ON A STABILITY PROPERTY OF NONLINEAR SYSTEMS WITH PERIODIC INPUTS HAVING SLOWLY VARYING AVERAGE

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Abstract: It is known that, if an equilibrium of a nonlinear system has a stability property when an external input is frozen, then the property is maintained under the input being slowly varying. In this paper, we show that the same stability property is preserved not only under slowly varying input but also under slowly-varying-average input (which is not actually slowly varying but its 'average' is slowly varying). The input is assumed to be periodic and to vary sufficiently fast. We prove the claim by the average theory and some previous results on the slowly varying inputs. Copyright © 2005 IFAC

Keywords: Average theory; Slowly varying input; Stability of nonlinear systems

1. INTRODUCTION

In the early 1990s, (Kelemen, 1986), (Lawrence and Rugh, 1990), and (Khalil and Kokotović, 1991) have presented a stability property of nonlinear systems with slowly varying inputs, which can now be found in a graduate textbook such as (Khalil, 1996). The goal of this paper is to extend the result of (Kelemen, 1986; Lawrence and Rugh, 1990; Khalil and Kokotović, 1991) in the sense that the same stability property is established under periodic inputs having slowly varying 'average.' In other words, we consider an input that may not be slowly varying itself, but whose average is slowly varying.

To be concrete, let us consider a dynamical system given by

$$\dot{x} = f(x, u), \tag{1}$$

where $f(\cdot, \cdot)$ is continuously differentiable, $x \in \mathbb{R}^n$, $u \in \Gamma \subset \mathbb{R}^m$ in which Γ is a connected compact set. Suppose that, for each *frozen* (i.e., constant) input $u \in \Gamma$, there exists a corresponding equilibrium $x^*(u)$ such that $f(x^*(u), u) = 0$.

Suppose also that, for any two (constant) inputs u_1 and u_2 contained in Γ, we want to drive the state $x(t)$, which is initially located in a neighborhood of $x^*(u_1)$, into a neighborhood of $x^*(u_2)$. This problem has been dealt with in (Kelemen, 1986; Lawrence and Rugh, 1990; Khalil and Kokotović, 1991), where it is proved that, if there exists a curve $\mathcal{U}$ in Γ that connects two points u_1 and u_2 and, for each fixed $\bar{u}$ on the curve $\mathcal{U}$, the corresponding equilibrium $x^*(\bar{u})$ is locally exponentially [1] stable, then, by changing $u(t)$ sufficiently slowly from u_1 to u_2 along the curve $\mathcal{U}$, the state $x(t)$ that is close to the point $x^*(u_1)$ can be driven into a small neighborhood of $x^*(u_2)$. In this paper, it is asserted that, for the same result, slowly varying $u(t)$ is not necessary but it is sufficient that the average of $u(t)$ is

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[1] Local exponential stability is easily relaxed to local asymptotic stability in (Khalil and Kokotović, 1991) once the region of attraction of each $x^*(\bar{u})$ does not vanish along the curve $\mathcal{U}$. The latter property has been coined as 'NvBA (Non-vanishing Basin of Attraction) stability' and a verification method of it is proposed in (Shim *et al.*, 2004).

slowly varying along the curve $\mathcal{U}$ if the periodic oscillation of $u(t)$ is sufficiently fast.

We note that, in practice, there are several occasions where the slowly-varying-average-input assumption is more suitable than the slowly-varying-input assumption. A classical example of the slowly-varying-average-input is the PWM (pulse-width modulation) control whose net effect is the average of instantaneous control inputs.

