ON A DESIGN METHOD OF SAMPLED-DATA DELAYED FEEDBACK CONTROLLER

Kentaro Hirata *,[1]

* *Graduate School of Engineering, Osaka Prefecture University, 1-1 Gakuen-cho, Sakai, Osaka, Japan,*
kent@ecs.ees.osakafu-u.ac.jp

Abstract: The design problem of a delayed feedback controller in the sampled-data setting is considered. Instead of the indirect method via the plant augmentation, we impose certain interpolation conditions on the free parameter of the stabilizing controller parametrization to embed the DFC structure into the controller directly. Since the underlying stabilization problem to be solved is independent of the length of the delay, the proposed method avoids the computational burden when the ratio of the periods of the target orbit and the sampling becomes larger.

Keywords: Delayed Feedback, Unstable Periodic Orbit, Parameterization, Interpolation

1. INTRODUCTION

Delayed Feedback Control (DFC) was proposed by Pyragas (Pyragas, 1992) to stabilize unstable periodic orbits (UPOs) embedded in a chaotic attractor. One theoretical advantage of the DFC is that it can stabilize the system to its unknown unstable equilibrium point or periodic orbit (Kokame *et. al.*, 2001a; Kokame *et. al.*, 2001b).

Under the continuous-time delayed feedback, the resulting closed-loop system becomes a delay-differential system. Since the corresponding state space is *infinite-dimensional*, some difficulties arise in its analysis and, especially in its synthesis. Although an exact stability analysis for retarded systems (Marshall *et. al.*, 1992) can be used for low order systems, for higher order cases, only sufficient conditions based on the robust stabilization via rational approximations (differential feedback (Kokame *et. al.*, 2000) or Padé approximation) are available. The continuous-time DFC and related issues provide mathematically interesting and challenging problems (Hirata and Kokame, 2003a; Hirata and Kokame, 2003b; Hirata and Kokame, 2003c; Hirata and Kokame, 2004).

Meanwhile, when it comes to the digital implementation stage after the design process, the delay element must be discretized. Therefore, it is natural to consider the sampled-data DFC from the practical viewpoint. In contrast to the continuous-time case, the analysis/synthesis of the discrete-time DFC is a *finite-dimensional* problem. This fact enables relatively thorough treatment as reported in (Konishi and Kokame, 1998; Yamamoto *et. al.*, 2001; Yamamoto *et. al.*, 2002). They are based on the well-known technique of the plant augmentation as used in the LQ-servo design or the H^∞ loop shaping. However, when one considers the stabilization of UPO, such an approach has the following potential drawback: As the ratio of the periods of the target orbit and the sampling becomes larger, the size of the problem

[1] Corresponding author. Partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists, 14750374, 2002-2004

to be solved also gets larger compared to the original problem. Strictly speaking, the UPO in the discrete-time is fictional and not related to a physical phenomenon. Therefore this problem can be addressed properly only in the sampled-data framework.

Recently a novel design procedure for the delay systems is investigated in (Meinsma *et. al.*, 2002; Mirkin, 2003) and others. They consider the problem that when a (stabilizing) H^∞ controller for delay-free system takes the form of the series connection of a causal part and pure delay. By pushing the delay element into the plant side, one can obtain a controller for the delay system. We employ a similar idea here and consider the problem that when a stabilizing controller takes the form of the series connection of a rational part and the state difference part. Via the stabilizing controller parametrization and the interpolation conditions on the free parameter, we derive a design method for the sampled-data DFC controller.

2. UNSTABLE PERIODIC ORBIT AND DELAYED FEEDBACK CONTROL

In the chaos oriented literature related to DFC, the UPO of nonlinear systems is considered in general. However, we restrict our attention here to the UPO generated by linear systems as depicted in Fig. 1. Let Σ_p and Σ_g denote an unstable plant

