FINITE-TIME STABILIZATION OF NONSMOOTHLY STABILIZABLE SYSTEMS

Bo Yang [*] **Wei Lin** [*,1]

[*] *Dept. of Electrical Engineering and Computer Science, Case Western Reserve University, Cleveland, OH 44106, USA*

Abstract: The problem of global *finite-time* stabilization is investigated in this paper. Using the finite-time Lyapunov stability theorem and the nonsmooth feedback design method developed recently, with a suitable twist, we prove that it is possible to achieve global finite-time stabilizability for a significant class of nonlinear systems which has been known not to be smoothly stabilizable, even locally, but can be globally *asymptotically* stabilized by Hölder continuous state feedback.

Keywords: Nonlinear Systems, Lyapunov Stability, Finite-Time Stabilization, Nonsmooth Feedback.

1. INTRODUCTION

We consider in this paper a family of nonlinear systems described by equations of the form

$$\begin{aligned} \dot{x}_1 &= x_2^{p_1} + f_1(x_1) \\ \dot{x}_2 &= x_3^{p_2} + f_2(x_1, x_2) \\ &\vdots \\ \dot{x}_n &= u^{p_n} + f_n(x_1, \cdots, x_n), \end{aligned} \tag{1.1}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and input, respectively, $p_i \geq 1$, $i = 1, \cdots, n$, are arbitrarily odd integers, and $f_i : \mathbb{R}^i \to \mathbb{R}$, $i = 1, \cdots, n$, are C^1 functions with $f_i(0, \cdots, 0) = 0$.

It has been recognized that (1.1) represents a significant class of nonlinear systems which is not stabilizable by any smooth state feedback, even locally, for the reason that the linearized system of (1.1) may contain an uncontrollable mode associated with eigenvalues on the right-half plane. As a consequence, stabilization of (1.1) can usually be achieved by *nonsmooth* state feedback.

Over the past fifteen years, the problem of feedback stabilization of nonsmoothly stabilizable nonlinear systems such as (1.1) has received considerable attention. For instance, the book (Bacciotti, 1992) and the papers (Kawski, 1989; Kawski, 1990; Dayawansa, 1992; Dayawansa *et al.*, 1990; Hermes, 1991*a*; Hermes, 1991*b*) studied the problem of local asymptotic stabilization via *continuous* but non-differentiable state feedback, for lower-dimensional (two or three-dimensional) systems with uncontrollable unstable linearization, while the works (Celikovsky and Aranda-Bricaire, 1999; Coron and Praly, 1991; Tzamtzi and Tsinias, 1999) investigated the *local* stabilization of the n-dimensional nonlinear system (1.1), using the idea of homogeneous approximation and Hermes' robust stability theorem for homogeneous systems (Hermes, 1991*a*; Hermes, 1991*b*; Hahn, 1967; Rosier, 1992).

More recently, it has been proved in (Qian and Lin, 2001*a*; Qian and Lin, 2001*b*) that *global* strong stabilization of the nonlinear system (1.1) is indeed possible by using nonsmooth state feedback. In particular, a Hölder continuous, globally stabilizing state feedback control law was explicitly constructed by the tool of adding a power integrator (Qian and Lin, 2001*a*; Qian and Lin, 2001*b*).

The main purpose of the paper is to prove that using Hölder continuous state feedback, global *finite-time* stabilization, instead of global asymptotic stabilization, can be achieved for the nonlinear system (1.1). The other objective of the paper is to demonstrate how the tool of adding a power integrator (Qian and Lin, 2001*a*; Qian and Lin, 2001*b*), with an appropriate twist, can be used to construct a Hölder continuous controller which renders the trivial solution $x = 0$ of (1.1) not only global stable but also convergent in *finite-time.*

The problem of finite-time stabilization arises naturally in many practical applications. A well-known example is the so-called *dead-beat* control system which has found wide applications in classical control engineering, for example, in process control and digital control, just to name a few. A more classical example is the time optimal control in which the concept of finite-time stability is automatically involved.

To be precise, consider the problem of time-optimal control for a double-integrator system. Using the well-known maximal principle, a time-optimal controller of bang-bang type can be derived, steering all the trajectories of the double-integrator system to the origin in a minimum time from any initial condition. The time-optimal control system thus obtained has a distinguished feature, namely, *finite-time convergence* rather than infinite settling time. Compared

[1] This work was supported in part by the U.S. National Science Foundation under Grants ECS-0400413 and DMS-0203387, and in part by the Air Force Research Laboratory under Grant FA8651-05-C-0110. Phone: (216)368-4493, Fax: (216)368-3123, E-mail: linwei@nonlinear.cwru.edu.

with the notion of asymptotic stability, finite-time stability requires essentially that a control system be stable in the sense of Lyapunov. Moreover, its trajectories converge to zero in finite time. Studying control systems that exhibit finite-time convergence is important for two reasons: 1) this class of systems usually has a faster convergent rate; 2) finite-time stable systems seem to perform better in the presence of uncertainties and disturbances (Bhat and Bernstein, 2000). Finally, it is worth noticing that the notion of finite-time stability also plays a key role in the design of sliding mode controllers (see, for instance, (Hirschorn, 2001)), whose strategy is to steer all the trajectories to the sliding surface in finite time.

The problem of finite-time stabilization has been studied, for instance, in the papers (Bhat and Bernstein, 1997; Bhat and Bernstein, 1998; Bhat and Bernstein, 2000; Ryan, 1979; Haimo, 1986; Hong *et al.*, 2001; Hong, 2002), which demonstrated that finite-time stable systems enjoy not only faster convergence but also better robustness and disturbance rejection properties. Notably, a fundamental result on finite-time stability was obtained in (Bhat and Bernstein, 2000), in which a Lyapunov theory for finite-time stability is presented. It provides a basic tool, within the finite-time framework, for analysis and synthesis of nonlinear control systems. The finite-time stability theory developed in (Bhat and Bernstein, 2000) was then employed to derive C^0 finite-time stabilizing state (Bhat and Bernstein, 1998) and output (Hong *et al.*, 2001) controllers, respectively, for the double integrator.