2. MAIN RESULT

To effectively describe the system under consideration, we suppose that the system is given by

$$\dot{x}(t) = f(x(t), u_a(t), u_f(\frac{1}{\epsilon}t)), \qquad (2)$$
$$u_a \in \Gamma \subset \mathbb{R}^m, \ u_f \in \mathbb{R}^l,$$

where $f(\cdot, \cdot, \cdot)$ and $u_a(\cdot)$ are continuously differentiable, the set Γ is connected and compact, and the piecewise continuous function $u_f(\cdot)$ is assumed to be periodic with a period T (thus, $u_f(\frac{1}{\epsilon}t)$ is ϵT-periodic). The small positive parameter ϵ (to be specified) indicates that the function u_f varies fastly. In fact, we are going to rely on three-time scale behavior of the system by specifying ϵ sufficiently small so that u_f oscillates sufficiently fast and by specifying the upper bound of the derivative of u_a (i.e., $\|\frac{du_a}{dt}\| \leq \kappa$ where κ is sufficiently small) so that u_a varies sufficiently slowly, relatively to the behavior of the system state $x(t)$.

Remark 1. A system having an input $u(t)$ whose average is slowly varying is regarded in this paper as

$$\dot{x}(t) = f(x(t), u(t)) = f(x(t), u_a(t) + u_f(\frac{1}{\epsilon}t))$$

where $u_a(t)$ is slowly varying and the periodic function $u_f(\cdot)$ has zero mean. In this way, this system is cast into (2).

A trick in our development is to regard $(\frac{1}{\epsilon}t)$ as an independent variable τ for now. Then, the system (2) is now rewritten as

$$\dot{x}(t) = f_{av}(x(t), u_a(t)) + f_p(x(t), u_a(t), u_f(\tau)) \qquad (3)$$

where f_{av} is defined as

$$f_{av}(x, u_a) = \frac{1}{T} \int_{\sigma}^{\sigma+T} f(x, u_a, u_f(s)) ds$$

with arbitrary σ, and $f_p(x, u_a, u_f) = f(x, u_a, u_f) - f_{av}(x, u_a)$. Note that, with this construction,

$$\int_{\sigma}^{\sigma+T} f_p(x, u_a, u_f(s)) ds = 0$$

for any x and u_a.

Now we formally assume the stability property of the averaged system with *frozen* input u_a as follows. Let $B_r(x)$ $(\bar{B}_r(x))$ denote the open (closed) ball centered at x with the radius $r > 0$.

Assumption 2. The averaged system

$$\dot{x} = f_{av}(x, u_a) \qquad (4)$$

has an isolated equilibrium $x^*(u_a)$ for each frozen input $u_a \in \Gamma$, that is, $f_{av}(x^*(u_a), u_a) = 0$, and there exist a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and a positive number r such that, for each $u_a \in \Gamma$ and each $x \in B_r(x^*(u_a))$,

$$\alpha_1(\|x - x^*(u_a)\|) \leq V(x, u_a) \leq \alpha_2(\|x - x^*(u_a)\|) \qquad (5)$$

$$\frac{\partial V}{\partial x} f_{av}(x, u_a) \leq -\alpha_3(\|x - x^*(u_a)\|), \quad \text{a.e.,} \quad (6)$$

where $\alpha_i(\cdot)$, $i = 1, 2, 3$, are class-$\mathcal{K}$ functions.

Theorem 3. Consider the system (2) under Assumption 2. Given a positive number ρ, there exist positive numbers ϵ^*, κ and δ such that, if

$$\left\| \frac{du_a}{dt}(t) \right\| \leq \kappa$$

so that the input $u_a(t)$ varies sufficiently slowly, and $0 < \epsilon < \epsilon^*$ so that the input $u_f(\frac{1}{\epsilon}t)$ varies sufficiently fastly, then the state $x(t)$ initiated in a neighborhood of $x^*(u_a(0))$ such that $\|x(0) - x^*(u_a(0))\| \leq \delta$ stays in a neighborhood of $x^*(u_a(t))$, that is,

$$\|x(t) - x^*(u_a(t))\| \leq \rho, \qquad \forall t \geq 0.$$

PROOF. The proof is taken from (Sanders and Verhulst, 1985) with modification.