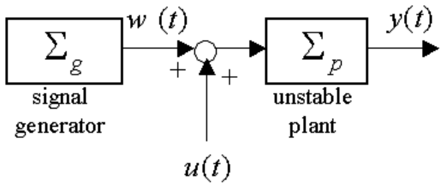

Fig. 1. Linear system with UPO

and an external signal generator, respectively. Both are assumed to be finite-dimensional, i.e., can be represented as

$$(\Sigma_g) \quad \dot{x}_g = A_g x_g, \quad w = C_g x_g, \quad x_g(0) = x_g^0 \neq 0,$$

$$(\Sigma_p) \quad \dot{x} = Ax + B(u + w), \quad y = Cx, \quad x(0) = x_0,$$

where $\max_i \mathbf{Re}[\lambda_i(A)] \geq 0$ and $\mathbf{Re}[\lambda_i(A_g)] = 0$, for $\forall i$. We assume that this system has an UPO in the following sense:

Assumption 1. This system generates a T-periodic output $y(t)$ and all oscillating modes are solely coming from Σ_g.

In this situation, the unstable poles of $P(s) := C(sI - A)^{-1}B$ must be cancelled out by the zeros of $G(s) := C_g(sI - A_g)^{-1}x_g^0$. Then there exist initial conditions x_g^0, x_0 such that the unstable modes of Σ_p do not appear in $y(t)$.

Obviously, such a cancellation is not allowed due to the internal stability requirement and infinitesimally small mismatch of the initial conditions results in divergence of the states. Hence this orbit is unstable.

Let $\omega_0 = 2\pi/T$ and w_k denote its harmonics $\omega_k = (k + 1)\omega_0$, $k = 1, 2, \cdots$. Note that the frequency components of $w(t)$ consist of $\{\omega_k\}$. Suppose that there exists a stabilizing output DFC controller $u(s) = C(s)(1 - e^{-Ts})y(s) = \hat{C}(s)y(s)$ with $|C(j\omega_k)| < \infty$. Then the closed-loop frequency response from w to y for $\{\omega_k\}$ are given by

$$\frac{P(j\omega_k)}{1 - C(j\omega_k)(1 - e^{-j\omega_k T})P(j\omega_k)} = P(j\omega_k),$$

$k = 0, 1, \cdots$. Thus the closed-loop behaviour against $w(t)$ is exactly the same as the open-loop virtual one. Note that this feature is achieved by the following facts:

a) $\hat{C}(s)$ is a stabilizing controller of $P(s)$.
b) $\hat{C}(s)$ has zeros [2] at $s = j\omega_k$, $k = 0, 1, \cdots$.

We consider the counterpart of these requirements in the case of sampled-data control systems. As is well-known, a) is equivalent to a discrete-time stabilization problem. For band-limited external signals, b) can be replaced by a finite number of interpolation conditions. Since $w(t)$ is generated by a finite-dimensional system Σ_g, assuming the band limited signal is not unrealistic. It is a contrast to the sampled-data H^∞ control case where the input signal may contain arbitrary frequency component. The detailed procedure is given in the next section.

3. DESIGN PROCEDURE

Let $\mathcal{S}$ and $\mathcal{H}$ denote the sampler and the zero-th order holder with the sampling period h. We consider a sampled-data feedback configuration in Fig. 2. To derive a compact formula, let $P(s)$ be a single-input and p-output plant with the following n-dimensional state space representation:

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with (C, A): observable, (A, B): controllable. The following assumptions are made.

Assumption 2.

a) The sampling period h is non-pathological.

[2] This is an interesting contrast to the internal model principle case, where the *poles* of the controller plays an important role to track the external signals.

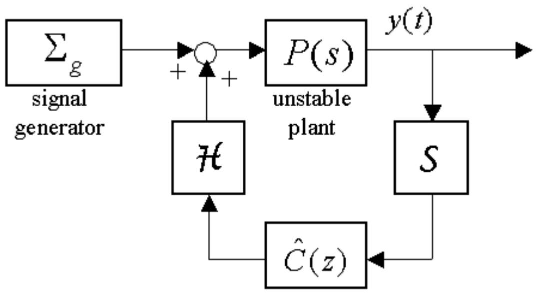

Fig. 2. Feedback Configuration

b) The target period T is an integer multiple of the sampling period h, i.e., $T = \ell h$, $\ell \in \mathbf{N}$.
c) None of $jk(2\pi/T)$, $k = 0, \cdots, \ell - 1$ corresponds to an eigenvalue of A.