It is worth pointing out that most of the aforementioned finite-time stabilization results are only applicable to lower dimensional control systems. Moreover, these results are local due to the use of homogeneous approximation. In the *higher-dimensional* case, the paper (Hong, 2002) considered primarily the *local* finite-time stabilization problem and proposed continuous finite-time stabilizers for a class of nonlinear systems such as triangular systems, using the homogeneous systems theory. Recently, we have addressed the problem of global finite-time stabilization for a family of uncertain nonlinear systems with controllable linearization. In particular, it was shown in (Huang *et al.*, 2004) that for the nonlinear system (1.1) with $p_i = 1, i = 1, \cdots, n$, global finite-time stabilization is achievable by Hölder continuous state feedback. In contrast to the standard backstepping design for global *asymptotic* stabilization, the feedback design method in (Huang *et al.*, 2004) is more subtle and delicate for the two reasons: 1) nonsmooth state feedback control laws, rather than the smooth ones, must be constructed at every step of the recursive design procedure; 2) to guarantee global finite-time stability of the closed-loop system, the derivative of the control Lyapunov function $V(x)$ along the trajectories of the closed-loop system must be not only negative definite but also less than $-cV^{\alpha}(x)$, for suitable real numbers $c > 0$ and $0 < \alpha < 1$.

In view of the work (Huang *et al.*, 2004), an interesting theoretical question arises naturally: can *global finite-time* stabilization be achieved for the nonlinear system (1.1) with *uncontrollable unstable linearization*?

In this work, we shall provide an affirmative answer to this theoretical issue. In particular, we show how the global finite-time stabilization result obtained in (Huang *et al.*, 2004) for feedback linearizable systems (i.e. system (1.1) with $p_i = 1, i = 1, \cdots, n$) can be extended to the nonlinear system (1.1). It should be emphasized that such an extension is by no means trivial. In fact, the finite-time feedback design method proposed in (Huang *et al.*, 2004) cannot be applied to (1.1), due to the presence of uncontrollable unstable linearization of (1.1). Therefore, one of the main contributions of this paper is to show how to find, based on the theory of homogeneous systems (particularly, the idea of *homogeneity with respect to a family of dilations* (Kawski, 1989; Kawski, 1990; Hermes, 1991*a*; Hermes, 1991*b*)), a control Lyapunov function and a finite-time global stabilizer simultaneously for the nonlinear system (1.1), so that global finite-time stabilization of the closed-loop system can be concluded from the finite-time stability theorem (Bhat and Bernstein, 2000).

2. FINITE-TIME STABILITY

In this section, we review some basic concepts and terminologies related to the notion of *finite-time stability* and the corresponding Lyapunov stability theory. We also introduce a number of useful inequalities to be used in the sequel.

It is known that the classical Lyapunov stability theory (e.g., see (Hahn, 1967)) is only applicable to a differential equation whose solution from any initial condition is unique. A sufficient condition for the existence of a unique solution of the autonomous system

$$\dot{x} = f(x), \qquad \text{with} \qquad f(0) = 0, \qquad x \in \mathbb{R}^n \tag{2.1}$$

is that the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous. Notably, solution trajectories of the locally Lipschitz continuous system (2.1) can have at most *asymptotic* convergent rate. However, nonsmooth or *non-Lipschitz continuous* autonomous systems might enjoy a finite-time convergent property. For instance, the non-Lipschitz continuous system

$$\dot{x} = -x^{\frac{1}{3}}, \qquad x(0) = x_0$$

has a unique solution

$$x(t) = \begin{cases} \operatorname{sgn}(x_0)\left(x_0^{\frac{2}{3}} - \dfrac{2}{3}t\right)^{3/2}, & 0 \le t < \dfrac{3}{2}x_0^{\frac{2}{3}}, \\ 0, & t \ge \dfrac{3}{2}x_0^{\frac{2}{3}}, \end{cases}$$

which converges to $x = 0$ in finite time. This simple example suggests that in order to achieve finite-time stabilizability, non-smooth or at least non-Lipschitz continuous feedback must be employed, even if the controlled plant $\dot{x} = f(x, u, t)$ is smooth. For the analysis and synthesis of non-Lipschitz continuous systems, new notions on stability and the corresponding Lyapunov stability theory must be introduced in the *continuous* framework.

In (Kurzweil, 1956), Kurzweil introduced the notion of *global strong stability (GSS)* for the continuous nonlinear system (2.1) and established Lyapunov stability theory, *without requiring uniqueness* of the solution trajectories of (2.1).

Definition 2.1. (pp. 69 in (Kurzweil, 1956)) The trivial solution $x = 0$ of (2.1) is said to be *globally strongly stable (GSS)* if there are two functions $B : (0, +\infty) \to (0, +\infty)$ and $T : (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ with B being increasing and $lim_{s\to 0} B(s) = 0$, such that $\forall \alpha > 0$ and $\forall \varepsilon > 0$, for every solution $x(t)$ of (2.1) defined on $[0, t_1)$, $0 < t_1 \le +\infty$ with $\|x(0)\| \le \alpha$, there is a solution $z(t)$ of (2.1) defined on $[0, +\infty)$ satisfying

(i) $z(t) = x(t),\ t \in [0, t_1)$;

(ii) $\|z(t)\| \le B(\alpha),\ \forall t \ge 0$;

(iii) $\|z(t)\| < \varepsilon,\ \forall t \ge T(\alpha, \varepsilon)$.

This definition is a natural extension of global asymptotic stability introduced by Lyapunov for the autonomous system (2.1) when it has a unique solution. Using the concept of GSS, Kurzweil proved (Kurzweil, 1956) that the Lyapunov's second theorem remains true as long as the vector field $f(x)$ is *continuous* (no local Lipschitz continuity is required).

