THE RATE OF CHANGE OF AN ENERGY FUNCTIONAL FOR AXIALLY MOVING CONTINUA

Kyung-Jinn Yang*, Keum-Shik Hong, and Fumitoshi Matsuno***

** Department of Mechanical Engineering and Intelligent Systems, University of Electro-Communications 1-5-1, Chofugaoka, Chofu, Tokyo 182-8585, Japan. Tel/Fax: +81-424-43-5403, Emails: {jinnky, matsuno}@hi.mce.uec.ac.jp*

*** School of Mechanical Engineering, Pusan National University, San 30 Jangjeon-dong Gumjeong-gu, Busan, 609-735, Korea. Tel.: +82-51-510-2454, Fax: +82-51-514-0685, Email: kshong@pusan.ac.kr*

Abstract: In this paper, with the utilization of a transport theorem and three-dimensional version of Leibniz's rule, the procedure for deriving the time rate of change of an energy functional for axially moving continua is investigated. In the control engineering, the correct solution of the time derivation of an energy functional is essential for designing an effective controller, especially, in the Lyapunov method. The key point to get the correct solution for axially moving continua is that the time derivation of an energy functional should be taken into account under Eulerian description with a physical concept. A novel way of deriving the time rate of change of the energy functional, then, is proposed.

Keywords: Axially moving continua, Leibniz's rule, transport theorem, Hamilton's principle, Eulerian and Lagrangian descriptions.

1. INTRODUCTION

Axially moving continua can be found in various engineering applications. Vibration control schemes for moving continua include references (Yang *et al*., 2004, 2005; Choi *et al*., 2004; Fung *et al*, 2002; Li *et al*., 2002; Zhu, 2002; Li and Rahn, 2000; Lee and Mote, 1999) among others. To design a suitable controller, most of above references have employed the Lyapunov method, in which an effective control law is established through the time derivation of the energy functional of the considering system. Thus, it is essential that the time differentiation of the energy functional considered should be exactly performed in a proper mathematical manner.

Renshaw *et al*. (1998) have suggested a derivation method in Eulerian description for the energy functionals of prototypical axially moving string and beam models, and have concluded that a conserved Eulerian functional is the Jacobi integral of the system and qualifies as a Lyapunov functional when it is positive definite. The conclusions have been accepted for calculating the time derivation of the energy functionals of axially moving systems in some papers.On the other hand, other papers have accepted the one-dimensional transport theorem or a differentiation method in Lagrangian description to get the time derivation of the energy functionals. Hence, the motivation of this paper is to establish a general theory for calculating the time differentiation of the energy functional of axially moving continua.

2. PROBLEM FORMULATION

Fig. 1 shows the schematic of an axially moving string with the fixed two support rolls. Let t be the time, x be the spatial coordinate along the longitude of motion, v be the traveling speed of the string, $w(x,t)$ be the transversal displacement of the string at spatial coordinate x and time t, and l be the length of the string from the left to the right supports. Also, let ρ be the mass per unit length and T_s be the tension applied to the string. Because the string travels at an axial speed, v , the total derivative operator (material derivative) with respect to time should be defined as

$$\frac{D(\cdot)}{Dt} = (\cdot)_t + v(\cdot)_x , \qquad (1)$$

where $(\cdot)_t = \partial(\cdot)/\partial t$ and $(\cdot)_x = \partial(\cdot)/\partial x$ denote the partial derivatives. The mechanical energy of the string between $x=0$ and $x=l$ is then given by

$$V_{string} = \frac{1}{2}\int_0^l \rho\left(\frac{Dw}{Dt}\right)^2 dx + \frac{1}{2}\int_0^l T_s w_x^2 \, dx . \qquad (2)$$

By using the extended Hamilton's principle for axially moving continua (see Section 4, below), the equations of motion and boundary conditions of the axially moving string are derived as follows:

$$\rho w_{tt} + 2\rho v w_{xt} + \rho v^2 w_{xx} - T_s w_{xx} = 0 , \ 0 < x < l ,$$
$$w(0,t) = w(l,t) = 0 . \qquad (3)$$