The design problem here is to find a controller $\hat{C}(z)$ with the following properties:

a') $\hat{C}(z)$ is a stabilizing controller of $P(s)$ in the sampled data setting (Fig. 2).
b') Every root of $1 - z^{-\ell} = 0$ is a blocking zero of $\hat{C}(z)$.

Theorem 3. Let $\{z_i\}$ denote the roots of $z^\ell - 1 = 0$ and

$$A_d = e^{Ah}, \quad B_d = \int_0^h e^{A(h-\xi)} B d\xi.$$

Given the feedback gains F, L and a ℓ-th stable polynomial $q(z)$:

$$q(z) = z^\ell + q_1 z^{\ell-1} + \cdots + q_{\ell-1} z + q_\ell,$$

set

$$R = \mathbf{diag}\left[\, q(z_1) \ \cdots \ q(z_\ell) \,\right],$$

$$V = \begin{bmatrix} 1 & z_1 & \cdots & z_1^{\ell-1} \\ \vdots & & & \vdots \\ 1 & z_\ell & \cdots & z_\ell^{\ell-1} \end{bmatrix},$$

$$\hat{w}(z) = \left[\begin{array}{cc|c} A_d & B_d F & 0 \\ 0 & A_d + B_d F & -L \\ \hline F & -F & 0 \end{array}\right], \quad (1)$$

$$W = \begin{bmatrix} \hat{w}(z_1) \\ \vdots \\ \hat{w}(z_\ell) \end{bmatrix}, \quad A_q = \begin{bmatrix} 0 & \cdots & 0 & -q_\ell \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & \vdots \\ \mathbf{0} & & 1 & -q_1 \end{bmatrix}.$$

Let $e_1 = [1 \ 0 \ \cdots \ 0], e_\ell = [0 \ \cdots \ 0 \ 1] \in \mathbf{R}^{1 \times \ell}$ and

$$A_c = \begin{bmatrix} A_d + B_L F + LC & B_L e_\ell \\ -B_q C_F & A_q - B_q D e_\ell \end{bmatrix},$$

where

$$B_L = B_d + LD, \ C_F = C + DF, \ B_q = V^{-1} R W.$$

Choose F and L as

R1 $A_d + B_d F$ and $A_d + LC$ are stable,
R2 the eigenvalues of A_c do not contain $\{z_i\}$.

Then a stabilizing ℓ-th DFC controller is given by

$$\hat{C}(z) = \left[\begin{array}{c|c} A_c & \begin{matrix} -L \\ B_q \end{matrix} \\ \hline F \ \ e_\ell & 0 \end{array}\right]. \quad (2)$$

Alternatively, by extracting the DFC structure $1 - z^{-\ell}$, $\hat{C}(z)$ can be expressed as

$$\hat{C}(z) = \tilde{C}(z)(1 - z^{-\ell}), \quad (3)$$

$$\tilde{C}(z) = \left[\begin{array}{c|c} A_c & \begin{matrix} -L \\ B_q \end{matrix} \\ \hline e_\ell \ \ e_\ell \tilde{T} + [F \ e_\ell] & 0 \end{array}\right], \quad (4)$$

where

$$A_0 = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \vdots & 0 \\ & \ddots & 0 & \vdots \\ \mathbf{0} & & 1 & 0 \end{bmatrix},$$

and $\tilde{T}$ is the unique solution of the Sylvester equation:

$$A_0 \tilde{T} - \tilde{T} A_c + e_1^T [F \ e_\ell] = 0. \quad (5)$$

PROOF. The zero-th order discretized plant of $P(s)$ is given by

$$\hat{P}(z) = \left[\begin{array}{c|c} A_d & B_d \\ \hline C & D \end{array}\right].$$

As is well understood, the closed-loop stability of the sampled-data systems in Fig. 2 is equivalent to the stability of the pair $(\hat{P}(z), \hat{C}(z))$ in the discrete-time sense. Due to Assumption 2-a, the observability and the controllability are preserved via this discretization, e.g., (Chen and Francis, 1995). Thus there exist F and L satisfying **R1** and all stabilizing controller parametrization for $\hat{P}(z)$ is given by

$$C_Q(z) = \{U(z) + M(z)Q(z)\}\{V(z) + N(z)Q(z)\}^{-1},$$

where $\forall Q(z) \in H^\infty(\mathbf{D})$ and

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} = \left[\begin{array}{c|cc} A_d + B_d F & B_d & -L \\ \hline F & I & 0 \\ C_F & D & I \end{array}\right].$$

