ON THE DESIGN OF DISCRETE-TIME FIXED-ORDER CONTROLLERS FOR PERSISTENT DISTURBANCE REJECTION

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Abstract: The focal point of this paper is the design of fixed-order controllers for persistent disturbance rejection. It is shown that optimal controller design can, in this case, be formulated as a generalized eigenvalue problems. The results in this paper rely on an extension of the concepts of equalized performance and superstability to the case of multi-input/multi-output systems described by transfer function matrices. The efficacy of the algorithm provided is illustrated via a numerical example.

Keywords: Disturbance Rejection, Linear Time Invariant Systems, Discrete-Time

1. INTRODUCTION

In this paper, we address the problem of fixed-order controller design for persistent disturbance rejection. More precisely, we provide a fixed-order controller design method for multi-input/multi-output (MIMO) discrete-time systems that minimizes the effect of perturbations, which are bounded in the ℓ_∞-norm, on the outputs. By fixing the controller order, we mean imposing constraints on the order of the polynomial matrices used in the polynomial matrix description of the controller. For an exposition on polynomial matrix descriptions see, for example, (Antsaklis and Michel, 1997).

The line of research in which this paper is integrated has its origin in the ℓ_1 control theory, which focuses directly on the time-domain specifications, e.g., see (Dahleh and Diaz-Bobillo, 1995). However, ℓ_1 control theory assumes zero initial conditions. Moreover, the resulting optimal controllers can have arbitrarily high order. These limitations lead to the development of the so-called *equalized performance* and *superstability* concepts, which were first introduced in (Blanchini and Sznaier, 1997) and (Polyak and Halpern, 1999). This preliminary work was followed by (Halpern and Polyak, 2000), (Sznaier *et al.*, 2002), (Polyak and Shcherbakov, 2002*a*), (Blanchini and Sznaier, 2000) and (Polyak and Shcherbakov, 2002*b*). In this work, procedures were developed for the design of controllers for persistent disturbance rejection for two cases: i) single-input/single-output (SISO) plants and ii) MIMO plants using static state feedback controllers. This new approach not only takes into account the effect of initial conditions but also, in the case of SISO systems, allows for restrictions on the order of the controller. Moreover, the problem of optimal controller design can be formulated as a generalized eigenvalue problem and, hence, easily solvable by currently available software.

However, these early results have limited applicability to MIMO systems. More precisely, early work on MIMO systems concentrated on the problem of minimizing the effect of disturbances on the states of the system. This leads to a performance measure that is realization dependent, i.e., a system might exhibit good disturbance rejection for a given state-space realization but not be able to mitigate the influence of the perturbation on the states if a different realization is chosen (Polyak and Halpern, 1999).

In this paper, we take a different approach to this problem. We extend previous definitions of *superstability*

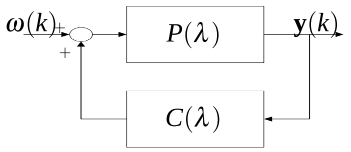

Fig. 1. Closed-loop System

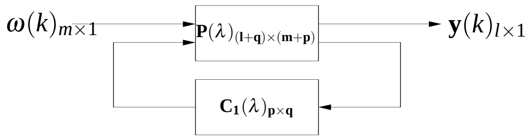

Fig. 2. Closed-loop System in General Case

and *equalized performance* for SISO systems in transfer function form to MIMO systems and, by relying on coprime factorizations over the field of polynomials, develop a procedure for fixed-order controller design for persistent disturbance rejection.

The paper is organized as follows: In Section 2, we provide the notation that is used throughout the paper. The definition of Input/Output superstability and equalized performance for Linear time invariant (LTI) MIMO systems is introduced in Section 3 where, also, some important properties of superstable systems are established. In Section 4, we address the problem of fixed-order controller design for LTI systems. Finally, in Section 5, we provide some concluding remarks and delineate some directions for further research.

2. NOTATION AND PRELIMINARY RESULTS

2.1 Notation

In the sequel, λ denotes the delay by one period, i.e. $\lambda e(k) = e(k-1)$. Also, $\|\cdot\|_1$ denotes the 1-norm of a matrix or of the coefficients of a polynomial. In other words, given a matrix $A = ((a_{ij}))_{n\times m} \in \mathbf{R}^{n\times m}$, $\|A\|_1 = \max_{1\le i\le n} \sum_{j=1}^{m} |a_{ij}|$; Given a polynomial $b(\lambda) = b_1\lambda + \cdots + b_m\lambda^m$, $\|\mathbf{b}\|_1 = \sum_{i=1}^{m} |b_i|$. The notation $\lceil x \rceil$ denotes the smallest integer larger than or equal to x. Finally, for a real number c, let $(c)_+ = \max\{0, c\}$.

