THE ODD WORLD OF PERIODIC POLYNOMIALS

S. Bittanti, P. Colaneri

Politecnico di Milano
Dipartimento di Elettronica e Informazione
Piazza Leonardo da Vinci 32, 20133 Milano (Italy)
bittanti@elet.polimi.it colaneri@elet.polimi.it

Abstract: The purpose of this paper is to set the basis for the study of polynomials with periodic coefficients. We define first the elementary algebra, and then we introduce fundamental notions such as those of adjoint polynomial and of zero of a polynomial. The we pass to periodic systems, for which we touch the problem of spectral factorization in discrete time.

Keywords: Periodic systems, Discrete time systems, Periodic polynomials, Zeros, Spectral factorization

1. INTRODUCTION

The research activity in periodic control extends over more than three decades, as witnessed by the papers presented at the IFAC workshops on Periodic Control Systems held in Cernobbio-Como (Italy) in 2001 (Bittanti, Colaneri, 2002) and in Yokohama (Japan) in 2004 (Katayama, Sano, 2004). However, not much is known about periodic polynomials in general, and about the polynomial approach to the analysis and control of periodic systems, see (Colaneri, Kucera, Longhi, 2003), (Mrabet, Bourles, 1998), (Bittanti, Colaneri, 2004) for early papers. Herein we propose to define an algebra of periodic polynomials, and use it for some basic questions, namely the definition of adjoint polynomial and the characterization of the notion of root (zero). This will lead to the definition of "characteristic system" and "lifted characteristic equation". Notice that this first part of our paper has some connections with the abstract theory of ideals in polynomial rings, (Lam, Leroy, 2000). Then we move towards periodic dynamical systems in discrete time, for which we treat the problem of spectral factorization. All these topics are of special interest for the analysis of seasonal time-series, see e. g. (Luetkepohl, 1993), (Lund, Seymour, 2002).

2. ELEMENTARY ALGEBRA OF PERIODIC POLYNOMIALS

Denote by $p_i(t)$, $i = 0, 1, \cdots n$, a set of real coefficients evolving periodically in time with period T, i.e. $p_i(t+T) = p_i(t)$, $\forall t$, and let

$$p(\sigma,t) = p_0(t)\sigma^n + p_1(t)\sigma^{n-1} + \cdots p_n(t) \quad (1)$$

be the associated T-periodic polynomial in the symbol σ, denoting the one-step ahead operator. In other words, the polynomial $p(\sigma,t)$ acts as an operator transforming a time signals $v(t)$ into another time signal $w(t)$ with the following rule:

$$w(t) = p(\sigma,t)v(t) = \sum_{i=0}^{n} p_i(t)v(t+n-i) \quad (2)$$

A more precise symbol could have been adopted in order to stress the operator character of $p(\sigma, t)$ (for instance $(p * w)(t)$). However, for the sake of simplicity, we will use the above notation throughout the paper. Furthermore, notice that $p(\sigma, t)$ is anticausal in that σ is the one-step ahead shift operator. The degree of the polynomial is defined as the T-periodic function $\rho(t)$ given by the power associated with the maximum-power coefficient which is non zero at time t. If $p_0(t) \neq 0$, $\forall t$, then $\rho(t) = n$, $\forall t$, and the polynomial is called *regular*. If $p_0(t) = 1$, $\forall t$, it is said to be *monic*.