Theorem 2.2. Suppose there is a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, which is positive definite and proper, such that $\dfrac{\partial V}{\partial x} f(x)$

is negative definite. Then, the trivial solution $x = 0$ of system (2.1) is globally strongly stable.

This theorem has been shown of paramount importance in establishing various global strong stabilization results by continuous state feedback for nonlinear control systems that are not smoothly stabilizable (Qian and Lin, 2001*a*; Qian and Lin, 2001*b*). Interestingly, Theorem 2.2 is analogous to Lyapunov's second theorem and recovers the case of the so-called global asymptotic stability when the solution of system (2.1) is unique.

Similar to the work of Bhat-Bernstein (Bhat and Bernstein, 2000), the concept of global strong stability for the continuous system (2.1) can also be extended to the case of finite-time stability. The following definition is a slight generalization of Definition 2.2 in (Bhat and Bernstein, 2000).

Definition 2.3. The system (2.1) is said to be *globally finite-time stable* at $x = 0$ if the following statements hold:

Global strong stability: The continuous system (2.1) is globally strongly stable at $x = 0$;

Finite-time convergence: There exists a function $T : \mathbb{R}^n \to [0, +\infty)$, called the *settling time*, such that every solution trajectory $\phi(t, x)$ of (2.1) starting from the initial condition $\phi(0, x) = x$ is well-defined on the interval $[0, T(x))$ and satisfies $\lim_{t\to T(x)} \phi(t, x) = 0$.

Based on the definition above, it is not difficult to prove the following Lyapunov's theorem on globally finite-time stability by using Theorem 2.2 and comparison principle (cf. Theorem 4.2 of (Bhat and Bernstein, 2000)).

Theorem 2.4. Suppose there exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$, which is positive definite and proper, such that

$$\dot{V} + cV^{\alpha} = \frac{\partial V}{\partial x} f(x) + cV^{\alpha} \leq 0 \qquad (2.2)$$

for some real numbers $c > 0$ and $\alpha \in (0, 1)$. Then, system (2.1) is globally finite-time stable at $x = 0$. Moreover, if T is the settling time, then $T(x) \leq \frac{1}{c(1-\alpha)} V(x)^{1-\alpha}, \ \ \forall x \in \mathbb{R}^n$.

In the case of global strong stability, the Kurzweil's stability theorem requires only $L_f V(x) < 0, \ \forall x \neq 0$. On the contrary, Theorem 2.4 on global finite-time stability requires a stronger condition, i.e. $\dot{V} \leq -cV^{\alpha}, \forall x \neq 0$. [2] For this reason, global *finite-time* stabilization is a more challenging problem than global *asymptotic* stabilization. Indeed, according to Theorem 2.4, in order to achieve finite-time stabilization, one must construct not only a non-Lipschitz continuous state feedback control law (because finite-time convergence is not possible in the case of either smooth or Lipschitz-continuous dynamics), but also a control Lyapunov function $V(x)$ satisfying the inequality (2.2). The latter is of course not easy to achieve and makes the construction of a control Lyapunov function much more subtle.

In summary, although Theorem 2.4 has provided a basic tool for testing global finite-time stability of nonlinear systems, how to employ it to design globally finite-time stabilizing controllers for nonlinear systems with uncontrollable unstable linearization such as (1.1) is a nontrivial problem that is certainly interesting from a theoretic point of view and will be addressed in this paper.

We conclude this section by introducing two useful lemmas to be used in the next section.

Lemma 2.5. For $x, y, p \in \mathbb{R}$ with $p > 0$, the following inequality holds:

$$(|x| + |y|)^p \leq \max(2^{p-1}, 1)(|x|^p + |y|^p). \qquad (2.3)$$

As a consequence, when $p = \frac{c}{r}$, where $c \geq 1$ and $r \geq 1$ are *odd* integers,

$$|x^p - y^p| \geq 2^{1-p}|x - y|^p, \qquad \text{if } p \geq 1; \qquad (2.4)$$

$$|x^p - y^p| \leq 2^{1-p}|x - y|^p, \qquad \text{if } p \leq 1. \qquad (2.5)$$

Lemma 2.6. Given positive real numbers x, y, m, n, a, b, the following inequality holds:

$$ax^m y^n \leq bx^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} y^{m+n}. \qquad (2.6)$$

The proofs of these lemmas can be found in (Qian and Lin, 2001*b*).

3. GLOBAL FINITE-TIME STABILIZATION BY HÖLDER CONTINUOUS FEEDBACK

Using the Lyapunov theory on finite-time stability introduced in the previous section, we can prove the following global finite-time stabilization result for the nonlinear system (1.1).

Theorem 3.1. For a chain of odd power integrators perturbed by a C^1 lower-triangular vector field (1.1), there is a state feedback control law $u = u(x)$ with $u(0) = 0$, which is Hölder continuous, such that the trivial solution $x = 0$ of (1.1) is globally finite-time stable.

Remark 3.2. Since the function $f_i(\cdot)$ in (1.1) is C^1 and vanishes at the origin, by the Taylor theorem there is a smooth function $\gamma_i(x_1, \cdots, x_i) \geq 0$ satisfying $(1 \leq i \leq n)$

$$|f_i(x_1, \cdots, x_i)| \leq (|x_1| + \cdots + |x_i|)\gamma_i(\cdot). \qquad (3.1)$$

This property will be used in the proof of Theorem 3.1.

Proof: The proof is based on the tool of *adding a power integrator* (Qian and Lin, 2001*a*; Qian and Lin, 2001*b*) with a suitable twist. In particular, by choosing an appropriate dilation and homogeneous degree, we simultaneously construct a C^1 control Lyapunov function satisfying the Lyapunov inequality $\dot{V} + cV^{\alpha} \leq 0$, as well as a Hölder continuous finite-time stabilizer.

For the convenience of the reader, we break up the proof in three steps.