2.2 Polynomial Matrix Description (PMD)

Central to the results of this paper is the coprime factorization of a plant over the field of polynomials. More precisely, given a transfer function matrix P, consider its polynomial matrix description (PMD) $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$, where the pair of polynomial matrices (N, D) is said to be a right coprime factorization of P and the pair of polynomial matrices $(\tilde{N}, \tilde{D})$ is a left coprime factorization of P. These factorizations satisfy the so-called Bezout's identity and doubly coprime factorization equality; i.e., there exist polynomial matrices X, Y, $\tilde{X}$ and $\tilde{Y}$ of appropriate dimensions, satisfying

$$XD + YN = I;\ \tilde{D}\tilde{X} + \tilde{N}\tilde{Y} = I;\ -Y\tilde{X} + X\tilde{Y} = 0. \quad (1)$$

See (Antsaklis and Michel, 1997) for a in depth exposition on PMDs.

2.3 Parametrization of All Stabilizing Controllers

Consider the setup depicted in Figure 1. It can be proven that all closed-loop transfer functions can be parameterized by the so-called Youla parameter, which we denote by Q, e.g., (Antsaklis and Michel, 1997). More precisely, any achievable stable closed-loop transfer function is of the form

$$H = (I - PC)^{-1}P = ND_Q^{-1}(D_QX - N_Q\tilde{N}) \quad (2)$$

where $Q = N_QD_Q^{-1} = \tilde{D}_Q^{-1}\tilde{N}_Q$ is any stable transfer function matrix. The corresponding stabilizing controller $C = \tilde{D}_C^{-1}\tilde{N}_C = N_CD_C^{-1}$ is given by

$$\begin{bmatrix}\tilde{D}_C & -\tilde{N}_C\end{bmatrix} = \begin{bmatrix}\tilde{D}_Q & \tilde{N}_Q\end{bmatrix} U;\ \begin{bmatrix}N_C \\ D_C\end{bmatrix} = U^{-1}\begin{bmatrix}-N_Q \\ D_Q\end{bmatrix}. \quad (3)$$

where N_C, D_C, N_Q, D_Q $\tilde{N}_C$, $\tilde{D}_C$, $\tilde{N}_Q$, $\tilde{D}_Q$ are polynomial matrices of appropriate dimensions and U is the unimodular polynomial matrix

$$U = \begin{bmatrix} X & Y \\ -\tilde{N} & \tilde{D}\end{bmatrix};\quad U^{-1} = \begin{bmatrix} D & -\tilde{Y} \\ N & \tilde{X}\end{bmatrix}.$$

where X, Y, $\tilde{X}$ and $\tilde{Y}$ satisfy the equations (1).

Remark 1. In order to use Q to parameterize all stable controllers of the plant, we need the open-loop system to be both observable and controllable. This will be assumed in the remainder of this paper.

The results above assume that the controller has access to all inputs and all outputs of the plant. However, in common cases, only part of the inputs and the outputs are available to the controller. The more general setup, which is depicted in Figure 2, can be addressed by requiring that the controller transfer function is of the form

$$C(\lambda) = \begin{bmatrix} 0_{m\times l} & 0_{m\times q} \\ 0_{p\times l} & C_1(\lambda)_{p\times q}\end{bmatrix}. \quad (4)$$

As it can be seen in Section 4, this will not significantly increase the complexity of the controller design method since it requires only the introduction of an additional set of linear equality constraints.

3. SUPERSTABILITY AND EQUALIZED PERFORMANCE

As mentioned in Section 1, currently available definitions of *superstability* and *equalized performance* for MIMO systems are realization dependent and, therefore, cannot be used as an intrinsic property of a system. To overcome this limitation, we now provide an alternative definition of *superstability* for MIMO systems. Consider a system described by

$$\begin{bmatrix} y_1(k) \\ \vdots \\ y_l(k) \end{bmatrix} = (1+\sum_{j=1}^{n} a_j)^{-1} \begin{bmatrix} \sum_{j=0}^{p_{11}} b_{11,j}\lambda^j & \dots & \sum_{j=0}^{p_{1m}} b_{1m,j}\lambda^j \\ \vdots & \ddots & \vdots \\ \sum_{j=0}^{p_{l1}} b_{l1,j}\lambda^j & \dots & \sum_{j=0}^{p_{lm}} b_{lm,j}\lambda^j \end{bmatrix} \begin{bmatrix} \omega_1(k) \\ \vdots \\ \omega_m(k) \end{bmatrix} \tag{5}$$

