OUTPUT SYNCHRONIZATION OF CHAOTIC OSCILLATORS AND PRIVATE COMMUNICATION [1]

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Abstract: A method for synchronizing chaos is presented. The approach exploits the well-known model-matching problem from nonlinear control theory, it is advantageously applied to achieve complete and output synchronization of identical and nonidentical chaotic oscillators, respectively. Some potential applications to private/secure communication schemes are also given.

Keywords: Chaos synchronization, model-matching problem, communications.

1. INTRODUCTION

Data security has been an issue of increasing importance in communications as the Internet and personal communication systems are being made accessible worldwide. On the other hand, numerous efforts have been made to use chaos for enhancing some features of communication systems. For this purpose, different approaches for chaos synchronization have been proposed (see e.g. Pecora and Carroll, 1990; Wu and Chua, 1993; Special Issue on Chaos, 1997; 2000; Chen and Dong, 1998; Cruz and Nijmeijer, 1999; 2000; Special Issue, 2000; Sira-Ramírez and Cruz, 2001; Aguilar and Cruz, 2002; López and Cruz, 2004). The aim of this work is to illustrate a method for synchronizing chaotic oscillators. The objective is achieved by using the model-matching problem from nonlinear control theory (Di Benedetto and Grizzle, 1994; Isidori, 1995). This approach presents the advantage that is systematic and useful to synchronize identical and nonidentical chaotic oscillators, further it uses unidirectional coupling, that let the coupling signal uses less transmission channels. Moreover, this approach makes that chaos synchronization has applications on some communication schemes. The attention is at first focused on chaos synchronization and finally, we illustrate the potential application to private/secure communication using different chaotic communication schemes.

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2. PROBLEM FORMULATION

Consider a dynamical system described by

$$P : \begin{cases} \dot{x} = f(x) + g(x)\,u, \\ y = h(x), \end{cases} \qquad (1)$$

where the state $x(t) \in \mathbb{R}^n$, the input $u(t) \in \mathbb{R}$, and the output $y(t) \in \mathbb{R}$, being $f(x)$ and

$g(x)$ smooth and analytical functions. In addition, consider another dynamical system described by

$$M : \begin{cases} \dot{x}_M = f_M(x_M) + g_M(x_M)\, u_M, \\ y_M = h_M(x_M), \end{cases} \tag{2}$$

where the state $x_M(t) \in \mathbb{R}^{n_M}$, the input $u_M(t) \in \mathbb{R}$, and the output $y_M(t) \in \mathbb{R}$. $f_M(x_M)$ and $g_M(x_M)$ are smooth and analytical functions. We assume that x° is an equilibrium point of system (1), i.e., $f(x^\circ) = 0$. Similarly, x_M° is an equilibrium point of system (2). Assume that dynamical systems (1) and (2) under certain conditions have chaotic behavior. Then, the chaotic oscillator (1) *synchronizes* with the chaotic oscillator (2), if

$$\lim_{t\to\infty} |y(t) - y_M(t)| = 0, \tag{3}$$

no matter which initial conditions $x(0)$ and $x_M(0)$ have, and for suitable input signals $u(t)$ and $u_M(t)$. Then, **output synchronization problem** between chaotic oscillators (1) and (2) is considered. In the next section we describe how to satisfy condition (3) from the perspective of the model-matching problem from nonlinear control theory.

3. MODEL-MATCHING PROBLEM

Now, consider the chaotic oscillators (1) and (2) like a *plant* P and *model* M, respectively. We want to design a feedback control law $u(t)$ for P which, irrespectively of the initial states of P and M, makes the output $y(t)$ asymptotically converges to the output $y_M(t)$ produced by M under an arbitrary input $u_M(t)$. This problem is the well-known *model-matching problem* (MMP) from nonlinear control theory (Di Benedetto and Grizzle, 1994; Isidori, 1995). In this work, we adopt the following solution: the MMP is reduced into a problem of decoupling the output of a suitable auxiliary system from the input $u_M(t)$ to the model M. This *auxiliary system* is defined as follows

