ludvigsen,H. (1998). Gloal Stabilization of Unstable Equilibrium Point of Pendulum, In *Proc. of the 37th IEEE Conf. Decision Control*, 4584-4585.

van der Shaft,A. & Schumacher,H. (2000). An Introduction to Hybrid Dynamical Systems (LNCIS251), London:Springer



a) Swinging b) Rotation

Fig. 6 Illustrations of swinging or rotation motions



a)Swinging motion b)Rotation motion

Fig. 7 State trajectories, input signals and energy transitions in swinging or rotation motions