Initial Step. Choose the Lyapunov function $V_1(x_1) = \frac{x_1^2}{2}$. Using (1.1) and (3.1), we have

$$\dot{V}_1(x_1) \leq x_1(x_2^{p_1} - x_2^{*p_1}) + x_1 x_2^{*p_1} + x_1^2 \gamma_1(x_1)$$
$$\leq x_1(x_2^{p_1} - x_2^{*p_1}) + x_1 x_2^{*p_1} + x_1^{2+d}\hat{\gamma}_1(x_1),$$

where d is a rational number whose denominator is an *even* integer while its numerator is an *odd* integer, such that

$$d \in \Big[-\frac{1}{R_n}, 0\Big) \triangleq \Big[-\frac{1}{p_{n-1}\cdots p_1 + \cdots + p_1 + 1}, 0\Big) \subset (-1, 0),$$

$$\text{and} \quad \hat{\gamma}_1(x_1) \geq x_1^{-d}\gamma_1(x_1), \qquad (3.2)$$

where $\hat{\gamma}_1(x_1) \geq 0$ is a smooth function.

It is of interest to note that such a rational number d always exists. In fact, one can simply pick $d = -\frac{2}{2R_n+1}$. Then, the virtual controller x_2^* defined by

$$x_2^{*p_1} = -x_1^{1+d}(n + \hat{\gamma}_1(x_1)),$$

or, equivalently,

$$x_2^* = -x_1^{\frac{1+d}{p_1}}(n + \hat{\gamma}_1(x_1))^{1/p_1} := -\xi_1^{q_2}\beta_1(x_1),$$

which is Hölder continuous, yields

$$\dot{V}_1(x_1) \leq -n\xi_1^{2+d} + \xi_1(x_2^{p_1} - x_2^{*p_1}),$$

where $\xi_1 = x_1$ and $\beta_1(x_1) = (n + \hat{\gamma}_1(x_1))^{1/p_1}$ is smooth.

Inductive Step. Suppose at step $k - 1$, there are a C^1 Lyapunov function $V_{k-1}(x_1, \cdots, x_{k-1})$, which is positive definite and proper, and a set of parameters

$$q_1 = 1, q_2 = \frac{q_1 + d}{p_1}, \cdots, q_k = \frac{q_{k-1} + d}{p_{k-1}}, \qquad (3.3)$$

and Hölder C^0 virtual controllers $x_1^*, \cdots, x_k^*$, defined by

[2] In (Bhat and Bernstein, 2000), it has been shown that the sufficient condition (2.2) is also necessary for a continuous autonomous system to be finite-time stable, under the condition that the settling-time function $T(x)$ is continuous at the origin.

$$
\begin{aligned}
&x_1^*=0, && \xi_1=x_1^{1/q_1}-x_1^{*1/q_1}\\
&x_2^*=-\xi_1^{q_2}\beta_1(x_1), && \xi_2=x_2^{1/q_2}-x_2^{*1/q_2}\\
&\vdots && \vdots\\
&x_k^*=-\xi_{k-1}^{q_k}\beta_{k-1}(x_1,\cdots,x_{k-1}), && \xi_k=x_k^{1/q_k}-x_k^{*1/q_k}
\end{aligned} \tag{3.4}
$$

with $\beta_1(x_1) > 0, \cdots, \beta_{k-1}(x_1, \cdots, x_{k-1}) > 0$, being *smooth*, such that

$$V_{k-1}(x_1,\cdots,x_{k-1}) \le 2(\xi_1^2+\cdots+\xi_{k-1}^2), \tag{3.5}$$

$$\dot V_{k-1}(x_1,\cdots,x_{k-1}) \le -(n-k+2)(\xi_1^{2+d}+\cdots+\xi_{k-1}^{2+d}) +\xi_{k-1}^{2-q_{k-1}}(x_k^{p_{k-1}}-x_k^{*p_{k-1}}). \tag{3.6}$$

From (3.2) and (3.3), it is easy to see that $1 = q_1 > q_2 > \cdots > q_k > 0$. In addition, using (2.3) and (3.4) yields

$$|x_k| \le |\xi_k|^{q_k} + |\xi_{k-1}|^{q_k}\beta_{k-1}(x_1,\cdots,x_{k-1}), \tag{3.7}$$

or, equivalently,

$$|x_k^{p_{k-1}}| \le (|\xi_k|^{q_k p_{k-1}} + |\xi_{k-1}|^{q_k p_{k-1}})\bar\beta_{k-1}(\cdot), \tag{3.8}$$

where $\bar\beta_{k-1}(x_1,\cdots,x_{k-1}) \ge 0$ is a smooth function.

Now, we claim that (3.5) and (3.6) also hold at step k. To prove the claim, consider

$$V_k(x_1,\cdots,x_k)=V_{k-1}(\cdot)+W_k(x_1,\cdots,x_k), \tag{3.9}$$

$$W_k(x_1,\cdots,x_k)=\int_{x_k^*}^{x_k}(s^{1/q_k}-x_k^{*1/q_k})^{2-q_k}\mathrm{d}s. \tag{3.10}$$

The Lyapunov function $V_k(x_1, \cdots, x_k)$ thus defined has several important properties listed in the following two propositions, whose proofs involve a tedious but straightforward calculation and hence are omitted.