$$E : \begin{cases} \dot{x}_E = f_E(x_E) + \hat{g}(x_E)\, u + \hat{g}_M(x_E)\, u_M, \\ y_E = h_E(x_E), \end{cases} \tag{4}$$

with state $x_E = (x, x_M)^T \in \mathbb{R}^{n+n_M}$, inputs $u(t)$ and $u_M(t)$, and

$$f_E(x_E) = \begin{pmatrix} f(x) \\ f_M(x_M) \end{pmatrix}, \quad \hat{g}(x_E) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix},$$

$$\hat{g}_M(x_E) = \begin{pmatrix} 0 \\ g_M(x_M) \end{pmatrix}, \; h_E(x_E) = h(x) - h_M(x_M).$$

The control objective of the model-matching problem is contained in the following definition.

Definition 1 (Model-matching problem, MMP): *Given the plant P and the model M around their respective equilibrium points x° and x_M° and a point x_E°, the MMP consists in finding $u(t) \in \mathbb{R}$ for system E such that, the output of the auxiliary system E (fedback by $u(t)$), $y_E(t) \to 0$ as $t \to \infty$.*

In the sequel the MMP will be treated in terms of a relative degree associated with the outputs $y(t)$ and $y_M(t)$.

Definition 2 (Relative degree [Isidori, 1995]): *The single-input single-output nonlinear oscillator* (1), *is said to have* **relative degree** *r at a point x° if*

(1) $L_g L_f^k h(x) = 0$

for all x in a neighborhood of x° and $k < r - 1$.

2. $L_g L_f^{r-1} h(x^\circ) \neq 0$.

In Definition 2, $L_f h(x) = \frac{\partial h(x)}{\partial x} f(x)$ and $L_g L_f^k h(x) = \frac{\partial \left(L_f^k h(x)\right)}{\partial x} g(x)$. A similar definition can be given for the relative degree of model (2), r_M near x_M°. Suppose that the output $y(t)$ of P and the output $y_M(t)$ of M have a finite relative degree r and r_M, respectively. It is well known that the MMP is locally solvable if, and only if (Isidori, 1995),

$$r \leq r_M. \tag{5}$$

Now, we show the representation of the auxiliary system E Eq. (4) in a different coordinate frame. In this paper, we restrict our results to fully linearizable plants P, i.e., $r = n$. From the definition of r and r_M; $h(x), \ldots, L_f^{n-1} h(x)$, and $h_M(x_M), \ldots, L_{f_M}^{n-1} h_M(x_M)$, are a set of independent functions from P and M, and can be chosen as new coordinates $\xi_i(x) = L_f^{i-1} h(x)$ and $\xi_{Mi}(x_M) = L_{f_M}^{i-1} h_M(x_M)$ with $i = 1, \ldots, n$, around x° and x_M°, respectively. Let us now consider the auxiliary system E and the new coordinates (Isidori, 1995):

$$(\zeta(x_E), x_M) = \phi(x_E) = \phi(x, x_M),$$

where $\zeta(x_E) = (\zeta_1(x_E), \ldots, \zeta_n(x_E))^T$ and $\zeta_i(x_E) = L_{f_E}^{i-1} h_E(x_E) = \xi_i(x) - \xi_{Mi}(x_M)$, $i = 1, \ldots, n$.

Thus, the closed-loop auxiliary system, using the following control law

$$u = \frac{1}{L_g L_f^{n-1} h(x)} \left(v - L_f^n h(x) + L_{f_M}^n h_M(x_M) + L_{g_M} L_{f_M}^{n-1} h_M(x_M)\, u_M\right), \tag{6}$$

takes the form

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, \qquad i = 1, \ldots, n-1, \\ \dot{\zeta}_n &= v = -c_0\, \zeta_1 - \ldots - c_{n-1}\, \zeta_n, \\ \dot{x}_M &= f_M(x_M) + g_M(x_M)\, u_M, \\ y_E &= \zeta_1. \end{aligned} \tag{7}$$