First of all, let us define a compact set of our interest,

$$D := \left\{ (x, u_a) \in \mathbb{R}^n \times \Gamma : x \in \bar{B}_r(x^*(u_a)) \right\}.$$

For each element $(x, u_a) \in D$, define $\mu(x, u_a, \tau)$ as

$$\mu(x, u_a, \tau) = \int_0^\tau f_p(x, u_a, u_f(s)) ds,$$

where $\tau \in \mathbb{R}$. Thus, obviously, it follows that

$$\frac{\partial \mu}{\partial x}(x, u_a, \tau) = \int_0^\tau \frac{\partial f_p}{\partial x}(x, u_a, u_f(s)) ds, \text{ and}$$

$$\frac{\partial \mu}{\partial u_a}(x, u_a, \tau) = \int_0^\tau \frac{\partial f_p}{\partial u_a}(x, u_a, u_f(s)) ds.$$

Note that the above integrals are conducted with x and u_a fixed. The norms of $\mu(x, u_a, \tau)$, $\frac{\partial \mu}{\partial x}(x, u_a, \tau)$, and $\frac{\partial \mu}{\partial x}(x, u_a, \tau)$ are bounded by some constant $c_m > 0$ for all $(x, u_a, \tau) \in D \times [0, \infty)$ since $f_p(x, u_a, u_f(\tau))$ and its partial derivatives with respect to x and u_a are T-periodic in τ and have zero mean.

We consider the change of variables

$$x = y + \epsilon \mu(y, u_a, t/\epsilon), \qquad (7)$$

where the new variable y is an n-dimensional vector. Note that the Jacobian of the map (7) is $\frac{\partial x}{\partial y}(y, u_a, \tau) = I + \epsilon\frac{\partial \mu}{\partial y}(y, u_a, \tau)$. Even though y is defined implicitly, the Jacobian is nonsingular, and the ratio of each leading principal minor of the Jacobian is strictly positive on $D \times [0, \infty)$ for sufficiently small ϵ since the partial derivative $\frac{\partial \mu}{\partial y}$ is bounded on $D \times [0, \infty)$ as stated earlier. Then, owing to (Kou *et al.*, 1973, Thm. 1), there exists an $\epsilon_1 > 0$ such that, for each $(u_a, t, \epsilon) \in \Gamma \times [0, \infty) \times [0, \epsilon_1]$, the map $y \mapsto x$ is C^1 and bijective for all $y \in \bar{B}_r(x^*(u_a))$. Differentiating both sides of the equation (7) with respect to time, we have

$$\dot{x}(t) = \dot{y}(t) + \epsilon\frac{\partial \mu}{\partial y}(y(t), u_a(t), t/\epsilon) \cdot \dot{y}(t)$$
$$+ \epsilon\frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \dot{u}_a(t)$$
$$+ \epsilon\frac{\partial \mu}{\partial \tau}(y(t), u_a(t), t/\epsilon) \cdot \frac{1}{\epsilon}.$$

We reformulate the above equation as follows

$$\left[I + \epsilon\frac{\partial \mu}{\partial y}(y(t), u_a(t), t/\epsilon)\right] \dot{y}(t)$$
$$= \dot{x}(t) - \epsilon\frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \dot{u}_a(t)$$
$$- \frac{\partial \mu}{\partial \tau}(y(t), u_a(t), t/\epsilon)$$
$$= f(x(t), u_a(t), u_f(t/\epsilon))$$
$$- \epsilon\frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \dot{u}_a(t)$$
$$- f_p(y(t), u_a(t), u_f(t/\epsilon))$$
$$= f_{av}(y(t), u_a(t)) - \epsilon\frac{\partial \mu}{\partial u_a}(y(t), u_a(t), t/\epsilon) \cdot \dot{u}_a(t)$$
$$+ f(x(t), u_a(t), u_f(t/\epsilon)) - f(y(t), u_a(t), u_f(t/\epsilon))$$
$$= f_{av}(y, u_a) + p_0(y, u_a, t, \epsilon)\dot{u}_a + p_1(y, u_a, u_f, t, \epsilon),$$