In the process of designing a controller using the Lyapunov method, it is essential to treat and analyze the time derivative of a Lyapunov function candidate, that is, the mechanical energy of the system considered. The Eulerian description of the mechanical energy of the span [0, l] in Fig. 1 is

$$V_{Eul} = \frac{1}{2}\int_0^l \left[\rho(w_t + v w_x)^2 + T_s w_x^2\right] dx . \qquad (4)$$

Alternatively, the Lagrangian description of the mechanical energy of the set of particles between $x = vt$ and $vt + l$ is

$$V_{Lag} = \frac{1}{2}\int_{vt}^{vt+l} \left[\rho(w_t + v w_x)^2 + T_s w_x^2\right] dx , \qquad (5)$$

where $V_{Lag} = V_{Eul}$ at $t=0$. But, note that dV_{Eul}/dt and dV_{Lag}/dt are distinct.

For dV_{Eul}/dt , the following result is derived:

$$\frac{dV_{Eul}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(w_t + v w_x)^2 + T_s w_x^2\right] dx$$
$$= \frac{1}{2}(T_s v - \rho v^3) w_x^2 \Big|_0^l , \qquad (6)$$

where the first equality has been derived using the one-dimensional Leibniz's rule of the form

$$\frac{d}{dt}\int_{A(t)}^{B(t)} f(t,x)dx = \int_{A(t)}^{B(t)} \frac{\partial}{\partial t} f(t,x)dx + f(t,B(t))\frac{dB(t)}{dt} - f(t,A(t))\frac{dA(t)}{dt} , \qquad (7)$$

and (3) has been used in deriving the second equality. Using (7), the first equality in (6) can also be derived from the equation

$$\frac{dV_{Eul}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(w_t + v w_x)^2 + T_s w_x^2\right] dx$$
$$+ \frac{1}{2}\left[\rho(w_t + v w_x)^2 + T_s w_x^2\right]\frac{dx}{dt}\Big|_0^l , \qquad (8)$$

where $dx/dt = 0$ is treated as zero at $x = 0, l$.

For dV_{Lag}/dt , conversely, the limits of integration are time dependent. Hence, the result using (7) is

$$\frac{dV_{Lag}}{dt} = \frac{1}{2}\int_{vt}^{vt+l} \frac{\partial}{\partial t}\left[\rho(w_t + v w_x)^2 + T_s w_x^2\right] dx$$
$$+ \frac{1}{2}\left[\rho(w_t + v w_x)^2 + T_s w_x^2\right]\frac{dx}{dt}\Big|_{vt}^{vt+l} = T_s v w_x^2 \Big|_{vt}^{vt+l} . \qquad (9)$$

It is observed that (9) is valid only at $t=0$ because (3) applies only when the material particles associated with V_{Lag} include the string span [0, l], and hence (9) is a valid energy functional only at $t=0$. From the reason, the followings have been concluded: A positive definite Lagrangian functional, even though it is a material derivative, cannot be used as a Lyapunov functional, because its time derivative is not valid for more than an instant.

However, strictly speaking, the way of evaluating the Eulerian functional (6) is not correct. In fact, the derivation methods given by (6) and (8) are only mathematics without considering a physical concept of axially moving continua. Therefore, the correct evaluation of the time derivation of the V_{string} in (2) should be given in the following form:

$$\frac{dV_{string}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(w_t + v w_x)^2 + T_s w_x^2\right] dx$$
$$+ \frac{1}{2} v \left[\rho(w_t + v w_x)^2 + T_s w_x^2\right]_0^l = T_s v w_x^2 \Big|_0^l , \qquad (10)$$

which will be explained in detail in the next section.

3. RATE OF CHANGE: THE CORRECT METHOD

To obtain the time derivation of an energy functional of axially moving continua, a three-dimensional version of the rate of change is derived in this section. To accomplish this, with noting that time-varying means moving and/or deforming while time-invariant means fixed, that is, neither moving nor deforming, the following symbols are first defined:

(A1) $\hat{U}(t)\big|_{open}^{m-mov}$ is the system volume as a collection of translating material particles comprising the part of translating continua passing through a specific region of interest, that is, a control volume. The specific region is supposed to be time-varying but not fixed, and then $\hat{U}(t)\big|_{open}^{m-mov}$ seems like an open system since the variation of the region means that the material particles comprising the system volume can be crossing the system boundaries.