Sum and Product

The sum of two T-periodic polynomials, say $p(\sigma, t)$ and

$$q(\sigma, t) = q_0(t)\sigma^n + q_1(t)\sigma^{n-1} + \cdots q_n(t)$$

is still a T-periodic polynomial, given by

$$p(\sigma, t) + q(\sigma, t) = \sum_{i=0}^{n} (p_i(t) + q_i(t))\sigma^{n-i}$$

The definition of product is based on the fulfilment of the concatenation rule concerning the subsequent application a two polynomial operators to a given signal. In this regard, consider first a generic periodic polynomial and an elementary polynomial constituted by a power os σ. One obtains:

$$[\sigma^k p(\sigma, t)]v(t) = \sigma^k w(t) = w(t + k) = \\ p(\sigma, t + k)v(t + k) = [p(\sigma, t + k)\sigma^k]v(t)$$

Therefore, these two polynomials do not commute. Precisely, for any integer k and signal $v(t)$:

$$[\sigma^k p(\sigma, t)]v(t) = [p(\sigma, t + k)\sigma^k]v(t)$$

The above property can be written as:

$$\sigma^k p(\sigma, t) = p(\sigma, t + k)\sigma^k \tag{3}$$

and is referred to as the *pseudo-commutative* property. Obviously, if k is a multiple of the period, then σ^k and $p(\sigma, t)$ commute.

The product of the two generic polynomials $p(\sigma, t)$ and $q(\sigma, t)$ (of the same degree n) is defined as follows:

$$p(\sigma, t)q(\sigma, t) = \sum_{i=0}^{2n} r_i(t)\sigma^{2n-i}$$

where

$$\begin{aligned} r_0(t) &= p_0(t)q_0(t + n) \\ r_1(t) &= p_0(t)q_1(t + n) + p_1(t)q_0(t + n - 1) \\ &\vdots \\ r_{2n-1} &= p_{n-1}(t)q_n(t + 1) + p_n(t)q_{n-1}(t) \\ r_{2n}(t) &= p_n(t)q_n(t) \end{aligned}$$

The generalization to the product of polynomials with different degrees is trivial.

The ring of periodic polynomials

Pursuant the previous definitions of sum and product, the set of T-periodic polynomials forms a non commutative ring. Obviously, the subset constituted by the polynomials in the symbol σ^T is a commutative ring.

In this ring we define the "unit" as follows. By following standard algebraic terminology, a T-periodic polynomial $p(\sigma, t)$ is said to be *unimodular* if there exists a T-periodic polynomial $q(\sigma, t)$ so that for each t,

$$p(\sigma, t)q(\sigma, t) = q(\sigma, t)p(\sigma, t) = 1$$

where 1 denotes the identity operator.

Contrary to the case of constant polynomials, there might exist unimodular T-periodic polynomials with degree different from zero at some time point. For instance, the 2-periodic polynomial

$$p(\sigma, t) = p_0(t)\sigma + p_1(t)$$

with $p_0(0) = 0$, $p_0(1) = 1$, $p_1(0) = 1$, $p_1(1) = -1$, is unimodular in that

$$(p_0(t)\sigma - p_1(t + 1))(p_0(t)\sigma + p_1(t)) = 1$$

Notice that the no regular T-periodic polynomial can be unimodular.

Adjoint polynomial

In order to define the adjoint polynomial, we preliminarily consider the Hilbert space of real bounded T-periodic functions, with

$$< v_1, v_2 > := \sum_{t}^{T-1} v_1(t)v_2(t)$$

as inner product. Correspondingly, the norm of a periodic signal v is defined as

$$\|v\|_2^2 = < v, v > := \sum_{t=0}^{T-1} v(t)^2$$

The inner product above defined induces the definition of *adjoint polynomial* $p^\sim(\sigma, t)$ of a given T-periodic polynomial $p(\sigma, t)$. The adjoint operator is defined so as to meet the condition that, for any pair of periodic signals v and w, the following inner products coincide:

$$< p(\sigma, t)v, w > = < v, p^\sim(\sigma, t)w >$$

Therefore, by exploiting the signal periodicity:

$$< p(\sigma,t)v, w >= \sum_{t=0}^{T-1}\sum_{k=0}^{n} p_k(t)v(t+n-k)w(t)$$
$$= \sum_{t=0}^{T-1}\sum_{k=0}^{n} v(t)p_k(t-n+k)w(t-n+k)$$

so that,

$$p^{\sim}(\sigma,t)w(t) = \sum_{k=0}^{n} p_k(t-n+k)w(t-n+k)$$

This means that, the adjoint polynomial is:

$$p^{\sim}(\sigma,t) = \sigma^{-n}p_0(t) + \sigma^{-n+1}p_1(t) + \cdots + p_n(t)$$