Proposition 3.3. $W_k(x_1, \cdots, x_k)$ is C^1. Moreover, $\frac{\partial W_k}{\partial x_k} = \xi_k^{2-q_k}$ and for $l = 1, \cdots, k-1$,

$$\frac{\partial W_k}{\partial x_l} = -(2-q_k)\frac{\partial (x_k^{*1/q_k})}{\partial x_l}\int_{x_k^*}^{x_k}(s^{1/q_k}-x_k^{*1/q_k})^{1-q_k}\mathrm{d}s.$$

Proposition 3.4. $V_k(x_1, \cdots, x_k)$ is C^1, positive definite and proper. Moreover, it can be bounded above by a quadratic function. In fact,

$$V_k(\cdot) \le 2(\xi_1^2+\cdots+\xi_k^2). \tag{3.11}$$

With the aid of Propositions 3.3 and 3.4, one can deduce from (3.6) that

$$
\begin{aligned}
\dot V_k(x_1,\cdots,x_k) &= \dot V_{k-1}(\cdot)+\frac{\partial W_k}{\partial x_k}\dot x_k+\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot x_l\\
&\le -(n-k+2)(\sum_{i=1}^{k-1}\xi_i^{2+d})+\xi_{k-1}^{2-q_{k-1}}(x_k^{p_{k-1}}-x_k^{*p_{k-1}})\\
&\quad+\xi_k^{2-q_k}(x_{k+1}^{p_k}-x_{k+1}^{*p_k})+\xi_k^{2-q_k}x_{k+1}^{*p_k}\\
&\quad+\xi_k^{2-q_k}f_k(x_1,\cdots,x_k)+\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot x_l.
\end{aligned} \tag{3.12}
$$

Next, we introduce additional two propositions which are quite useful when estimating the last two terms in inequality (3.12).

Proposition 3.5. There are non-negative smooth functions $\tilde\gamma_k(x_1, \cdots, x_k)$ and $\bar\gamma_k(x_1, \cdots, x_k)$ such that

$$|f_k(\cdot)| \le (|\xi_1|^{q_k}+\cdots+|\xi_k|^{q_k})\tilde\gamma_k(\cdot) \tag{3.13}$$

$$|\dot x_k| \le (|\xi_1|^{q_k+d}+\cdots+|\xi_{k+1}|^{q_k+d})\bar\gamma_k(\cdot). \tag{3.14}$$

Proposition 3.6. There is a non-negative smooth function $C_{k,l}(x_1, \cdots, x_k)$ such that for $l = 1, \cdots, k-1$,

$$\left|\frac{\partial (x_k^{*1/q_k})}{\partial x_l}\right| \le (|\xi_{k-1}|^{1-q_l}+\cdots+|\xi_{l-1}|^{1-q_l})C_{k,l}(\cdot). \tag{3.15}$$

The proofs of Propositions 3.5 and 3.6 are omitted due to the limited space.

Using Proposition 3.5 and Lemma 2.6, one has

$$
\begin{aligned}
|\xi_k^{2-q_k}f_k(\cdot)| &\le |\xi_k|^{2-q_k}(\sum_{i=1}^{k}|\xi_i|^{q_k+d})\hat\gamma_k(x_1,\cdots,x_k)\\
&\le \frac{1}{3}(\sum_{i=1}^{k-1}\xi_i^{2+d})+\xi_k^{2+d}\tilde\rho_k(x_1,\cdots,x_k),
\end{aligned} \tag{3.16}
$$

where $\hat\gamma_k(x_1, \cdots, x_k) \ge (\xi_1^{-d}+\cdots+\xi_k^{-d})\tilde\gamma(x_1, \cdots, x_k)$ and $\tilde\rho_k(x_1, \cdots, x_k)$ are non-negative smooth functions.

Using Lemmas 2.5 and 2.6 and observing that $p_{k-1} = (q_{k-1}+d)/q_k$, we obtain

$$
\begin{aligned}
|\xi_{k-1}^{2-q_{k-1}}(x_k^{p_{k-1}}-x_k^{*p_{k-1}})| &\le 2|\xi_{k-1}|^{2-q_{k-1}}|\xi_k|^{q_{k-1}+d}\\
&\le \frac{\xi_{k-1}^{2+d}}{3}+c_k\xi_k^{2+d}
\end{aligned} \tag{3.17}
$$

for a suitable $c_k > 0$.

To estimate the last term in (3.12), we observe from Proposition 3.6 that for $l = 1, \cdots, k-1$,

$$
\begin{aligned}
\left|\frac{\partial W_k}{\partial x_l}\right| &=(2-q_k)\left|\frac{\partial (x_k^{*1/q_k})}{\partial x_l}\right|\left|\int_{x_k^*}^{x_k}(s^{1/q_k}-x_k^{*1/q_k})^{1-q_k}\mathrm{d}s\right|\\
&\le(2-q_k)(\sum_{i=l-1}^{k-1}|\xi_i|^{1-q_l})C_{k,l}(\cdot)|x_k^*-x_k||\xi_k|^{1-q_k}\\
&\le 2(2-q_k)(\sum_{i=l-1}^{k-1}|\xi_i|^{1-q_l})C_{k,l}(\cdot)|\xi_k|.
\end{aligned} \tag{3.18}
$$

Then, combining (3.18) and (3.14) yields

$$
\begin{aligned}
\left|\sum_{l=1}^{k-1}\frac{\partial W_k}{\partial x_l}\dot x_l\right| &\le\sum_{l=1}^{k-1}\Big[2(2-q_k)(\sum_{i=l-1}^{k-1}|\xi_i|^{1-q_l})C_{k,l}(\cdot)\\
&\qquad\times|\xi_k|(\sum_{i=1}^{l+1}|\xi_i|^{q_l+d})\bar\gamma_k(\cdot)\Big]\\
&\le\frac{1}{3}(\sum_{i=1}^{k-1}\xi_i^{2+d})+\xi_k^{2+d}\bar\rho_k(x_1,\cdots,x_k).
\end{aligned} \tag{3.19}
$$

The last inequality follows from Lemma 2.6.

Substituting the estimates (3.16), (3.17) and (3.19) into (3.12), we arrive at

$$
\begin{aligned}
\dot V_k &\le -(n-k+1)(\sum_{i=1}^{k-1}\xi_i^{2+d})+\xi_k^{2-q_k}(x_{k+1}^{p_k}-x_{k+1}^{*p_k})\\
&\quad+\xi_k^{2-q_k}x_{k+1}^{*p_k}+\xi_k^{2+d}(c_k+\tilde\rho_k(\cdot)+\bar\rho_k(\cdot)).
\end{aligned}
$$

From (3.2), it follows that $q_k + d > 0$.