(A2) On the other hand, if the specific region of interest is fixed, the mass of the part of translating continua is constant. Hence, such case seems like a closed system, and then the system volume is described as $\hat{U}(t)\big|_{clos}^{m-mov}$.

(A3) When the system is a stationary continua, that is, not axially moving, then from (A1) and (A2), $\hat{U}(t)\big|_{open}^{m-fix}$ is used as the system volume for an open system with the varying specific region of interest while $\hat{U}\big|_{clos}^{m-fix}$ is for an closed system with the fixed specific region of interest.

$U(t)\big|_{open}^{v-mov}$ denotes the control volume for (A1), and the external boundary of which is the control surface defined as $S(t)\big|_{open}^{v-mov}$. The control volume for (A2) is time-invariant due to the fixed specific region of interest, and then defined as $U\big|_{open}^{v-fix}$. The control surface that encloses $U\big|_{open}^{v-fix}$ is used as $S\big|_{open}^{v-fix}$.For the stationary continua systems in (A3), $U(t)\big|_{open}^{v-mov}$ and $U\big|_{clos}^{v-fix}$ are defined as the control volumes for $\hat{U}(t)\big|_{open}^{m-fix}$ and $\hat{U}\big|_{clos}^{m-fix}$, respectively.

Note that, to derive the three-dimensional version of the rate of change, the translating continua system have to be analyzed in view of Eulerian description since our attention is focused on *what happens on the moving continua (i.e., system volume) in the specific region of interest (i.e., control volume) as time passes*. Hence, now consider the system with $\hat{U}(t)\big|_{open}^{m-mov}$ in (A1) which is a typical case for the continua systems. Let $\vec{u}$ be *the traveling velocity of the axially moving continua*, and $\vec{u}_S$ be *the moving velocity of time-varying control surface* bounding the time-varying control volume.

At first, it is necessary to deal with the function $\varphi(t)$ defined by an integral of the form

$$\varphi(t) = \int_{\hat{U}(t)\big|_{open}^{m-mov}} \rho\zeta(\vec{x},t)\, dU = \int_{\hat{U}(t)\big|_{open}^{m-mov}} \psi(\vec{x},t)\, dU \,, \quad (11)$$

where $\vec{x}$ is the position vector relative to a chosen origin, ρ is the continua density, and the quantity ζ denotes the property of interest per unit mass, and then ψ represents the continua property that occurs in $\hat{U}(t)\big|_{open}^{m-mov}$ such as fluxes of mass, linear momentum, angular momentum, internal energy, and kinetic energy, et cetera. In (11), the volume integral is a triple integral.

Note that the position vector $\vec{x}$ is a time-independent variable in Eulerian description whereas, in Lagrangian description, the vector $\vec{x}$ is a time-dependent variable. Hence, in Eulerian description, $\vec{x}$ describing a material point at time t is called the field coordinate, and *the velocity given in field coordinates is* $\partial\vec{x}/\partial t = 0$ even though the material velocity (Cartesian velocity) is given as $d\vec{x}/dt = \vec{u}$.

Now, an expression for the time rate of change $d\varphi/dt$ for $\varphi(t)$ in (11), that is, a three-dimensional version of axially moving continua, is derived in Eulerian description. By introducing a physical notion such as fluid dynamics, the time derivation $d\varphi/dt$ for the system (A1) is obtained as

$$\frac{d}{dt}\int_{\hat{U}(t)\big|_{open}^{m-mov}} \psi\, dU = \int_{U(t)\big|_{open}^{v-mov}} \frac{\partial}{\partial t}\psi\, dU + \oint_{S(t)\big|_{open}^{v-mov}} \psi\, \vec{u}\cdot\hat{n}\, dS \,. \quad (12)$$

Applying (12), the time rate of change for other systems in (A2) and (A3) can be easily obtained.