From (7) we identify two subsystems:

(1) The subsystem described by

$$\dot{x}_M = f_M(x_M) + g_M(x_M)\,u_M,$$

which represents the dynamics of M, and

(2) The subsystem described by

$$\dot{\zeta} = A^*\zeta$$

with

$$A^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{n-1} \end{bmatrix},$$

which represents the dynamics of $y_E(t)$. The model M is stable by assumption and if we chose the control law $u(t)$ so that the eigenvalues of A^* have real part negative, then the closed-loop system will be exponentially stable and the **output synchronization** condition (3) holds.

Since $y_E(t) = \zeta_1(t) = \xi_1(x) - \xi_{M1}(x_M) \to 0$, notice that $\xi(x)$ and $\xi_M(x_M)$ are diffeomorphisms. So, if P and M are **identical** chaotic oscillators, $\xi(x) \to \xi_M(x_M)$ and, if the mappings have the same structure and tends to be equals, then the arguments too, i.e., $x(t) \to x_M(t)$. Moreover, from the control law (6) we can see that, $u(t) \to u_M(t)$, to decouple $u_M(t)$ from E. Thus, for identical chaotic oscillators, **complete synchronization** is achieved. For nonidentical chaotic oscillators only output synchronization is guarantied.

4. CHAOS SYNCHRONIZATION

In this section, we make use of the previous material to synchronize identical and nonidentical chaotic oscillators. Figure 1 shows the block diagram of chaos synchronization using model-matching approach. Controller C has like input signals to $x(t)$, $x_M(t)$, and $v(t)$. It has like output signal to $u(t)$ that is the input signal of the plant, and $e(t) = y_E(t) = y(t) - y_M(t)$ is the *output synchronization error* between the output signals of P and M. Rössler system and Lorenz system are used in order to illustrate chaos synchronization, although the proposed approach can be applied to any chaotic oscillator that holds (5) and for all plant P with a strong relative degree.

4.1 Complete synchronization

Consider the Rössler system given by (Rössler, 1976):

$$\begin{aligned} \dot{x}_1 &= -(x_2 + x_3), \\ \dot{x}_2 &= x_1 + \widehat{\alpha} x_2, \\ \dot{x}_3 &= \widehat{\alpha} + x_3(x_1 - \mu). \end{aligned} \tag{8}$$

With the parameter values $\widehat{\alpha} = 0.2$ and $\mu = 7$, the Rössler system (8) exhibits chaotic dynamics. We can write it in the form (1) by means of adding a control law $u(t)$ into some equation, we choose rewrite it as follows

$$P: \begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_1 + \widehat{\alpha} x_2 \\ \widehat{\alpha} + x_3(x_1 - \mu) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \\ y = x_2. \end{cases} \tag{9}$$

The system (9) will be called *the plant P*. The relative degree of P is $r = 3$. Let us propose a reference model M for P, using another Rössler system writing it in the form (2) and taking the same relative degree $r_M = 3$. Notice that, both systems have the same relative degree, $r = r_M$, that is, (5) holds, and there exists solution to the MMP, so we can achieve output synchronization between systems (9) and (10), i.e., the condition (3) is satisfied. So, we have

$$M: \begin{cases} \begin{pmatrix} \dot{x}_{M1} \\ \dot{x}_{M2} \\ \dot{x}_{M3} \end{pmatrix} = \begin{pmatrix} -x_{M2} - x_{M3} \\ x_{M1} + \widehat{\alpha} x_{M2} \\ \widehat{\alpha} + x_{M3}(x_{M1} - \mu) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_M, \\ y_M = x_{M2}. \end{cases} \tag{10}$$