Then, it is clear that the C^0 virtual controller

$$x_{k+1}^{*p_k} = -\xi_k^{q_k+d}[n-k+1+c_k+\tilde\rho_k(\cdot)+\bar\rho_k(\cdot)],$$

or, equivalently,

$$x_{k+1}^* = -\xi_k^{q_{k+1}}\beta_k(x_1,\cdots,x_k) \tag{3.20}$$

with $\beta_k(\cdot) = [n-k+1+c_k+\tilde\rho_k(\cdot)+\bar\rho_k(\cdot)]^{1/p_k} > 0$ being smooth and $0 < q_{k+1} = \frac{q_k+d}{p_k} < q_k$, is such that

$$\dot V_k \le -(n-k+1)(\xi_1^{2+d}+\cdots+\xi_k^{2+d})+\xi_k^{2-q_k}(x_{k+1}^{p_k}-x_{k+1}^{*p_k}).$$

This completes the inductive proof.

Last Step. According to the inductive argument above, we conclude at the n-th step that there are a Hölder continuous state feedback control law of the form

$$u = x_{n+1}^* = -\xi_n^{q_{n+1}}\beta_n(x_1,\cdots,x_n) \tag{3.21}$$

with $\beta_n(\cdot) > 0$ being smooth, and a C^1 positive definite and proper Lyapunov function $V_n(x_1, \cdots, x_n)$ of the form (3.10), such that

$$V_n(x_1,\cdots,x_n) \le 2(\xi_1^2+\cdots+\xi_n^2), \tag{3.22}$$

$$\dot V_n(x_1,\cdots,x_n) \le -(\xi_1^{2+d}+\cdots+\xi_n^{2+d}). \tag{3.23}$$

Finally, pick $\alpha = \frac{2+d}{2} \in (0,1)$. Then, it is straightforward to show that

$$\dot V_n + \frac{1}{4}V_n^\alpha \le -\frac{1}{2}(\xi_1^{2+d}+\cdots+\xi_n^{2+d}) \le 0. \tag{3.24}$$

By Theorem 2.4, it is concluded from (3.24) that the trivial solution $x=0$ of the closed-loop system (1.1)–(3.21) is globally finite-time stable. □

Remark 3.7. If we choose $d=0$ in the proof of Theorem 3.1, it is deduced from (3.3) that $q_1=1$, $q_2=\frac{1}{p_1},\cdots$, $q_n=\frac{1}{p_1\cdots p_{n-1}}$. Hence, our proof is degenerated to the proof of Theorem 3.7 in (Qian and Lin, 2001*a*) when $d=0$. However, as indicated in (Qian and Lin, 2001*a*), the choice of $d=0$ results in only an *asymptotic* stabilizer rather than a *finite-time* stabilizer because only negative definiteness of $\dot V_n$ can be guaranteed, not $\dot V_n + \frac{1}{4}V_n^\alpha \le 0$.

Remark 3.8. Theorem 2 of (Bhat and Bernstein, 1997) states that for a homogeneous system, finite-time stability is equivalent to asymptotic stability plus a *negative* homogeneous degree. For the non-homogeneous system (1.1), d in the proof above can be viewed as a degree-like parameter and $q_1,\cdots,q_{n+1}$ are dilation-like parameters, which are determined by (3.3). Therefore, one of the most crucial points in proving Theorem 3.1 is to determine the degree-like parameter d, which must be *negative* due to *finite-time stability.* Once the assignments of d and $q_1,\cdots,q_{n+1}$ are completed, one can easily use the adding a power integrator technique to construct a non-Lipschitz continuous, finite-time stabilizer, as illustrated in the proof of Theorem 3.1.

In the rest of this section, we use a simple yet nontrivial example given in (Qian and Lin, 2001*b*) to demonstrate how a global finite-time stabilizer can be explicitly constructed, by suitably choosing the degree d and dilation q_1, q_2 and by employing the adding a power integrator technique.

Example 3.9. Consider the planar system

$$\begin{aligned}\dot x_1 &= x_2^3 + x_1 e^{x_1}\\ \dot x_2 &= u.\end{aligned} \tag{3.25}$$

whose Jacobian linearization is given by

$$(A,B) = \left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right),$$

and contains an uncontrollable mode associated with a positive eigenvalue.

As shown in (Qian and Lin, 2001*b*), the planar system (3.25) is not smoothly stabilizable but can be globally asymptotically stabilized by Non-Lipschitz continuous state feedback.

By Theorem 3.1, we now know that (3.25) is also globally finite-time stabilizable. To find a finite-time stabilizer, one can choose (as done in the proof of Theorem 3.1)

$$R_2 = 4, \quad d = -\frac{2}{9} < 0, \quad q_1 = 1, \quad q_2 = \frac{7}{27},$$

and a C^1, positive definite and proper Lyapunov function

$$V_2(x_1,x_2) = \frac{x_1^2}{2} + \int_{x_2^*}^{x_2} (s^{\frac{27}{7}} - x_2^{*\frac{27}{7}})^{\frac{47}{27}} \mathrm{d}s$$

with $x_2^* := -x_1^{\frac{7}{27}}\left[2+e^{x_1}(1+x_1^2)^{\frac{1}{9}}\right]^{\frac{1}{3}}$.

A direct computation gives a Hölder continuous controller

$$u = -\xi_2^{\frac{1}{27}}\left(1.8 + 3C_1(x_1) + 0.6\tilde C_1(x_1)^{\frac{16}{9}}\right) \tag{3.26}$$

with $\xi_2 = x_2^{\frac{27}{7}} - x_2^{*\frac{27}{7}}$,

$$C_1(x_1) = \left[2+e^{x_1}(1+x_1^2)^{\frac{1}{9}}\right]^{\frac{9}{7}} + \left[2+e^{x_1}(1+x_1^2)^{\frac{1}{9}}\right]^{\frac{2}{7}} e^{x_1}(1+x_1^2)^{\frac{10}{9}},$$

and

$$\tilde C_1(x_1) = 6[1+e^{x_1}(1+x_1^2)^{\frac{1}{9}}]C_1(x_1),$$

such that

$$\dot V_2 + \frac{1}{4}V_2^{\frac{8}{9}} \le -\frac{1}{2}(x_1^{\frac{16}{9}} + \xi_2^{\frac{16}{9}}) \le 0. \tag{3.27}$$

By Theorem 2.4, it follows immediately from (3.27) that the trivial solution $x=0$ of the closed-loop system (3.25)–(3.26) is globally finite-time stable.