Obviously, σ^{-1} is the one-step delay operator. In view of the pseudo-commutative property, the adjoint operator can be equivalently rewritten as

$$p^{\sim}(\sigma,t) = p_0(t-n)\sigma^{-n} + p_1(t-n+1)\sigma^{1-n} + p_2(t-n+2)\sigma^{2-n} + \cdots + p_n(t)$$

The action of this polynomial in σ^{-1} on a signal $v(t)$ is as follows:

$$w(t) = p^{\sim}(\sigma,t)v(t) = p_0(t-n)v(t-n) + + p_1(t-n+1)v(t-n+1) + \cdots + p_n(t)v(t)$$

Characteristic equation and zeros

For a T-periodic polynomial, the notion of *zero* can be introduced by making reference to a "periodic signal blocking property" as follows. Consider the polynomial as a filter fed by a periodic signal, say $\bar{y}(t)$, of the same period of the coefficients. The zeros of the polynomial correspond to those situations for which the output of the filter is zero. Precisely, for a given (real or complex) λ, consider the following equation:

$$p(\lambda\sigma,t)\bar{y}(t) = \sum_{i=0}^{n} p_i(t)\lambda^{n-i}\bar{y}(t+n-i)$$

By writing this equation for $t = 0, 1, \cdots, T-1$ we obtain a system of T equations in $T+n$ unknowns $\bar{y}(0), \bar{y}(1), \cdots, y(T+n-1)$. Restrict now the attention to T-periodic functions $\bar{y}(\cdot)$. By imposing periodicity, such system (referred to as *characteristic system*) becomes a system of T equations in the T unknowns $\bar{y}(0), \bar{y}(1), \cdots, y(T-1)$. It can be written in the form $A(\lambda)\underline{y} = 0$ where $\underline{y}$ is the vector of the values $\bar{y}(0), \bar{y}(1), \cdots, y(T-1)$. Obviously, the system admits a nontrivial solution if and only if the $det[A(\lambda)] = 0$. The equation $det[A(\lambda)] = 0$, referred to as *lifted characteristic equation*, has a distinctive feature: due to the peculiar structure $A(\lambda)$, only powers of λ^T appear in the $det[A(\lambda)]$. Therefore the lifted characteristic equation admits n real or complex solutions λ^T.

The solutions (in λ^T) of the lifted characteristic equation are named zeros of the periodic polynomial $p(\sigma,t)$. Therefore a regular n-degree T-periodic polynomial admits n zeros, as in the time-invariant case. If $T = 1$ (time-invariant polynomial) the definition of zero and characteristic equation now given reduce to the well known standard definitions.

Finally, observe that the zeros of the adjoint polynomial are the reciprocals of the zeros of the original one.

Lifted periodic polynomials and zeroes

Given a T-periodic polynomial $p(\sigma,t)$, the associated lifted polynomial $P(\sigma^T,t)$ is a $T\times T$ polynomial matrix, with T-periodic coefficients, uniquely defined by the condition

$$P(\sigma^T,t)\begin{bmatrix}1\\ \sigma\\ \vdots\\ \sigma^{T-1}\end{bmatrix} = \begin{bmatrix}1\\ \sigma\\ \vdots\\ \sigma^{T-1}\end{bmatrix} p(\sigma,t) \qquad (4)$$

A number of basic properties of the lifted polynomial are in order:

(i) *Time-recursion of* $P(\sigma^T,t)$
For any given T-periodic polynomial $p(\sigma,t)$, the associated lifted polynomial matrix $P(\sigma,t)$ satisfies the following recursive formula:

$$P(\sigma^T,t{+}1) = \begin{bmatrix}0 & I_{T-1}\\ \sigma^T & 0\end{bmatrix} P(\sigma^T,t)\begin{bmatrix}0 & \sigma^{-T}\\ I_{T-1} & 0\end{bmatrix}$$