The same parameter values for P and M are considered. To solve the MMP, and with this, the original output synchronization problem, we need take an auxiliary system (4) and thus we reduce the problem described before to disturbance decoupling problem. Then we take $u_M(t)$ like a "disturbance" signal and we seek the control law (6) for system E that is given by

$$\begin{aligned} u = & -v + \left(\widehat{\alpha}^2 - 1\right)(x_1 - x_{M1}) \\ & + \widehat{\alpha}\left(\widehat{\alpha}^2 - 2\right)(x_2 - x_{M2}) + \widehat{\alpha}(x_3 - x_{M3}) \\ & + x_3(x_1 - \mu) - x_{M3}(x_{M1} - \mu) + u_M. \end{aligned} \tag{11}$$

The auxiliary system (4), after a change of coordinates: $\zeta_1 = x_2 - x_{M2}$, $\zeta_2 = x_1 - x_{M1} + \widehat{\alpha}(x_2 - x_{M2})$, and $\zeta_3 = \widehat{\alpha}(x_1 - x_{M1}) + \left(\widehat{\alpha}^2 - 1\right)(x_2 - x_{M2}) - (x_3 - x_{M3})$, takes the form (7), with $v = -C\zeta = -c_0\zeta_1 - c_1\zeta_2 - c_2\zeta_3$, and $C = \begin{pmatrix} 27 & 27 & 9 \end{pmatrix}$. Some simulations were done. The initial conditions $x(0)$ and $x_M(0)$ were $(1, 1, 1)$

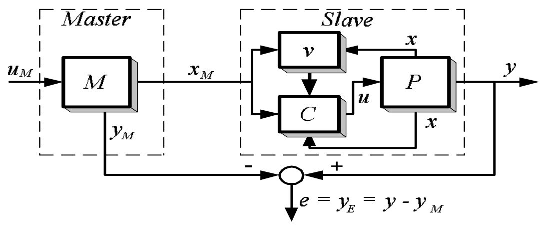

Fig. 1. Block diagram of chaos synchronization using model-matching approach.

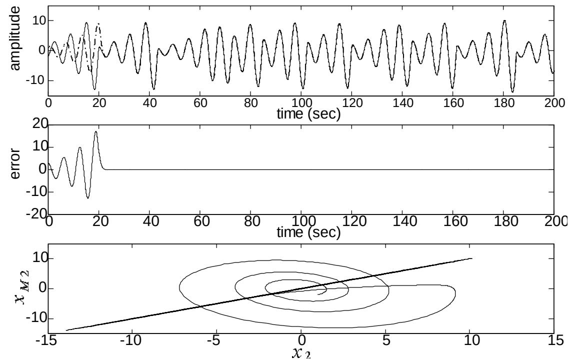

Fig. 2. Rössler-Rössler synchronization. Solid line $y_M = x_{M2}$, dashed line $y = x_2$ (top of figure). Error signal $e = y_E = y - y_M$ (middle of figure). Output synchronization between x_{M2} and x_2 (bottom of figure). Control u takes action when $t = 20$ sec.

and $(2, -2, 2)$, respectively. Figure 2 shows the output of the plant $x_2(t)$ following the output of the model $x_{M2}(t)$ (top of figure), the error signal $e(t) = y_E(t) = y(t) - y_M(t)$ (middle of figure), and the typical phase plot confirming synchronization between the outputs $y(t)$ and $y_M(t)$ (bottom of figure). The control $u(t)$ takes action after 20 seconds.

4.2 Output synchronization

Now consider the coupling between two nonidentical chaotic oscillators as P and M; for instance, a Lorenz system (Lorenz, 1963) like a model with relative degree $r_M = 3$ (for all x_M such that $x_{M1} \neq 0$):

$$M\!:\begin{cases}\begin{pmatrix}\dot{x}_{M1}\\ \dot{x}_{M2}\\ \dot{x}_{M3}\end{pmatrix}=\begin{pmatrix}\sigma\left(x_{M2}-x_{M1}\right)\\ \hat{r}x_{M1}-x_{M2}-x_{M1}x_{M3}\\ x_{M1}x_{M2}-bx_{M3}\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}u_M,\\ \qquad y_M = x_{M1}.\end{cases} \tag{12}$$