For the stationary continua systems with $\hat{U}(t)\big|_{open}^{m-fix}$ and $\hat{U}\big|_{clos}^{m-fix}$, since the material velocity $\vec{u}$ is zero, and then the time derivations of the systems are given as, respectively,

$$\frac{d}{dt}\int_{\hat{U}(t)\big|_{open}^{m-fix}} \psi\, dU = \int_{U(t)\big|_{open}^{v-mov}} \frac{\partial}{\partial t}\psi\, dU \,, \quad (13)$$

$$\frac{d}{dt}\int_{\hat{U}\big|_{clos}^{m-fix}} \psi\, dU = \int_{U\big|_{clos}^{v-fix}} \frac{\partial}{\partial t}\psi\, dU \,. \quad (14)$$

Remark 1: Note that, from (6)-(8) and (12), it is seen that the calculation method used in (6)-(8) in spite of acted in Eulerian description is only Leibniz's rule without considering the physical idea, that is,

$$\frac{d}{dt}\int_{U(t)}\psi\, dU = \left[\int_{U(t)}\frac{\partial}{\partial t}\psi\, dU + \oint_{S(t)}\psi\, \vec{u}_S\cdot\hat{n}\, dS\right]. \quad (15)$$

Hence, the controller designed by using (15) might bring about an erroneous stability result in actual translating systems, and such a situation is shown in Section 5, below.

From the above results, with noting that the form of (12) for (A2) is the same as Reynolds transport theorem, the followings are observed:

(C1) Calculating the rate of change of any property for axially moving continua needs a flux term with the translating speed of moving continua, regardless of the moving velocity of varying control surface bounding the varying control volume.

(C2) From (12) which denotes the classic Reynolds transport theorem in the case of (A2), it can be asserted that (12) is a general transport theorem since (12) has been extended to a translating system with a varying control volume (and a varying control surface).

(C3) From (13), it is seen that, even in the case of stationary continua with the varying control volume, Leibniz's rule in (15) cannot be directly employed to this system. Indeed, Leibniz's rule in (15) is purely mathematical not stemming from a physical concept such as fluid dynamics, and hence, ψ in (15) is not identified with any material property.

(C4) As mentioned in (C1), when evaluating the time derivation for translating continua, a flux term with the traveling speed of continua is always contained. This means that the time derivative used in the translating continua can be treated as the material derivative in (1), that is,

$$\frac{d}{dt}\int_{U(t)}\psi\, dU = \frac{D}{Dt}\int_{U(t)}\psi\, dU = \int_{U(t)}\frac{\partial}{\partial t}\psi\, dU + \oint_{S(t)}\psi\vec{u}\cdot\hat{n}\, dS \,,$$

which is also obtained from (12). Hence, in analyzing the behavior of the translating continua system, the relationship of $d(\cdot)/dt = D(\cdot)/Dt$ can be employed, and which is useful to analyze the system.

Finally, for the one-dimensional axially moving string introduced in (3), the material velocity $\vec{u}$ in (12) is $\vec{v}$. Note that in this system the material points at $x = 0, l$ still have the material velocity $\vec{v}$ despite the fixed boundary positions, which means the fixed control volume, i.e., $\vec{u}_S = 0$. Hence, the velocity term in (12) is given as $\vec{u} = \vec{v}$, and then $\vec{v}\cdot\hat{n}$ at the boundaries is given as $-v$ at $x = 0$ and $+v$ at $x = l$ because $\hat{n}$ is the outwardly positive unit normal vector to $S(t)$. Thus, using (12), the time derivation of V_{string} in (2) is obtained as

$$\dot{V}_{string} = \frac{dV_{string}}{dt} = \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho(w_t + vw_x)^2 + T_s w_x^2\right]dx + \frac{1}{2}v\left[\rho(w_t + vw_x)^2 + T_s w_x^2\right]_0^l = T_s v w_x^2\Big|_0^l . \quad (16)$$

Note that (16) is the same as (10) and entirely different from the results in (6) and (9).

4. ENERGY CONSERVATION: HAMILTON'S PRINCIPLE

In this section, it is investigated if the energy of the translating continua expressed by (12) is conservative via Hamilton's principle since which has been expanded through the principle of conservation of mechanical energy to dynamic problems (see Benaroya and Wei, 2000).