An important consequence of the recursive formula is that the determinant of a lifted polynomial matrix is a polynomial in the powers of σ^T with *constant* coefficients, i.e.

$$det[P(\sigma^T,t)] = \bar{p}(\sigma^T), \quad \textit{independent of } t$$

This allows one to properly define the *zeros* of a T-periodic polynomial $p(\sigma,t)$ as the roots of $det[P(\sigma^T,t)]$. Obviously, contrary to the case of constant coefficient polynomials, one can construct infinitely many T-periodic monic polynomials with the same set of zeros.

This definition of zeros of a periodic polynomial is equivalent to the one previously given. Indeed, it can be easily proven that

$$P(\lambda^T,0) = \Delta A(\lambda)\Delta^{-1}$$

where $\Delta = diag\{1, \lambda, \cdots, \lambda^{T-1}\}$ and $A(\lambda)$ is the matrix previously defined. Hence,

$$det[P(\lambda^T,0)] = det[A(\lambda)]$$

(ii) *Upper-triangularity of* $P(0,t)$

Another significant property of a lifted polynomial matrix $P(\sigma^T,t)$ is that $P(0,t)$ is upper triangular for each t in the period. Indeed,

$$P(0,t) = \begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}, \quad \forall t$$

The set of T-periodic polynomial matrices in σ^T satisfying the recursion in point (i) and the upper triangularity condition in point (ii) is in one-to-one correspondence with the set of T-periodic polynomials. Precisely, given a T periodic polynomial $p(\sigma,t)$, the lifted T-periodic polynomial matrix is uniquely defined through (4). Conversely, given a lifted polynomial matrix, i.e. a T-periodic polynomial matrix in σ^T satisfying the recursion in point (i) and the upper triangularity condition in point (ii), the T-periodic polynomial can be computed as

$$p(\sigma,t) = \left[P(\sigma^T,t)\right]_{(1)} \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{T-1} \end{bmatrix}$$

where $[P(\sigma,t)]_{(1)}$ denotes the first row of $P(\sigma,t)$. This expression is an obvious consequence of (4). Of course, the set of lifted polynomials is closed with respect to the sum and the product. This means that the lifted polynomial associated with the sum or the product of two T-periodic polynomials $p_1(\sigma,t)$ and $p_2(\sigma,t)$ is given by the sum or, respectively, the product of the two lifted polynomials $P_1(\sigma^T,t)$ and $P_2(\sigma^T,t)$. This claim is easily verified and the details are therefore omitted.

3. RATIONAL PERIODIC OPERATORS

In this section we introduce the notion of rational operator with periodic coefficients. For, consider the formal power series in the symbol σ^{-1}:

$$g(\sigma,t) = \sum_{i=0}^{\infty} g_i(t)\sigma^{-i} \tag{5}$$

where $g_i(t)$ are T-periodic coefficients.

This formal series can be seen as an operator which processes a time-signal to supply another time-signal. From this viewpoint, σ^{-1} is intended to be the unit delay operator. Accordingly, $g(\sigma,\cdot)$ is a causal operator. One can rewrite (5) as follows

$$g(\sigma,t) = \sum_{k=0}^{T-1} \left[\sum_{j=0}^{\infty} g_{k+jT}(t)\sigma^{-jT}\right]\sigma^{-k} \tag{6}$$

We say that $g(\sigma,t)$ is *rational* if the functions

$$\hat{g}_k(\sigma^T,t) = \sum_{j=0}^{\infty} g_{k+jT}(t)\sigma^{-jT} \tag{7}$$

are rational in σ^T, for each t in the period. If this is the case, it is possible to write

$$\hat{g}_k(\sigma^T,t) = \frac{\hat{n}_k(\sigma^T,t)}{\hat{d}_k(\sigma^T,t)} \tag{8}$$