Let us to consider again the same plant described by (9). Thus, the control law $u(t)$ for output synchronization between (9) and (12) is given by

$$\begin{aligned} u = -\Big\{ & v - \Big[\left(\left(\widehat{\alpha}^2 - 1\right) x_1 + \widehat{\alpha}\left(\widehat{\alpha}^2 - 2\right) x_2 - \widehat{\alpha} x_3 \right.\\ & -\widehat{\alpha} - x_3(x_1 - \mu)\Big] + \sigma[\sigma(\sigma + \hat{r} - x_{M3})(x_{M2} - x_{M1}) \\ & - (\sigma + 1)\left(\hat{r}x_{M1} - x_{M2} - x_{M1}x_{M3}\right) \\ & -x_{M1}\left(x_{M1}x_{M2} - bx_{M3}\right)] - \sigma x_{M1}u_M \Big\}. \end{aligned} \tag{13}$$

The results are exemplified by numerical simulations. Initial conditions for P and M are $x(0) = (3, 1, 1)$ and $x_M(0) = (1, 1.5, 0.1)$, respectively. Parameter values are $\sigma = 10$, $\hat{r} = 28$, $b = 8/3$, $\widehat{\alpha} = 0.2$, and $\mu = 7$. Figure 3 shows Lorenz-Rössler output synchronization: a) $y_M = x_{M1}$, b) x_2 versus x_{M1}, c) $y = x_2$, and d) error signal $e = y_E = y - y_M$. Control law takes action after 25 sec.

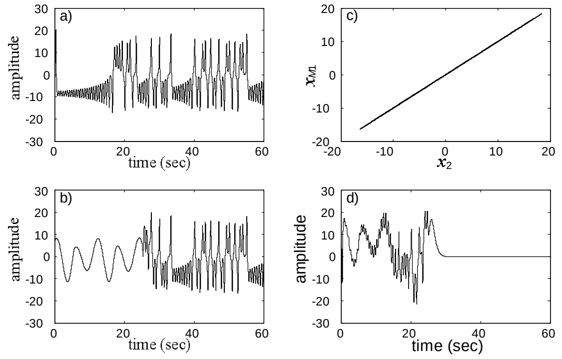

Fig. 3. Lorenz-Rössler output synchronization. a) $y_M = x_{M1}$, b) x_2 versus x_{M1}, c) $y = x_2$, and d) error signal $e = y_E = y - y_M$. Control takes action after 25 sec.

Remark 1. In this case, unlike the previous one, synchronization between the outputs of both systems was only obtained. No other state of the plant synchronizes with those of the model.

5. PRIVATE COMMUNICATION

This section does not pretend to propose secure chaos-based communication systems. It tries to illustrate the flexibility of the model-matching approach for chaotic communication. Nevertheless, certain security properties are found.

5.1 Chaotic communication using two channels

In order to illustrate the proposed approach to transmit private information signals, a chaotic communication scheme using two transmission channels is now designed. It is based on the output synchronization between identical or non-identical chaotic oscillators. To this purpose, consider that $u(t)$ can be separated in $u = \gamma_2(x)\left[u_1(x_M) + u_2(x)\right]$, with

$$\begin{aligned} u_1(x_M) &= v_1(x_M) + L_{f_M}^n h_M(x_M) \\ &\quad + L_{g_M} L_{f_M}^{n-1} h_M(x_M)\, u_M, \\ u_2(x) &= v_2(x) - L_f^n h(x), \\ \gamma_2(x) &= \frac{1}{L_g L_f^{n-1} h(x)}, \\ v_1(x_M) &= c_0\, \xi_{M1}(x_M) + \ldots + c_{n-1}\, \xi_{Mn}(x_M), \\ v_2(x) &= -c_0\, \xi_1(x) - \ldots - c_{n-1}\, \xi_n(x), \end{aligned}$$

as we can see from (6).