Alternatively, $C_Q(z)$ is given in the LFT form as

$$C_Q = \mathbf{LFT}(J, Q) \quad (6)$$

where

$$J = \left[\begin{array}{c|cc} A_d + B_L F + LC & -L & B_L \\ \hline F & 0 & I \\ -C_F & I & -D \end{array}\right]. \quad (7)$$

(See (Zhou *et.al.*, 1996), for example.) Choose $Q(z)$ as

$$Q(z) = \frac{1}{q(z)}[r_1(z) \ \cdots \ r_p(z)]$$

$$r_i(z) = r_i^{(1)} z^{\ell-1} + \cdots + r_i^{(\ell-1)} z + r_i^{(\ell)}.$$

Then, provided Assumption 2-c [3], one can express the interpolation condition $C_Q(z_i) = 0$, $i = 1, \ldots, \ell$ as

$$-M^{-1}U(z_i) = Q(z_i). \quad (8)$$

A state space manipulation shows that the left hand-side of (8) is equal to $\hat{w}(z_i)$ in (1). On the other hand, the right hand-side of (8) is expressed as

$$\frac{1}{q(z_i)}[1 \ z_i \ \cdots \ z_i^{\ell-1}] \begin{bmatrix} r_1^{(\ell)} & \cdots & r_p^{(\ell)} \\ \vdots & & \vdots \\ r_1^{(1)} & \cdots & r_p^{(1)} \end{bmatrix}. \quad (9)$$

Denote the last matrix in (9) by B_q. Then by stacking up the condition (8), we obtain $RW = VB_q$. Because $z_i \neq z_j$ when $i \neq j$, the Vandermonde matrix V is non-singular. Therefore B_q is given by $B_q = V^{-1}RW$. Since $Q(z)$ is realized as

$$Q(z) = \left[\begin{array}{c|c} A_q & B_q \\ \hline e_\ell & 0 \end{array}\right],$$

substitution into (6) yields (2). By **R2**, the assigned zeros are not canceled by the poles of $\hat{C}(z)$. The existence of the unique solution of (5) is guaranteed since $\lambda_i(A_0) \neq \lambda_j(A_c)$, $\forall i, j$, e.g., Lemma 13.2 of (Kodama and Suda, 1978).

Note 1. The order of the controller (2) is $n+\ell$. In contrast, the design via the plant augmentation usually gives a controller of the order $n + 2\ell$ (including $1-z^{-\ell}$). Although (3) looks like having the order $n+2\ell$, it is not true since stable pole-zero cancellation at $z = 0$ with multiplicity ℓ occurs.

Note 2. Since every complex interpolation data appears in a pair with its conjugate, the coefficient matrix B_q is always real. Let

$$M_{RW} = \begin{bmatrix} I & I \\ -jI & jI \end{bmatrix} \begin{bmatrix} RW \\ \overline{RW} \end{bmatrix},$$

$$M_V = \begin{bmatrix} I & I \\ -jI & jI \end{bmatrix} \begin{bmatrix} V \\ \overline{V} \end{bmatrix},$$

where $\overline{(\cdot)}$ denote the complex conjugate. Then one may solve

$$M_{RW} = M_V B_q \quad (10)$$

instead of $RW = VB_q$ for the computational accuracy. (Since both M_{RW} and M_V are real matrices, one can avoid the inverse operation of the complex matrix.) Because

$$\mathbf{rank}\, M_V = \mathbf{rank} \left[M_V \ M_{RW} \right] = \ell,$$

(10) has a unique solution (e.g., Theorem 3.7 in (Kodama and Suda, 1978)).