Recall now that the symbol σ^T commutes with T-periodic functions. This means that it is possible to perform the product of the functions $\hat{d}_k(\sigma^T,t)$ and $\hat{n}_k(\sigma^T,t)$ in any possible order. Letting

$$\bar{d}(\sigma^T,t) = \prod_{k=0}^{T-1} \hat{d}_k(\sigma^T,t)$$

$$\bar{n}_i(\sigma,t) = \prod_{k\neq i,k=0}^{T-1} \hat{d}_k(\sigma^T,t)\hat{n}_i(\sigma^T,t)$$

it follows that $g(\sigma,t)$ can be written as

$$g(\sigma,t) = d_L(\sigma,t)^{-1}n_L(\sigma,t) = n_R(\sigma,t)d_R(\sigma,t)^{-1}$$

where

$$\begin{aligned} d_L(\sigma,t) &= \sigma^{T-1}\bar{d}(\sigma^T,t) \\ n_L(\sigma,t) &= \sum_{i=0}^{T-1} \bar{n}_i(\sigma^T,t+T-1)\sigma^{T-1-i} \\ d_R(\sigma,t) &= d(\sigma,t)\sigma^{T-1} \\ n_R(\sigma,t) &= \sum_{i=0}^{T-1} \bar{n}_i(\sigma^T,t)\sigma^{T-1-i} \end{aligned}$$

Therefore, a causal rational operator $f(\sigma,t)$ can be written in the above left or right factorized form, where the four polynomials obviously satisfy

$$d_L(\sigma,t)n_R(\sigma,t) = n_L(\sigma,t)d_R(\sigma,t)$$

and, moreover, the degree of $d_L(\sigma,t)$ ($d_R(\sigma,t)$) is greater than that of $n_L(\sigma,t)$ ($n_R(\sigma,t)$), for each t.

Example 3.1. Consider the periodic polynomials of periodic 3:

$$a(\sigma,t) = a_0(t)\sigma^2 + \sigma - 2, \quad b(\sigma,t) = b_0(t)\sigma$$

where $a_0(0) = a_0(1) = 0$, $a_0(2) = 1$, $b_0(0) = 1$, $b_0(1) = 0$, $b_0(2) = 2$. Now consider the rational left factorized operator

$$g(\sigma,t) = a(\sigma,t)^{-1}b(\sigma,t) = c(\sigma,t)d(\sigma,t)^{-1}$$

In order to find $c(\sigma,t)$ and $d(\sigma,t)$ one has to solve

$$a(\sigma,t)c(\sigma,t) = b(\sigma,t)d(\sigma,t)$$

where, in order to match the polynomial degrees,

$$\begin{aligned} d(\sigma,t) &= d_0(t)\sigma^2 + d_1(t)\sigma + d_2(t) \\ c(\sigma,t) &= c_0(t)\sigma + c_1(t) \end{aligned}$$

Therefore $c_1(t) = 0$ and

$$\begin{aligned} a_0(t)c_0(t+2) &= b_0(t)d_0(t+1) \\ a_0(t)c_1(t+2) + c_0(t+1) &= b_0(t)d_1(t+1) \\ c_1(t+1) - 2c_0(t) &= b_0(t)d_2(t+1) \end{aligned}$$

One feasible solution of this system is

$$c(\sigma,t) = c_0(t)\sigma, \quad d(\sigma,t) = d_1(t)\sigma + d_2(t)$$

where $c_0(1) = c_0(2) = 0$, $c_0(0) = 1$, $d_2(0) = d_2(2) = 0$, $d_2(1) = -2$, $d_1(0) = 0.5$, $d_1(1) = d_1(2) = 0$.

It is also important to point out that if the polynomial at the denominator is not regular, the system may not be properly causal even if the degree of the denominator is not smaller than that of the numerator for each t. The following example clarifies this issue.