where $p_0(y, u_a, t, \epsilon) = -\epsilon\frac{\partial \mu}{\partial u_a}$ and $p_1(y, u_a, u_f, t, \epsilon) = [f(y + \epsilon\mu, u_a, u_f) - f(y, u_a, u_f)]$. The function p_1 is written as

$$p_1(y, u_a, u_f, t, \epsilon) = \bar{f}(y, u_a, u_f, \epsilon\mu) \cdot \epsilon\mu,$$

where $\bar{f}$ is a matrix of continuous functions whose existence follows from the continuous differentiability of f. It can be seen easily that

$$\|p_0(y, u_a, t, \epsilon)\| \le \epsilon c_m \qquad (8)$$
$$\|p_1(y, u_a, u_f, t, \epsilon)\| \le \epsilon c_f c_m, \qquad (9)$$

for all $(y, u_a) \in D$, $t \in [0, \infty)$ and $\epsilon \in [0, \epsilon_1]$, where

$$c_f = \sup_{(y, u_a) \in D, 0 \le \tau, 0 \le \epsilon\mu \le \epsilon_1 c_m} \|\bar{f}(y, u_a, u_f(\tau), \epsilon\mu)\|$$

($c_f < \infty$ since $\bar{f}$ is continuous and u_f is piecewise continuous).

We use the function V in Assumption 2 to analyze the transformed system. The time-derivative of V along the system in y-coordinates is given by

$$\dot{V}(y, u_a) = \frac{\partial V}{\partial y}\dot{y}(t) + \frac{\partial V}{\partial u_a}\dot{u}_a(t)$$
$$= \frac{\partial V}{\partial y}\left[I + \epsilon\frac{\partial \mu}{\partial y}\right]^{-1}(f_{av} + p_0\dot{u}_a + p_1) + \frac{\partial V}{\partial u_a}\dot{u}_a(t)$$
$$= \frac{\partial V}{\partial y}f_{av} + \frac{\partial V}{\partial y}\left(\left[I + \epsilon\frac{\partial \mu}{\partial y}\right]^{-1} - I\right)f_{av}$$
$$+ \frac{\partial V}{\partial y}\left[I + \epsilon\frac{\partial \mu}{\partial y}\right]^{-1}(p_0\dot{u}_a + p_1) + \frac{\partial V}{\partial u_a}\dot{u}_a(t).$$

Since the function V is locally Lipschitz, its partial derivatives $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial u_a}$ are bounded on D almost everywhere, and we let

$$c_v = \text{ess.sup}_{(y, u_a) \in D} \left\|\frac{\partial V}{\partial y}(y, u_a)\right\|,$$
$$c_w = \text{ess.sup}_{(y, u_a) \in D} \left\|\frac{\partial V}{\partial u_a}(y, u_a)\right\|,$$
$$c_a = \text{ess.sup}_{(y, u_a) \in D} \|f_{av}(y, u_a)\|.$$

In addition, for $\epsilon < \frac{1}{c_m}$, we have

$$\left\|\left[I + \epsilon\frac{\partial \mu}{\partial y}\right]^{-1}\right\| \le \frac{1}{1 - \epsilon c_m}, \text{ and}$$
$$\left\|\left[I + \epsilon\frac{\partial \mu}{\partial y}\right]^{-1} - I\right\| \le \frac{\epsilon c_m}{1 - \epsilon c_m}.$$

Consequently, from Assumption 2, (8) and (9), it follows that

$$\dot{V}(y(t), u_a(t)) \le -\alpha_3(\|y(t) - x^*(u_a)\|)$$
$$+ c_v\frac{\epsilon c_m}{1 - \epsilon c_m}c_a + c_v\frac{1}{1 - \epsilon c_m}\epsilon c_m\|\dot{u}_a\|$$
$$+ c_v\frac{1}{1 - \epsilon c_m}\epsilon c_f c_m + c_w\|\dot{u}_a\|,$$

on D, almost everywhere.