or, equivalently,

$$\mathbf{y}(k) = (1+a(\lambda))^{-1} \begin{bmatrix} b_{11}(\lambda) & \dots & b_{1m}(\lambda) \\ \vdots & \ddots & \vdots \\ b_{l1}(\lambda) & \dots & b_{lm}(\lambda) \end{bmatrix} \boldsymbol{\omega}(k) \tag{6}$$

where m and l are the number of inputs and outputs respectively and n, p_{ij} are integer numbers. Moreover, for $i = 1,2,\dots,l$ and all k, we define the vector containing the last n samples of y_i

$$\mathbf{Y}_i(k) \doteq \left[y_i(k)\ y_i(k-1)\ \cdots\ y_i(k-n+1) \right].$$

Definition 1. (Input/Output (I/O) Superstability). The system (5) is said to be *I/O superstable* (or superstable for short) if $q = \|\mathbf{a}\|_1 < 1$.

When $m = l = 1$, the definition above reduces to the definition given in (Blanchini and Sznaier, 1997) and (Polyak and Halpern, 1999) for the SISO case. A superstable system has many distinguished features. Some of them are given in the remainder of this section. The results provided below are natural extensions to the MIMO case of the ones presented in (Blanchini and Sznaier, 1997) and (Polyak and Shcherbakov, 2002*a*).

Lemma 1. Consider a superstable system H of form (5) and assume that no input is applied, i.e.,

$$\boldsymbol{\omega}(k) = \left[\omega_1(k)\ \cdots\ \omega_m(k) \right]' = 0,$$

then, for all time instants $k > 0$, all $i = 1,\cdots,l$ and initial conditions $\mathbf{Y}_i(-1)$,

$$|y_i(k)| \le q^{\lceil (k+1)/n \rceil} \|\mathbf{Y}_i(-1)\|_\infty .$$

PROOF. Since $\boldsymbol{\omega}(k) = 0$, then, for all $i = 1,\cdots,l$,

$$\begin{aligned} |y_i(k)| &= |a(\lambda)y_i(k)| \le \|\mathbf{a}\|_1 \|\mathbf{Y}_i(k-1)\|_\infty \\ &= q\|\mathbf{Y}_i(k-1)\|_\infty \le q^2 \|\mathbf{Y}_i(k-1-n)\|_\infty \\ &\le \cdots \le q^{\lceil (k+1)/n \rceil} \|\mathbf{Y}_i(-1)\|_\infty . \end{aligned}$$

A superstable system also has an important characteristic: the so-called *equalized performance* whose SISO version was first introduced in (Blanchini and Sznaier, 1997). A generalization of this property for MIMO systems, which we refer to as *I/O equalized performance*, is given below.

Definition 2. (I/O Equalized Performance). A super–stable system H with transfer function matrix of the form (5) is said to have *I/O equalized performance* (or *equalized performance* for short) less than μ if given any initial condition $\|\mathbf{Y}_i(-1)\|_\infty \le \mu$, $i = 1,2,\dots,l$, and any bounded input $\|\boldsymbol{\omega}(j)\|_\infty \le 1$, $j = -\max\{p_{is}\}, -\max\{p_{is}\}+1,\dots$, $i = 1,2,\dots,l$ and $s = 1,\dots,m$ then for all $k \ge 0$ and all $i = 1,2,\dots,l$

$$|y_i(k)| \le \mu.$$

There is an immediate way to check whether the equalized performance of a system is less than or equal to a given level $\mu \ge 0$.

Theorem 2. Consider a plant H of the form (5) and, for $i = 1,2,\dots,l$, define $\mathbf{b}_i \doteq \left[\|b_{i1}\|_1\ \dots\ \|b_{im}\|_1 \right]$. and recall that $q = \|\mathbf{a}\|_1$. Let $\mu \ge 0$ be given. Then, the plant H has *equalized performance* less than μ if and only if for all $i = 1,2,\dots,l$,

$$\mu q + \|\mathbf{b}_i\|_1 \le \mu. \tag{7}$$

Therefore, the smallest equalized performance level μ^* achievable by the plant H is

$$\mu^* = \max_i \frac{\|\mathbf{b}_i\|_1}{1-q}. \tag{8}$$

PROOF. We first prove necessity. Proceeding by contradiction, assume that the equalized performance condition (7) does not hold, then there exists an i such that $\mu q + \|\mathbf{b}_i\|_1 > \mu$. Equation (5) implies that

$$|y_i(0)| = \left| a(\lambda)y_i(0) + \sum_{j=1}^{m} b_{ij}(\lambda)\,\omega_j(0) \right|.$$