Remark 3.10. According to Theorem 2.4, it is concluded that the four-dimensional, underactuated unstable two degree of freedom mechanical system (Rui *et al.*, 1997; Qian and Lin, 2001*a*)

$$\dot x_1 = x_2,\ \dot x_2 = x_3^3 + \frac{g}{l}\sin x_1,\ \dot x_3 = x_4,\ \dot x_4 = u$$

is also globally finite-time stabilizable by nonsmooth state feedback. A globally finite-time stabilizing, nonsmooth state feedback controller can be explicitly designed, in a fashion similar to Example 3.9.

4. FURTHER EXTENSION AND DISCUSSION

In this section, we briefly discuss how the global finite-time stabilization result derived so far can be extended to a more general class of nonlinear systems of the form

$$\begin{aligned}\dot x_1 &= d_1(t)x_2^{p_1} + f_1(t,x_1,x_2)\\ &\vdots\\ \dot x_{n-1} &= d_{n-1}(t)x_n^{p_{n-1}} + f_{n-1}(t,x_1,\cdots,x_n)\\ \dot x_n &= d_n(t)u^{p_n} + f_n(t,x_1,\cdots,x_n,u),\end{aligned} \tag{4.1}$$

where $x=(x_1,\cdots,x_n)^T \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the system state and input, respectively. For $i=1,\cdots,n$, $p_i \ge 1$ is an odd integer, $f_i : \mathbb{R}\times\mathbb{R}^{i+1} \to \mathbb{R}$, is a C^0 function with $f_i(t,0,\cdots,0)=0$, $\forall t \in \mathbb{R}$, and $d_i(t)$ is a C^0 function of time t, which represents an unknown time varying parameter.

The following assumptions characterize a subclass of nonlinear systems (4.1).

Assumption 4.1. For $i=1,\cdots,n$, there are positive real numbers λ_i and μ_i such that

$$0 < \lambda_i \le d_i(t) \le \mu_i.$$

Assumption 4.2. For $i=1,\cdots,n$,

$$f_i(t,x_1,\cdots,x_i,x_{i+1}) = \sum_{j=0}^{p_i-1} x_{i+1}^j a_{i,j}(t,x_1,\cdots,x_i), \tag{4.2}$$

where $x_{n+1}=u$. Moreover, there is a smooth function $\gamma_{i,j}(x_1,\cdots,x_i) \ge 0$ such that for $j=0,\cdots,p_i-1$,

$$|a_{i,j}(t,x_1,\cdots,x_i)| \le (|x_1|+\cdots+|x_i|)\gamma_{i,j}(\cdot). \tag{4.3}$$

Remark 4.3. In the case when $a_{i,j}(t,x_1,\cdots,x_i)$ is independent of t (i.e. $a_{i,j}(t,x_1,\cdots,x_i) \equiv a_{i,j}(x_1,\cdots,x_i)$), if the function $a_{i,j}(x_1,\cdots,x_i)$ is C^1 and $a_{i,j}(0,\cdots,0)=0$, there always exists a smooth function $\gamma_{i,j}(x_1,\cdots,x_i) \ge 0$ satisfying the property (4.3), as explained in Remark 3.2.

The following result on global finite-time stabilizability is an extension of Theorem 3.1.

Theorem 4.4. For a family of uncertain nonlinear systems (4.1) satisfying Assumption 4.1–4.2, there is a Hölder continuous controller $u=u(x)$ with $u(0)=0$, which renders the trivial solution $x=0$ of (4.1) globally finite-time stable.

The proof can be carried out in the spirit similar to that of Theorem 3.1, with a more subtle design. It is omitted for the sake of space.

The significance of Theorem 4.4 can be illustrated by the following example, which clearly indicates the generality of Theorem 4.4 over Theorem 3.1.

Example 4.5. Consider an affine system of the form

$$\dot{x}_1 = x_2^p + x_2^{p-2} a_{p-2}(x_1) + \cdots + x_2 a_1(x_1) + a_0(x_1)$$
$$\dot{x}_2 = v \tag{4.4}$$

where $a_0(x_1), a_1(x_1), \cdots, a_{p-2}(x_1)$ are smooth functions, with $a_0(0) = a_1(0) = \cdots = a_{p-2}(0) = 0$, and $p \geq 1$ is an integer.

First of all, it is clear that global finite-time stabilization of system (4.4) is not solvable by Theorem 3.1 because (4.4) is not in the triangular form (1.1).

On the other hand, it is straightforward to verify that the planar system (4.4) satisfies Assumption 4.2. By Theorem 4.4, global finite-time stabilization of (4.1) is achievable by Hölder continuous state feedback as long as p is an odd integer. Moreover, a finite-time stabilizer can be designed in a manner similar to Example 3.9.

It is of interest to note that (4.4) is representative of a class of two-dimensional affine systems. In fact, Jakubczyk and Respondek (Jakubczyk and Respondek, 1990) proved that every smooth affine system in the plane, i.e., $\dot{\xi} = f(\xi) + g(\xi)u$, is feedback equivalent to (4.4) if $g(0)$ and $\mathrm{ad}_f^p g(0)$ are linearly independent. A more general result was proved in (Cheng and Lin, 2003) later on, showing that (4.4) is indeed a special case of the so-called "p-normal form" (Cheng and Lin, 2003). In other words, (4.4) is a *normal form* of two-dimensional affine systems when $\mathrm{rank}[g(0), \mathrm{ad}_f^p g(0)] = 2$. In (Qian and Lin, 2001*a*), it was proved that global asymptotic stabilization of (4.4) is possible by non-Lipschitz continuous state feedback, although there may not exist any smooth stabilizer for system (4.4). Now, using Theorem 4.4 we have further concluded that system (4.4) is not only globally asymptotically stabilizable but also globally finite-time stabilizable.