The classic Hamilton's principle for this system can be obtained by integrating d'Alembert's principle for a continua system with respect to time over an internal t_0 to t_f. However, this is not generally the case where the system is comprised with translating material elements, which denotes a translating continua system. Supposing the material elements comprising the translating continua are moving at the speed of $\vec{u}$, then the Cartesian velocity is given as $d\vec{x}/dt = \vec{u}$ in Eulerian description. This means that on the occasion of translating continua systems, the system configurations at two distinct times can not be readily prescribed. That is, the two end points of $\vec{w}(\vec{x},t)$ at the beginning and ending times are varied, not fixed, due to $d\vec{x}/dt = \vec{u}$. Thus, a novel Hamilton's principle extended to such systems is required to generalize the analysis to include the translating continua systems.

Now, consider the translating system with $\hat{U}(t)\big|_{open}^{m\_mov}$ in (A1) again. Then, for the system, d'Alembert's principle is given as

$$\delta L_{\hat{U}(t)|_{open}^{m\_mov}} + \delta W_{n.c.} - \frac{d}{dt}\left[\int_{\hat{U}(t)|_{open}^{m\_mov}} \left[\left(\rho\frac{D\vec{w}(\vec{x},t)}{Dt}\right)\cdot\delta\vec{w}(\vec{x},t)\right]dU\right] = 0 , \quad (17)$$

where $L_{\hat{U}(t)|_{open}^{m\_mov}}$ denotes the Lagrangian of the time-varying system volume. Hence, by applying (12) to the last term in (17), the following is obtained

$$\delta L_{U(t)|_{open}^{v\_mov}} + \delta W_{n.c.} - \int_{U(t)|_{open}^{v\_mov}} \frac{\partial}{\partial t}\left[\left(\rho\frac{D\vec{w}(\vec{x},t)}{Dt}\right)\cdot\delta\vec{w}(\vec{x},t)\right]dU - \oint_{S(t)|_{open}^{v\_mov}} \left[\left(\rho\frac{D\vec{w}(\vec{x},t)}{Dt}\right)\cdot\delta\vec{w}(\vec{x},t)\right]\vec{u}\cdot\hat{n}\,dS = 0 , \quad (18)$$

where $L_{U(t)|_{open}^{v\_mov}}$ denotes the Lagrangian of the time-varying control volume. Note that comparing (17) and (18), it is observed that the variation of the property in the system volume can be rewritten in terms of that in the control volume. The advantage of this approach is that the system configuration in the control volume is prescribed at all times since the rate of change of the volume integral in (18) has been derived under the condition of $d\vec{x}/dt = 0$. Thus, now integrating (18) with respect to time, the extended form of Hamilton's principle for a translating continua system is given as

$$\delta\int_{t_0}^{t_f} L_{U(t)|_{open}^{v\_mov}}\,dt + \int_{t_0}^{t_f} \delta W_{n.c.}\,dt \quad (19a)$$

$$-\int_{t_0}^{t_f}\oint_{S(t)|_{open}^{v\_mov}} \left[\left(\rho\frac{D\vec{w}}{Dt}\right)\cdot\delta\vec{w}\right]\vec{u}\cdot\hat{n}\,dS\,dt = 0 . \quad (19b)$$

Now, the principle of conservation of mechanical energy is established by using the extended Hamilton's principle in (19b) as well as the derivation method in (12). Following (Benaroya and Wei, 2000), for translating continua system, let the virtual displacement $\delta\vec{w}$ of the translating material particle coincide with the actual displacement $D\vec{w}$. Hence, from considering the relationship of $\delta\vec{w} = Dt\cdot D\vec{w}/Dt$, the variational operator can be defined with the material derivative operator such as $\delta(\cdot) = Dt\cdot D(\cdot)/Dt$. Let $W_{n.c.}$ is zero, then replacing the variation with the material differential and eliminating the common Dt factor, (18) yields

$$\frac{DL_{U(t)|_{open}^{v\_mov}}}{Dt} - \int_{U(t)|_{open}^{v\_mov}} \frac{\partial}{\partial t}\left[\left(\rho\frac{D\vec{w}}{Dt}\right)^2\right]dU - \oint_{S(t)|_{open}^{v\_mov}} \left[\left(\rho\frac{D\vec{w}}{Dt}\right)^2\right]\vec{u}\cdot\hat{n}\,dS = 0 . \quad (20)$$