Example 3.2. Consider a 2-periodic system with transfer operator

$$g(\sigma,t) = (p_0(t)\sigma + 1)^{-1}$$

where $p_0(0) = 0$, $p_0(1) = 1$. The denominator degree fluctuates periodically between 0 and 1, whereas the numerator has constant degree equal to zero. Notice that the denominator is unimodular so that it does not possess finite singularities. This implies that the transfer operator does not correspond to a proper dynamical system. To see this, it suffices to left-multiply both sides of

$$(p_0(t)\sigma + 1)y(t) = u(t)$$

by $p_o(t)\sigma - 1$, so that $y(t) = -p_0(t)u(t+1) + u(t)$. This is not a dynamical system since the output at odd time points depends on the value of future inputs.

It is obviously possible to define the concept of the *adjoint* of the rational operator

$$g(\sigma,t) = d_L(\sigma,t)^{-1}n_L(\sigma,t) = n_R(\sigma,t)d_R(\sigma,t)^{-1}$$

as

$$g^{\sim}(\sigma,t) = n_L^{\sim}(\sigma,t)d_L^{\sim}(\sigma,t)^{-1} = d_R^{\sim}(\sigma,t)^{-1}n_R^{\sim}(\sigma,t)$$

A T-periodic rational operator $f(\sigma,t)$ can be given a lifted reformulation, induced by the reformulation already defined for polynomials. Indeed, it is easy to show that the rational transfer matrix

$$\begin{aligned} G(\sigma^T,t) &= D_L(\sigma^T,t)^{-1}N_L(\sigma^T,t) \\ &= N_R(\sigma^T,t)D_R(\sigma^T,t)^{-1} \end{aligned}$$

is the so-called lifted system at time t, where, as obvious, $D_L(\sigma^T,t)$, $N_L(\sigma^T,t)$, $N_R(\sigma^T,t)$ and $D_R(\sigma^T,t)$ are the lifted reformulation of the T-periodic polynomials $d_L(\sigma,t)$, $n_L(\sigma,t)$, $n_R(\sigma,t)$ and $d_R(\sigma,t)$, respectively. As already pointed out, the matrix $G(\sigma^T,t)$ is constituted by rational functions in σ^T. However, recall that all matrices $D_L(\sigma^T,t)$, $N_L(\sigma^T,t)$, $N_R(\sigma^T,t)$ and $D_R(\sigma^T,t)$ have constant determinants.

From the lifted reformulation it becomes clear whether a T-periodic transfer operator $G(\sigma,t)$ corresponds to a dynamical system: this happens when its lifted reformulation $G(\sigma^T,t)$ corresponds to a time-invariant proper dynamical system.

Example 3.3. Consider again the transfer operator defined in Example (3.2). Then,

$$D(\sigma^2,t) = \begin{bmatrix} 1 & p_0(t) \\ p_0(t+1)\sigma^2 & 1 \end{bmatrix}, \qquad N(\sigma,t) = I_2$$

and the lifted transfer function is

$$G(\sigma^2,t) = \begin{bmatrix} 1 & -p_0(t) \\ -p_0(t+1)\sigma^2 & 1 \end{bmatrix}$$

Matrix $G(\sigma^2,0)$ is the transfer function from the lifted input signal $u_L^0(k) = [u(2k)\ u(2k+1)]'$ to the lifted output signal $y_L^0(k) = [y(2k)\ y(2k+1)]'$. As apparent, $G(\sigma^2,0)$ is a polynomial matrix, so that $g(\sigma,t)$ does not correspond to any T-periodic dynamical systems.