5. CONCLUSION

We have studied the problem of global finite-time stabilization for a class of nonlinear systems. The systems under consideration usually involve an uncontrollable unstable Jacobian linearization (i.e., the uncontrollable modes have eigenvalues on the right half plane), and therefore are extremely difficult to be controlled. Using the Lyapunov theory on finite-time stability (Bhat and Bernstein, 2000), together with the tool of adding a power integrator (Qian and Lin, 2001*a*; Qian and Lin, 2001*b*) with an appropriate modification (in particular, by subtly choosing a homogeneous degree d and a family of dilations $q_1, q_2, \cdots, q_n$), we have presented a systematic feedback design method for the explicit construction of globally finite-time stabilizing, Hölder continuous state feedback control laws, as well as a C^1 control Lyapunov function that satisfies the Lyapunov finite-time stability inequality $\dot{V}(x) \leq -cV^{\alpha}(x)$, which is a stronger requirement than the traditional Lyapunov inequality $\dot{V} < 0$, $\forall x \neq 0$.

While the results presented in this paper are theoretical in nature, it would be interesting to see some practical applications of the proposed finite-time stabilization theory. This is certainly an important subject to be investigated in the next phase of research. We hope that this work would generate interest in the control community and make the control engineer be aware of the difficulty, subtlety and power of the finite-time control strategy, and in turn stimulate research activities in the areas of finite-time nonlinear control theory and applications.

REFERENCES

Bacciotti, A. (1992). *Local stabilizability of nonlinear control systems*. World Scientific.

Bhat, S. P. and D. S. Bernstein (1997). Finite-time stability of homogeneous systems. In: *Proc. ACC'97*. pp. 2513–2514.

Bhat, S. P. and D. S. Bernstein (1998). Continuous finite-time stabilization of the translational and rotational double integrators. *IEEE TAC* **43**, 678–682.

Bhat, S. P. and D. S. Bernstein (2000). Finite-time stability of continuous autonomous systems. *SICON* **38**, 751–766.

Celikovsky, S. and E. Aranda-Bricaire (1999). Constructive non-smooth stabilization of triangular systems. *SCL* **36**, 21–37.

Cheng, D. and W. Lin (2003). On p—normal forms of nonlinear systems. *IEEE TAC* **48**, 1242–1248.

Coron, J. M. and L. Praly (1991). Adding an integrator for the stabilization problem. *SCL* **17**, 89–104.

Dayawansa, W. (1992). Recent advances in the stabilization problem for low dimensional systems. In: *Proc. IFAC NOLCOS'92*. Bordeaux. pp. 1–8.

Dayawansa, W., C. Martin and G. Knowles (1990). Asymptotic stabilization of a class of smooth two dimensional systems. *SICON* **28**, 1321–1349.

Hahn, W. (1967). *Stability of Motion*. Springer-Verlag.

Haimo, V. T. (1986). Finite-time controllers. *SICON* **24**, 760–770.

Hermes, H. (1991*a*). Asymptotic stabilization of planar nonlinear systems. *SCL* **12**, 437–445.

Hermes, H. (1991*b*). Homogeneous coordinates and continuous asymptotically stabilizing feedback controls. In: *Differential Equations Stability and Control* (S. Elaydi, Ed.). pp. 249–260. Marcel Dekker.

Hirschorn, R. M. (2001). Singular sliding-mode control. *IEEE TAC* **46**, 276–285.

Hong, Y. (2002). Finite-time stabilization of a class of controllable systems. *SCL* **46**, 231–236.

Hong, Y., Y. Xu and J. Huang (2001). On an output feedback finite-time stabilization problem. *IEEE TAC* **46**, 305–309.

Huang, X., W. Lin and B. Yang (2004). Finite-time stabilization in the large for uncertain nonlinear systems. In: *Proc. ACC'04*. Boston, MA. pp. 1073–1078.

Jakubczyk, B. and W. Respondek (1990). Feedback equivalence of planar systems and stability. In: *Robust Control of Linear Systems and Nonlinear Control* (M.A. Kaashoek et al., Ed.). pp. 447–456. Birkhäuser.

Kawski, M. (1989). Stabilization of nonlinear systems in the plane. *SCL* **12**, 169–175.

Kawski, M. (1990). Homogeneous stabilizing feedback laws. *Contr. Theory & Advanced Tech.* **6**, 497–516.

Kurzweil, J. (1956). On the inversion of Lyapunov's second theorem on the stability of motion. *Amer. Math. Soc. Trans.* **24**, 19–77.

Qian, C. and W. Lin (2001*a*). A continuous feedback approach to global strong stabilization of nonlinear systems. *IEEE TAC* **46**, 1061–1079.

Qian, C. and W. Lin (2001*b*). Non-lipschitz continuous stabilizer for nonlinear systems with uncontrollable unstable linearization. *SCL* **42**, 185–200.

Rosier, L. (1992). Homogeneous Lyapunov functions for homogeneous continuous vector field. *SCL* **19**, 467–473.

Rui C., M. Reyhanoglu, I. Kolmanovsky, S. Cho and N.H. McClamroch (1997). Nonsmooth stabilization of an underactuated unstable two degrees of freedom mechanical system, In: *Proc. 36th IEEE CDC*. San Diego. pp. 3998-4003.

Ryan, E. P. (1979). Singular optimal controls for second-order saturating system. *Int. J. Contr.* **36**, 549–564.

Tzamtzi, M. and J. Tsinias (1999). Explicit formulas of feedback stabilizers for a class of triangular systems with uncontrollable linearization, *SCL,* **38**, 115-126.