Then, there exist $|y_i(k)| \le \mu$ and $\left|\omega_j(k)\right| \le 1$, with $j = 1,\dots,m$ and $k = -n,\dots,0$, such that

$$\begin{aligned} |y_i(0)| &= q\,\|\mathbf{Y}_i(-1)\|_\infty + \|\mathbf{b}_i\|_1 \|\boldsymbol{\omega}(0)\|_\infty \\ &= \mu q + \|\mathbf{b}_i\|_1 > \mu \end{aligned}$$

which contradicts the initial assumption.

We now prove sufficiency by induction. If the equalized performance condition (7) holds, then equation (5) implies that for all $i = 1,\dots,l$,

$$\begin{aligned} |y_i(0)| &= \left| a(\lambda)y_i(0) + \sum_{j=1}^{m} b_{ij}(\lambda)\,\omega_j(0) \right| \\ &\le q\,\|\mathbf{Y}_i(-1)\|_\infty + \|\mathbf{b}_i\|_1 \|\boldsymbol{\omega}(0)\|_\infty \\ &\le \mu q + \|\mathbf{b}_i\|_1 \le \mu. \end{aligned}$$

Now, to complete the proof, assume that $|y_i(k-1)| \le \mu, \quad k \ge 0$, then

$$\begin{aligned} |y_i(k)| &= \left| a(\lambda)y_i(k) + \sum_{j=1}^{m} b_{ij}(\lambda)\,\omega_j(k) \right| \\ &\le \|\mathbf{a}\|_1 \|\mathbf{Y}_i(k-1)\|_\infty + \|\mathbf{b}_i\|_1 \|\boldsymbol{\omega}(k)\|_\infty \\ &\le \mu q + \|\mathbf{b}_i\|_1 \le \mu. \end{aligned}$$

3.1 *Equalized Performance and Plant Order*

Since the concept of equalized performance accounts for the effect of initial conditions, one should use the "true" transfer function of the plant to compute it; i.e., one should use the transfer function which corresponds to the difference equation that describes the plant, without performing any pole/zero cancellations. As an example, consider a plant $H_{min}(\lambda) = (1-0.1\lambda)/(1-0.8\lambda)$. This plant has equalized performance $\mu^* = 5.5$. However, if one considers its non-minimal realization $H(\lambda) = (1-0.01\lambda^2)/(1-0.7\lambda-0.08\lambda^2)$ then the smallest equalized level is $\mu^* = 4.5909$. One can also obtain worse performance with another non-minimal plant $H(\lambda) = (1+0.8\lambda-0.09\lambda^2)/(1+0.1\lambda-0.72\lambda^2)$ which has equalized performance $\mu^* = 10.5$. Hence, the results presented in this paper should be applied to the original transfer function of the plant without performing any pole/zero cancellation.

3.2 *Performance Under Arbitrary Initial Conditions*

The analysis above assumes that the initial conditions are bounded by μ, i.e. for $i = 1\ldots l$, $\|\mathbf{Y}_i(-1)\|_\infty \le \mu$. However, if the condition above is not satisfied, one can still provide bounds on the output of the system. This result is provided below.

Theorem 3. Consider a superstable system of the form (5), with initial conditions $\|\mathbf{Y}_i(-1)\|_\infty \in \mathbf{R}^n$, $i = 1\ldots l$, and bounded disturbance $|\omega_i(k)| \le 1$, $k \ge 0$. Then,

$$|y_i(k)| \le \mu_i + q^{\lceil (k+1)/n \rceil}(\|\mathbf{Y}_i(-1)\|_\infty - \mu_i)_+ \qquad (9)$$

for all $i = 1,\ldots,l$, where $\mu_i = \frac{\|\mathbf{b}_i\|_1}{1-q}$.