Now applying the derivation method (12) and material derivative operator to (20) yields

$$\frac{d}{dt}\left(K_{\hat{U}(t)|_{open}^{m\_mov}} + P_{\hat{U}(t)|_{open}^{m\_mov}}\right) = 0 , \quad (21)$$

where $K_{\hat{U}(t)|_{open}^{m\_mov}}$ and $P_{\hat{U}(t)|_{open}^{m\_mov}}$ denote the kinetic and potential energies of the system volume. From (21), it is observed that the total mechanical energy of the translating continua system is constant, and which is a statement of the principle of conservation of mechanical energy.

5. NUMERICAL SIMULATIONS

In this section, comparing two control systems of the axially moving string designed using the derivation method proposed and the Leibniz's rule, respectively, the correctness of the proposed method is demonstrated through a mathematical analysis and numerical simulations. Fig. 2 shows the considered axially moving string with a hydraulic touch roll actuator in the right boundary. Let the mass and damping coefficients of the hydraulic actuator be m_c and d_c, respectively. The control force $f_c(t)$ is applied to the touch rolls to suppress the transverse vibrations of the axially moving string.

5.1. Vibration Control: Proposed method (12)

By employing the extended Hamilton's principle in (19b), the governing equation and boundary conditions of the axially moving string are derived as

$$\rho w_{tt} + 2\rho v w_{xt} + \rho v^2 w_{xx} = T_s w_{xx}\ ,\ 0 < x < l\ , \quad (22)$$

$$w(x,0) = w_0(x)\ ,\ w_t(x,0) = w_{t0}(x)\ ,\ w(0,t) = 0, \quad (23)$$

$$-f_c(t) = m_c w_{tt}(l) + d_c w_t(l) + T_s w_x(l)\ . \quad (24)$$

To suppress the vibration energy of the axially moving string as well as to attenuate the effect of the disturbance at the output of the controller, a boundary controller is proposed as follows:

$$\dot{\xi}_1(t) = \omega_1 \xi_2(t)\ , \quad (25)$$

$$\dot{\xi}_2(t) = -\omega_1 \xi_1(t) + \dot{w}(x,t)|_{x=l}\ , \quad (26)$$

$$f_c(t) = -d_c w_t(l) + k_1 \dot{w}(x,t)|_{x=l} + m_c v_s w_{xt}(l) + k_2 \xi_2(t)\ , \quad (27)$$

where $\xi_i \in R$ for $i = 1,2$ are the controller states with the material velocity feedback at the right boundary as the input signal, i.e., $\dot{w}(x,t)|_{x=l} = w_t(l) + v_s w_x(l)$, and where ω_1 , k_1 , and k_2 denote the control parameters with positive values.

Now, a positive definite functional $V(t)$, as the total energy of the moving string system including the actuator, is defined as follows:

$$V(t) = V_{string}(t) + V_{actuator}(t)\ , \quad (28)$$

where $V_{string}(t) = \frac{1}{2}\int_0^l \rho (w_t + v_s w_x)^2\, dx + \frac{1}{2}\int_0^l T_s w_x^2\, dx$,

$$V_{actuator}(t) = \frac{1}{2} m_c \{ w_t(l) + v_s w_x(l) \}^2 + \frac{1}{2} k_2 (\xi_1^2 + \xi_2^2)\ .$$

Note that the position of the actuator is fixed at the right boundary of the string span while the string is axially moving. Hence, for $V_{actuator}(t)$, both of $\bar{u}$ and $\bar{u}_S$ in (12) should be set as all zero. Thus, by using (12) and Example 1, the time derivative of the Lyapunov function candidate $V(t)$ is obtained by

$$\begin{aligned} \dot{V}(t) &= \dot{V}_{string}(t) + \dot{V}_{actuator}(t) \\ &= \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho (v_s w_x + w_t)^2 + T_s w_x^2\right] dx \\ &\quad + \frac{1}{2} v_s \left[\rho (v_s w_x + w_t)^2 + T_s w_x^2\right]_0^l \\ &\quad + \frac{1}{2}\frac{\partial}{\partial t}\left[m_c \{ w_t(l) + v_s w_x(l) \}^2 + k_2 (\xi_1^2 + \xi_2^2) \right] \\ &= -v_s T_s w_x^2(0) - k_1 (w_t(l) + v_s w_x(l))^2 \le 0\ . \end{aligned} \quad (29)$$