This let us to propose the following coupling scheme with two transmission channels shown in Fig. 4, in which $u_1(\cdot)$ is the output from a new

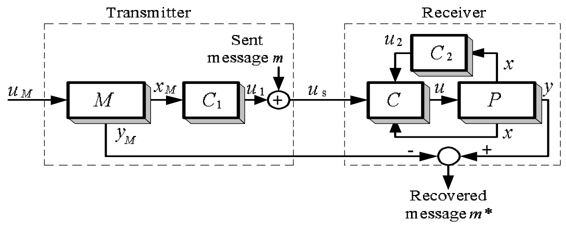

Fig. 4. Analog communication scheme using two transmission channels.

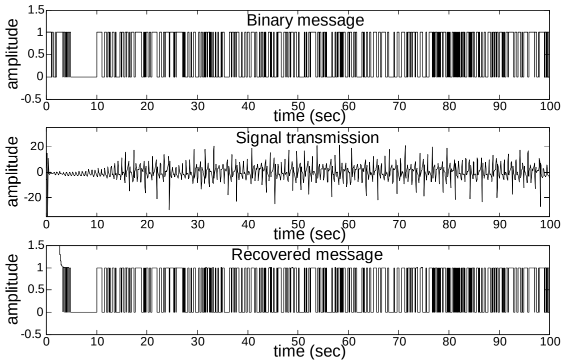

Fig. 5. Transmission and recovering of a binary message using Lorenz-Rössler output synchronization.

control block C_1, $u_2(\cdot)$ is the output of C_2 and $u(t)$ is the output of C. A remarkable feature is that, in the proposed scheme, the signal that is sent to obtain synchronization is a nonlinear function of the state $x_M(t)$, but is not the own state. So, with this scheme we obtain high privacy because it is possible to hide a message through the coupling signal $u_s(\cdot) = u_1(\cdot) + m(t)$, and with this to increase the transmission security, because $y_M(t)$ does not contain any message. Then, a third person cannot recover the hidden message with the reported methods in (Short, 1994; 1996). This message is recovered by comparison between the output $y(t)$ and $y_M(t)$ at the receiver end, i.e., $m^*(t) = k * (y(t) - y_M(t))$, with a gain k. Figure 5 shows a binary signal obtained from a picture like the private message (top of figure), the transmitted coupling signal including the hidden binary message (middle of figure), and the recovered message at the receiver end (bottom of figure), using Lorenz-Rössler output synchronization.

5.2 Chaotic communication using a single channel

The scheme in Fig. 6 uses a single transmission channel. The message $m(t)$ is injected into the transmitter through the input signal $u_M(t)$. The output signal of the transmitter is a nonlinear function $u_1(\cdot)$ whereas it is possible to take like output of receiver to $u(t)$, which, when synchronization is achieved between the outputs of P and M, then $u(t) \to u_M(t) = m(t)$, and thus we

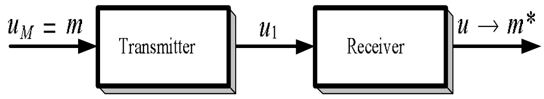

Fig. 6. Analog communication system using a single transmission channel.

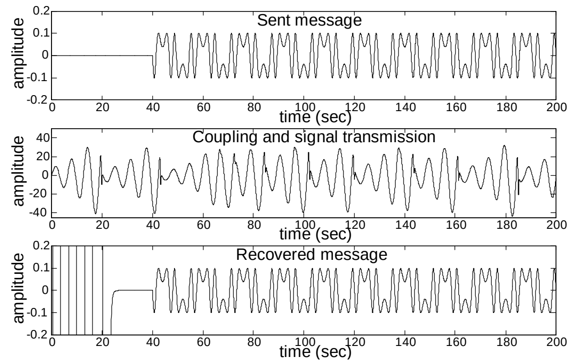

Fig. 7. Transmission of private information using a single channel.

obtain the recovered message $m^*(t)$. This scheme is only useful for identical oscillators because in this case all states of P synchronizes with those of M and $u(t)$ has not to compensate any asynchronous states, so that $u(t) \to u_M(t)$. Figure 7 shows the transmission through a single transmission channel using *Rössler-Rössler* synchronization, in which, control $u(t)$ takes action after 20 seconds and the private message is sent after 40 seconds, when complete synchronization has been achieved. Since, this scheme does not send any single chaotic signal, but it sends the nonlinear function $u_1(\cdot)$, any chaotic attractor can be reconstructed in order to extract the hidden message by means of the reported existing methods in (Short, 1994; 1996).