4. NUMERICAL EXAMPLES

4.1 Stabilization of Autonomous Oscillation of Ball & Beam System

Consider a ball & beam system depicted in Fig. 3. The control input u is the angle reference ϕ_{ref} of the servo motor and the output y is the position of the ball z. Thus this is a SISO system. With the physical dimensions of the experimental apparatus in our laboratory, its transfer function is given by $P(s) = 5.05/s^2$. To satisfy Assumption 2-c, a minor feedback $u = -y/10.1 + \tilde{u}$ is applied. Then the transfer function from $\tilde{u}$ to y is $P'(s) = 5.05/(s^2 + 0.5)$. One can construct an external signal $w^*(t)$ which generates a periodic orbit with $T = 1.5$ for the open-loop system. However, since $P'(s)$ has poles on the imaginary axis, the target orbit is not stable. Thus it cannot be achieved when the initial value error exists. (Fig. 4). Let $h = 0.05$ and $\ell = 30$. With

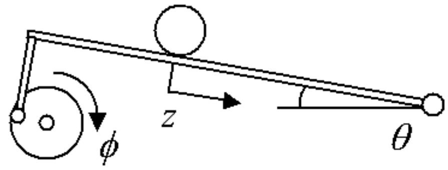

Fig. 3. Ball & Beam System

$q(z) = z^\ell - 2^{-\ell}$, $\mathbf{eig}(A_d + B_d F) = \{0.6, 0.8\}$ and $\mathbf{eig}(A_d + LC) = \{0.9, 0.8\}$, one can compute $\hat{C}(z)$ by Theorem 3. Fig. 5 shows that the target orbit is successfully stabilized via DFC. In Fig. 6, the output of the external signal generator (dashed line) and DFC input (solid line) are plotted. It can be seen that the DFC input tends to be zero as the system enters into a steady state. Note that if we take the plant augmentation approach, the dimension of the stabilization problem to be solved blows up from 2 to 32.

[3] Let z_u be an unstable poles of $\hat{P}$. Since $\hat{P} = NM^{-1}$, $M(z_u) = 0$ and hence we cannot assign z_u as a zero of C_Q by adjusting Q via $U + MQ$. Assumption 2-c excludes this situation. Practically, it guarntees each element of $\hat{w}(z_i)$ in (1) takes finite value.

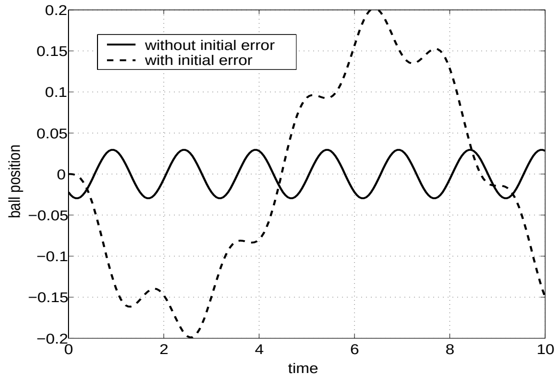

Fig. 4. Open-loop time response

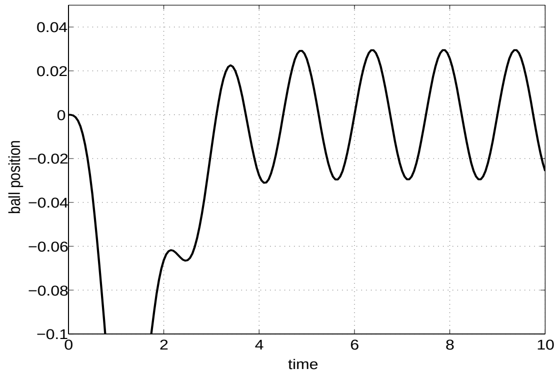

Fig. 5. Closed-loop time response (output)

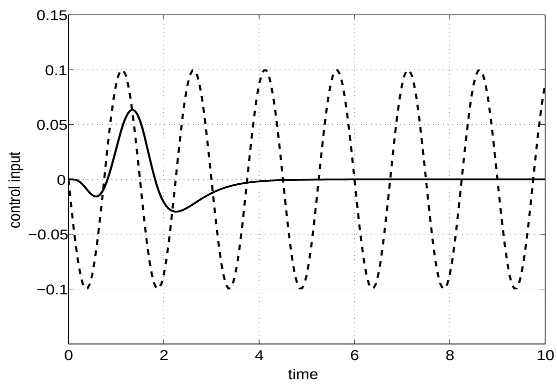

Fig. 6. Closed-loop time response (control input)