From (29), it is concluded that all the signals in the closed loop system are bounded. By using LaSalle's invariance principle, it is concluded that the solutions of the closed loop system asymptotically tend to the zero solution. Further, the controller given by (25)-(27) eliminates the effect of the disturbance $n(t)$ at the output of the controller, and the asymptotic stability of the closed loop system is still guaranteed in this case. For details of the above results in this subsection, we can refer (Yang *et al.*, 2005).

5. 2 Vibration Control: Leibniz's rule (15)

In this subsection, the string system in Fig. 2 is analyzed from a purely mathematical standpoint, i.e., $d\bar{x}/dt = 0$ here. Hence, we face a basic problem such as which Hamilton's principle among (19a) and (19b) should be treated in this case. Depending on the employment of (19a) and (19b), the boundary condition of the string system and boundary controller to suppress the vibration energy of the axially moving string as well as to attenuate the effect of the disturbance at the output of the controller are distinctly given, respectively, as:

(S1) By employing the classic type (19a):

$$-f_c(t) = m_c w_{tt}(l) + (d_c - \rho v w)_t(l) + (T_s - \rho v^2) w_x(l)\ , \quad (30)$$

$$\dot{\xi}_1(t) = \omega_1 \xi_2(t)\ , \quad (31)$$

$$\dot{\xi}_2(t) = -\omega_1 \xi_1(t) + w_t(l)\ , \quad (32)$$

$$f_c(t) = -d_c w_t(l) + k_1 w_t(l) + k_2 \xi_2(t)\ . \quad (33)$$

(S1) By employing the extended type (19b):

$$-f_c(t) = m_c w_{tt}(l) + d_c w_t(l) + T_s w_x(l)\ , \quad (34)$$

$$\dot{\xi}_1(t) = \omega_1 \xi_2(t)\ , \quad (35)$$

$$\dot{\xi}_2(t) = -\omega_1 \xi_1(t) + w_t(l)\ , \quad (36)$$

$$f_c(t) = -d_c w_t(l) + k_1 w_t(l) + k_2 \xi_2(t) - \rho v (w_t(l) + v w_x(l))\ . \quad (37)$$

The stability of the closed-loop systems (S1) and (S2) can be easily proved by following the procedure in Subsection 5.1. However, the stability is proved using Leibniz's rule given by (15) in this subsection.

Following (6)-(8), a conserved Eulerian functional is considered, and then a positive definite functional $V_{Eul}(t)$, as the total energy of the moving string system, is defined as:

$$\begin{aligned} V_{Eul}(t) &= \frac{1}{2}\int_0^l \rho w_t^2\, dx + \frac{1}{2}\int_0^l (T_s - \rho v_s^2) w_x^2\, dx \\ &\quad + \frac{1}{2} m_c w_t^2(l) + \frac{1}{2} k_2 (\xi_1^2 + \xi_2^2)\ , \end{aligned} \quad (38)$$

Using Leibniz's rule given as (15) (or (7)), the time derivation of $V_{Eul}(t)$ in (38) is obtained as

$$\begin{aligned} \dot{V}_{Eul}(t) &= \frac{1}{2}\int_0^l \frac{\partial}{\partial t}\left[\rho w_t^2 + (T_s - \rho v_s^2) w_x^2\right] dx \\ &\quad + \frac{1}{2}\frac{\partial}{\partial t}\left[m_c \{ w_t(l) + v_s w_x(l) \}^2 + k_2 (\xi_1^2 + \xi_2^2) \right] \\ &= -k_1 w_t^2(l) \le 0\ . \end{aligned} \quad (39)$$

From (39), it is also concluded the same stability result as that of Subsection 5.1.