5.3 Chaotic switching

In the scheme shown in Fig. 8, we have proposed p like the parameters of P. The same way, p and p' have been proposed like the parameters for controller C_1. During both P and C_1 are on p then there exists synchronization or, at least, output synchronization and during C_1 is on p' there exists an error different from zero. This scheme commonly is known like chaotic switching or chaos shift keying. Figure 9 shows the transmission of binary information using *nonidentical* oscillators: Lorenz-Rössler output synchronization. To make this possible consider that $y_E(t) \to 0$ when $m = 0$ and $y_E(t) \nrightarrow 0$ when $m = 1$, interpreting $y_E(t) = 0$ like "0" logical and $y_E(t) \neq 0$ like "1" logical. In this example the parameter $\hat{r}$ of Lorenz system (12) is switching in C_1 between two values: $p = \hat{r} = 28$ when $m = 0$ and $p' = \hat{r}' = 29$ when $m = 1$ in accordance with $p^* = \hat{r} + m$, with $p^* = (p, p')$. The message is recovered faithfully

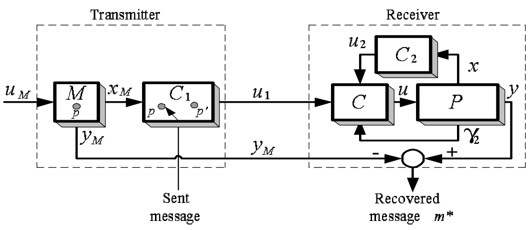

Fig. 8. Digital communication system of private information by chaotic switching.

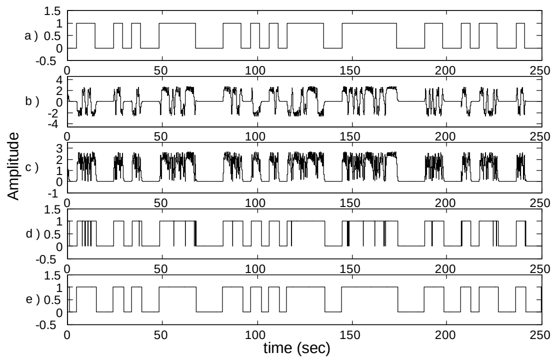

Fig. 9. Transmission of a binary signal by chaotic switching using Lorenz-Rössler output synchronization: a) original message, b) recovered message at the receiver by output synchronization error detection, c) absolute magnitude of the error signal, d) rounding and iterative signal processing, and e) recovered binary message.

after a brief iterative signal processing. Since, this scheme does not switch between two chaotic attractors of identical oscillators, but it switches a controller parameter, it is a secure cryptosystem, where the hidden message through the coupling signal cannot be reconstructed by means of the reported existing methods in literature (see e.g. Pérez and Cerdeira, 1995).

6. CONCLUDING REMARKS

A systematic method to synchronize chaotic oscillators (in continuous-time) is presented. In particular, we used the MMP from the nonlinear control theory (see (Aguilar and Cruz, 2002) for the discrete-time context). We have obtained *complete synchronization* for identical oscillators and *output synchronization* for nonidentical oscillators. In addition, some communication schemes based on complete and output synchronization are proposed, using: two transmission channels, a single transmission channel, and chaotic switching. The advantages over other cited approaches are: The approach is systematic, it uses unidirectionally coupled oscillators, gains for controller are small and synchronization is obtained after a short transient behavior. Moreover, the transmission schemes are secure.

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