Now, without loss of generality, suppose that the given number $\rho \le r$. We can take a pair of positive numbers (ϵ_2, κ) such that

$$\frac{\epsilon_2 c_m}{1 - \epsilon_2 c_m}c_v(c_a + \kappa + c_f) + c_w\kappa \le (\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1)(\frac{\rho}{2}).$$

Then,

$$\dot{V}(y(t), u_a(t)) \le -\alpha_3(\alpha_2^{-1}(V)) + (\alpha_3 \circ \alpha_2^{-1} \circ \alpha_1)(\frac{\rho}{2}),$$

which implies that, if $V(y(0), u_a(0)) \le \alpha_1(\rho/2)$, then $V(y(t), u_a(t)) \le \alpha_1(\rho/2)$ for all $t \ge 0$. Therefore, letting $\delta = (\alpha_2^{-1} \circ \alpha_1)(\frac{\rho}{2})$, we have

$$\|y(t) - x^*(u_a(t))\| \le \frac{\rho}{2},$$

for all $t \ge 0$ if $\|y(0) - x^*(u_a(0))\| \le \delta$ and $\epsilon < \min\{\epsilon_1, \epsilon_2, \frac{1}{c_m}\}$.

To complete the proof, the analysis is given in x-coordinate. From (7), $x(0) = y(0)$. Thus, if $\|x(0) - x^*(u_a(0))\| \le \delta$, then $\|y(t) - x^*(u_a(t))\| \le \frac{\rho}{2}$ and $(y(t), u_a(t)) \in D$ for all $t \ge 0$, which implies that

$$\|x(t) - x^*(u_a(t))\| \leq \|x(t) - y(t)\| + \|y(t) - x^*(u_a)\|$$
$$\leq \|\epsilon\mu(y(t), u_a(t), t/\epsilon)\| + \frac{\rho}{2}$$
$$\leq \epsilon c_m + \frac{\rho}{2}.$$

Therefore, we finally set

$$\epsilon^* = \min\{\epsilon_1, \epsilon_2, \frac{1}{c_m}, \frac{\rho}{2c_m}\},$$

and we have

$$\|x(t) - x^*(u_a(t))\| \leq \rho,$$

for all $t \geq 0$ provided that $\epsilon < \epsilon^*$.

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3. CONCLUSION

In this note, we use the features of the slowly-varying periodic input and the averaging theory to prove that if each equilibrium point for the frozen system of our concern is asymptotically stable, its stability property is not severely disturbed by inputs having slowly-varying average. Finally, it seems that the result in this note can be extended using 'general' averaging theory applicable to the case that $u_f(t)$ may not be periodic.

REFERENCES

Kelemen, M. (1986). A stability property. *IEEE Trans. Automat. Contr.* **31**, 766–768.

Khalil, H. (1996). *Nonlinear Systems.* 2nd ed.. Prentice-Hall.

Khalil, H. and P. Kokotović (1991). On stability properties of nonlinear systems with slowly varying inputs. *IEEE Trans. Automat. Contr.* **36**, 229.

Kou, S., D. Elliott and T. Tarn (1973). Observability of nonlinear systems. *Information and Control* **22**, 89–99.

Lawrence, D. and W. Rugh (1990). On a stability theorem of nonlinear systems with slowly varying inputs. *IEEE Trans. Automat. Contr.* **7**, 860–864.

Sanders, J. and F. Verhulst (1985). *Averaging methods in nonlinear dynamical systems.* Springer-Verlag.

Shim, H., H. Chang and J. Seo (2004). Nonvanishing basin of attraction with respect to a parametric variation and center manifold. *IEEE Conf. on Decision and Control, pp. 1048-1054.*