4.2 Tracking to Equilibrium Shift of Inverted Pendulum System

Next example is a SIMO plant, an inverted pendulum system placed on an unknown slope (Hirata *et. al.*, 2001). As shown in Fig. 7, z, ϕ and θ denote the cart position, the angle of pendulum and slope, respectively. Let μ, p_g, l and I be the viscous coefficient at the pivot, the center of gravity, the distance from the pivot to p_g and the moment of inertia around p_g, respectively. Since the cart is driven by a velocity servo unit, the dynamics from input u to $\dot{z}$ can be approximated by a first order system with gain K_z and time constant T_z. Then a state space model is given by

$$\dot{x} = Ax + B_1 u + B_2 \theta, \quad y = Cx, \tag{11}$$

$$x = \begin{bmatrix} z \\ \dot{z} \\ \phi \\ \dot{\phi} \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -T_z & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & mlT_z/I & mlg/I & -\mu/I \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ K_z \\ 0 \\ -mlK_z/I \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -mlg/I \end{bmatrix},$$

where $m = 0.023$, $l = 0.2$, $I = 4ml^2/3$, $g = 9.8$, $K_z = 80\pi$, $T_z = 95$. Our objective is to stabilize this system at its equilibrium point via the sampled-data DFC without the information of θ. Note that the target orbit is a biased fixed point and not periodic in the usual sense. Since the open-loop system has a pole at $s = 0$, as in the previous example, we apply the following minor feedback:

$$u = \alpha z + \tilde{u}, \tag{12}$$

with $\alpha = 0.1$. Thus the system is modified into

$$\dot{x} = \tilde{A}x + B_1 \tilde{u} + B_2 \theta. \tag{13}$$

We will design $\hat{C}(z)$ for $h = 0.01$, $\ell = 10$ and $q(z) = z^\ell - 0.7^\ell$. The gain F is determined so as to achieve the same pole location as continuous-time LQ regulator with the following performance index

$$J = \int_0^\infty x^T Q x + \tilde{u}^T R u \ dt,$$

with $Q = \mathbf{diag}[10^3 \ 1 \ 1 \ 1]$ and $R = 1$, under the ZOH discretization. Since L can be interpreted as an observer gain, by following the rules of thumb, it is determined so that $A_d + LC$ have the eigenvalues 10% smaller (faster) than that of $A_d + B_d F$ in magnitude. The time responses against the step change in the slope angle θ are illustrated in Fig. 8 and 9. For comparison, the result of the case $Q(z) = 0$ corresponding to the LQG design is also shown. The LQG controller yields a steady state error in the cart position as

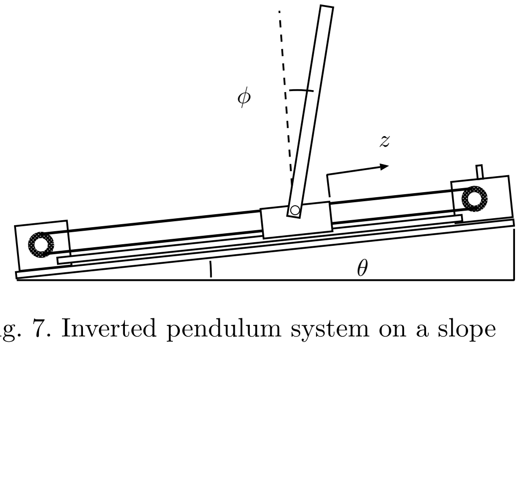

Fig. 7. Inverted pendulum system on a slope

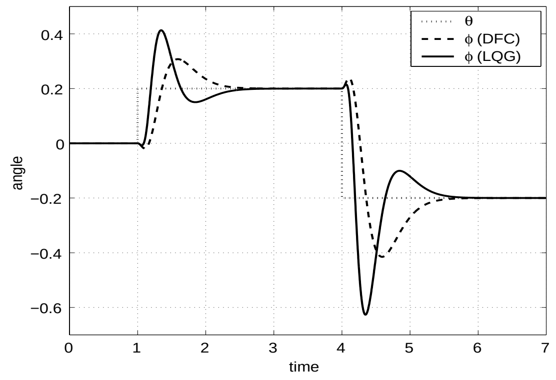

Fig. 8. Time Response (angle)