4. SPECTRAL FACTORIZATION

The problem of spectral factorization can be stated as follows. Let a T-periodic rational operator $\gamma(\sigma,t)$ be given, and assume that it satisfies the assumptions that

$$\gamma^{\sim}(\sigma,t) = \gamma(\sigma,t), \quad \gamma(\sigma,t) > 0$$

Therefore, the rational operator γ is auto-adjoint. Then, the problem consists in finding a minimum factor of $\gamma(\sigma,t)$, i.e a T-periodic rational operator $\hat{g}(\sigma,t)$, stable with stable inverse and such that

$$\hat{g}(\sigma,t)\hat{g}^{\sim}(\sigma,t) = \gamma(\sigma,t)$$

This problem is complicated in the periodic case, since the ring of scalar rational operators is not commutative. Of course, in the state space framework, one can resort to the theory related to the use of periodic Riccati equations.

When a factor $g(\sigma,t)$ is already given, the problem of spectral factorization consists in finding a stable, with stable inverse, transfer operator $\hat{g}(\sigma, t)$ such that

$$\hat{g}(\sigma, t)\hat{g}^{\sim}(\sigma, t) = g(\sigma, t)g^{\sim}(\sigma, t)$$

This means that $g(\sigma, t)$ and $\hat{g}(\sigma, t)$ share the same spectral properties. Of course, it is possible to perform the spectral factorization iff the given system does not have zeros or poles on the unit circle. As for the computation of a spectral factor, at least in the scalar case, one can resort to operator manipulations, as illustrated in the example below.

Example 4.1. Consider the system with the rational T-periodic operator with period $T = 2$:

$$g(\sigma, t) = (\sigma + a(t))^{-1}$$

and assume that the system is unstable, i.e. $a(0)a(1) \geq 1$. We want to find a rational T-periodic operator $\hat{g}(\sigma, t)$ such that

$$g^{\sim}(\sigma, t) = (\sigma^{-1} + a(t))^{-1}$$

This factor must be stable, invertible, with stable inverse. To this end, a parametric form reflecting these requirement is

$$\hat{g}(\sigma, t) = (\sigma + b(t))^{-1}c(t)\sigma$$

with $|b(0)b(1)| < 1$. Simple computations show that

$$c(t) = \frac{1}{a(t)}\sqrt{\frac{1 + a(t)^2}{1 + a(t+1)^2}}, \quad b(t) = a(t)c(t)^2 \quad (9)$$

Notice that $c(0)c(1) = b(0)b(1) = (a(0)a(1))^{-1}$.

The spectral factorization is very useful to solve the so-called *Wiener filtering problem.* Assume that w_1 and w_2 are white independent gaussian noises with unit covariances, and consider a T-periodic system described by the transfer operator $g(\sigma, t)$ and output y. Moreover, denote by s the to-be estimated signa, i.e.

$$y = g(\sigma, t)w_1 + w_2, \quad s = g(\sigma, t)w_1$$

an estimate

$$\hat{s} = f(\sigma, t)y$$

such that the covariance of the error $s - \hat{s}$ is minimized. This correspond to the H_2 filtering problem in the deterministic setting. The solution can be easily given a closed-loop formula by usual spectral factorization and square completing. It turns out that the optimal filter is

$$f_{ott}(\sigma, t) = \left[h(\sigma, t) - (h^{\sim}(\sigma, t))^{-1}\right]_s h(\sigma, t)^{-1}$$

where $[\cdot]_s$ denotes the stable proper part and $h(\sigma, t)$ is a stable, invertible, with stable inverse factor of $[g(\sigma, t)\ 1]$, i.e.

$$h(\sigma, t)h^{\sim}(\sigma, t) = g(\sigma, t)g^{\sim}(\sigma, t) + 1$$

Since $h(\sigma, t)^{-1}$ is stable, $(h^{\sim}(\sigma, t))^{-1}$ is unstable and its stable part corresponds to the direct input-output term. Hence, denoting this term as $\gamma(t)$, it follows that

$$f_{ott}(\sigma, t) = 1 - \gamma(t)h(\sigma, t)^{-1}$$

5. CONCLUSIONS

This paper contributes to periodic control theory with a view towards the polynomial approach (in discrete time) in the solution of the spectral factorization and Wiener filtering problems.

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