5.3. Simulation Results

In this subsection, the correctness of the stability results is verified by numerical simulations. Let $\rho = 1$ kg/m, $v = 0.5$ m/sec, $m_c = 0.1$ kg, $d_c = 0.02$ N/m/sec, $T_s = 1$ N, and $l = 1$ m. Figs. 3 and 4 draw the displacement of the whole string under the disturbance, in which $k_1 = 1$, $k_2 = 10$, and $\omega_1 = 10$ are given. Fig. 3 shows the simulation results for the moving string with the boundary controller (25)-(27), whereas Fig. 4 depicts those with the boundary controller (31)-(33) (or (35)-(37)).

It is seen in Fig. 3, in the case of the moving string controlled by (25)-(27), the initial vibrations dissipate asymptotically despite the disturbance. However, as shown in Fig. 4, the string vibrations under the boundary control action with (31)-(33) (or (35)-(37)) do not dissipate to zero and diverge even though the mathematical analysis guaranteed the asymptotic stability despite the disturbance.

The main reason of the unstability result of closed-loop systems (S1) and (S2) is due to the time derivation method of the Lyapunov function candidate V_{Eul} in (38). Since the time differentiation was performed in a not correct mathematical manner at the first step in control design, the following designed controller as well as the mathematical analysis all have lead to such unsuitable results.

6. CONCLUSIONS

In order to get the correct solution of the time derivation of the energy functional for axially moving continua, Eulerian description should be taken into account, but surely with a physical concept. As shown here, if a controller designed from only a purely mathematical standpoint is put into operation, an erroneous stability might be brought about in actual working, especially, axially translating continua systems. On the basis of the contents here, various controller design and system analysis for axially moving continua can be established in the correct mathematical manner.

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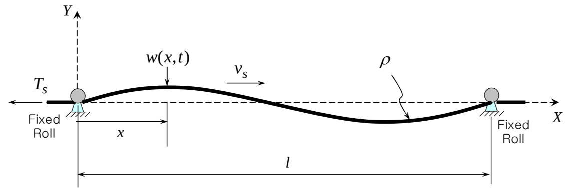

Fig.1 Axially moving string with fixed boundaries.

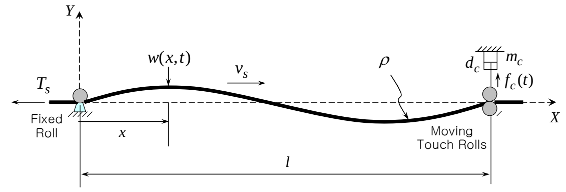

Fig. 2 Axially moving string with a hydraulic actuator at the right boundary

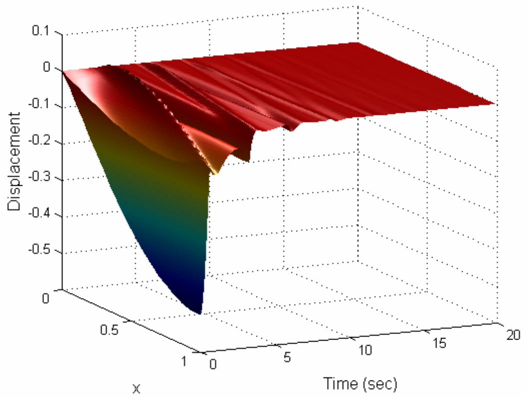

Fig. 3 Displacement of the moving string controlled by (25)-(27): $k_1 = 1$, $k_2 = 10$, $\omega_1 = 10$, $n(t) = \cos 10t$.

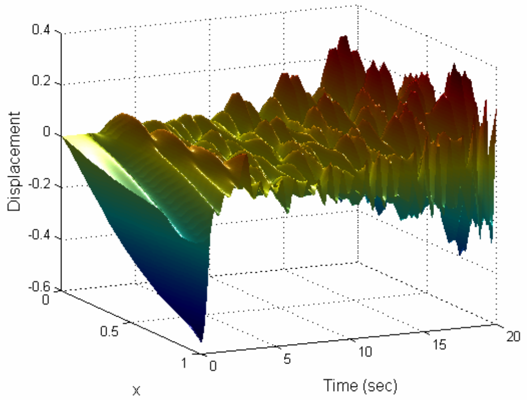

Fig. 4 Displacement of the moving string controlled by (31)-(33) (or (35)-(37)): $k_1 = 1$, $k_2 = 10$, $\omega_1 = 10$, $n(t) = \cos 10t$.