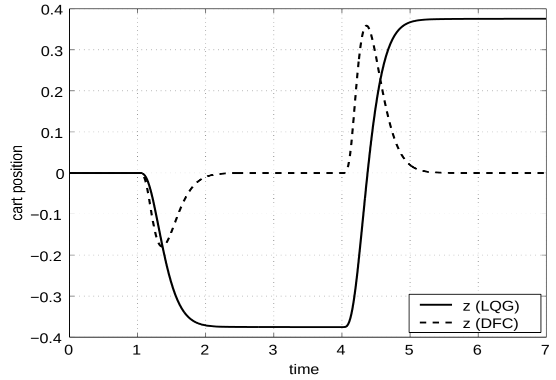

Fig. 9. Time Response (cart position)

a side effect of the equilibrium shift due to the change in θ. In contrast, there is no such a steady state error in the DFC case.

5. CONCLUSIONS

A design method of the sampled-data delayed feedback controller is proposed. It is based on the stabilizing controller parametrization and the interpolation conditions on the free parameter $Q(z)$. Two illustrative examples are shown.

Essentially the MIMO case can be treated along the same line because the requirement for the DFC structure is simply given by the blocking zero conditions. The remaining difficulty is how to express the controller in a concise state space form. In addition, it is desired to obtain a realization independent formula for SIMO case presented here.

To improve the intersample behaviour beyond the mere stability, one should take the performance measure of the sampled-data control systems such as H^∞ norm into account. Since the sampled-data H^∞ controller is also given in a LFT form, such an extension may be possible by employing the Nevanlinna-Pick type interpolation.

REFERENCES

Chen, T. and Francis, B. (1995). Optimal Sampled-Data Control Systems, Springer.

Hirata, K. *et. al.* (2001). Observer-based Delayed Feedback Control of Continuous-time systems, *Proc. of ACC2001*.

Hirata, K. and Kokame, H. (2003a). Stability analysis of linear systems with state jump, *Proc. of CCA2003* 949–953.

Hirata, K. and Kokame, H. (2003b). Delayed feedback control of linear systems with state jump, *Proc. of IFAC TDS2003*.

Hirata, K. and Kokame, H. (2003c). Stability analysis of retarded systems via lifting technique, *Proc. of CDC2003*.

Hirata, K. and Kokame, H. (2004). On numerical computation of the spectrum of a class of convolution operators related to delay systems, *Proc. of MTNS2004*.

Kodama, S. and Suda, N. (1978). Matrix Theory for Systems & Control (in Japanese), Corona.

Kokame, H. *et. al.* (2000). Stabilization via the delayed feedback with a large delay, *Proc. of IFAC LTDS2000*, 15–20.

Kokame, H. *et. al.* (2001a). State difference feedback for stabilizing uncertain steady states of nonlinear systems, *Int. J. Cntr.*, **74**, 537–546.

Kokame, H. *et. al.* (2001b). Difference Feedback Can Stabilize Uncertain Steady States, *IEEE Trans. Automat. Contr.*, **46**, 1908-1913.

Konishi, K. and Kokame, H. (1998). Observer-based delayed-feedback control for discrete-time chaotic systems, *Phys. Lett. A*, **248**, 359–368.

Marshall, J. *et. al.* (1992). Time-Delay Systems: -Stability and Performance Criteria with Applications-, Ellis Horwood.

Meinsma, G. *et. al.* (2002). Control of Systems with I/O Delay via Reduction to a One-Block Problem, *IEEE Trans. Automat. Contr.*, **47**, 1890-1895.

Mirkin, L. (2003). On the Extraction of Dead-Time Controllers and Estimators from Delay-Free Parametrizations, *IEEE Trans. Automat. Contr.*, **48**, 543-553.

Pyragas, K. (1992). Continuous control of chaos by self-controlling feedback, *Physics Letter A*, **170**, 421–428.

Yamamoto, S. *et. al.* (2001). Dynamic Delayed Feedback Controllers for Chaotic Discrete-Time Systems, *IEEE Trans. Circuits and Systems*, **48**, 785–789.

Yamamoto, S. *et. al.* (2002). Delayed Feedback Control with Minimal-Order Observer for Stabilizaiton of Chaotic Discrete-Time Systems, *Int. J. Bifurcation and Chaos*, **12**, 1047–1055.

Zhou, K. *et.al.* (1996). Robust and Optimal Control, Prentice-Hall.