PROOF. We now proceed by induction. First, consider the case $k = 0$,

$$\begin{aligned} |y_i(0)| &= \left| a(\lambda)y_i(0) + \sum_{j=1}^{m} b_{ij}(\lambda)\,\omega_j(0) \right| \\ &\le \|\mathbf{a}\|_1 \|\mathbf{Y}_i(-1)\|_\infty + \|\mathbf{b}_i\|_1 \|\boldsymbol{\omega}(0)\|_\infty \\ &\le \|\mathbf{Y}_i(-1)\|_\infty q + \|\mathbf{b}_i\|_1 \\ &= \mu_i + q(\|\mathbf{Y}_i(-1)\|_\infty - \mu_i) \\ &\le \mu_i + q(\|\mathbf{Y}_i(-1)\|_\infty - \mu_i)_+ \end{aligned}$$

Now, to complete the proof, assume that equation (9) is valid for $0,\ldots,k-1$. Then,

$$\begin{aligned} |y_i(k)| &\le \|\mathbf{a}\|_1 \|\mathbf{Y}_i(k-1)\|_\infty + \|\mathbf{b}_i\|_1 \|\boldsymbol{\omega}(k)\|_\infty \\ &\le q\|\mathbf{Y}_i(k-1)\|_\infty + \|\mathbf{b}_i\|_1 \\ &\le q(\mu_i + q^{\lceil (k-n+1)/n \rceil}(\|\mathbf{Y}_i(-1)\|_\infty - \mu_i)_+) + \|\mathbf{b}_i\|_1 \\ &\le \mu_i + q^{\lceil (k+1)/n \rceil}(\|\mathbf{Y}_i(-1)\|_\infty - \mu_i)_+ \end{aligned}$$

4. FIXED ORDER CONTROLLER DESIGN

It turns out that optimizing the performance of closed-loop plants of the form (5) can be recasted as a generalized eigenvalue problem which can be easily solved by available numerical tools. To see this, represent the Youla parameter Q in the form $Q(\lambda) = \frac{1}{d_Q(\lambda)} N_Q(\lambda)$, where $d_Q(\lambda)$ is a polynomial. Now, if one uses (2) to compute the closed-loop transfer function matrix and puts it in the form (5), it can be seen that the denominator of the closed-loop plant is $d_Q(\lambda)$ and the numerator of the closed loop plant is a linear function of $d_Q(\lambda)$ and $N_Q(\lambda)$. Moreover, (3) indicates that the factorization of the controller is a linear function of d_Q and N_Q. Hence, constraints in the order of the controller can be represented as linear constraints on the coefficients of d_Q and N_Q. Therefore, given a $\mu \ge 0$, the problem of determining a fixed order controller that achieves an equalized performance level μ can be formulated as a linear program involving the coefficients of $d_Q(\lambda)$ and $N_Q(\lambda)$. We now elaborate on this.

Consider an open-loop plant P with $m+p$ inputs and $l+q$ outputs as depicted in Figure 2, with left and right coprime factorizations $P = ND^{-1} = \tilde{D}^{-1}\tilde{N}$. Let X and Y be polynomial matrices satisfying Diophantine equations (1). Now, as above, the Youla parameter Q is represented in the following form

$$Q = \frac{1}{d_Q} N_Q \qquad (10)$$

where d_Q is a n-th order polynomial and N_Q is a polynomial matrix of appropriate dimensions. Using (2), one can see that the closed-loop system is given by

$$\begin{aligned} H(\lambda,Q) &= N\,{d_Q}^{-1}(d_Q X - N_Q \tilde{N}) \\ &= \frac{1}{\sum_{i=0}^{n} e_i(Q)\lambda^i} \begin{bmatrix} \sum_{j=0}^{n} f_{11,j}(Q)\lambda^j & \cdots & \sum_{j=0}^{n} f_{1\,m+p,j}(Q)\lambda^j \\ \vdots & \ddots & \vdots \\ \sum_{j=0}^{n} f_{l+q\,1,j}(Q)\lambda^j & \cdots & \sum_{j=0}^{n} f_{l+q\,m+q,j}(Q)\lambda^j \end{bmatrix} \end{aligned} \qquad (11)$$

where $\sum_{i=0}^{n} e_i(Q)\lambda^i = d_Q(\lambda)$ and $f_{ik,j}(Q)$ are affine functions of the coefficients of d_Q and N_Q. Moreover, no pole zero cancellation is performed in system (11). In other words, the closed-loop plant considered has an order equal to the summation of the order of the open-loop plant and the order of the controller. Furthermore, the left factorization and the right factorization of the controller are given by

$$\left[\tilde{D}_Q,\ \tilde{N}_Q\right] = \left[\tilde{D}_C,\ -\tilde{N}_C\right] U^{-1}, \quad \begin{bmatrix} -N_Q \\ D_Q \end{bmatrix} = U \begin{bmatrix} N_C \\ D_C \end{bmatrix}$$

where U is a unimodular matrix that depends only on the open-loop plant. In order to make Q in form (10), $\tilde{D}_Q$ and D_Q should be diagonal polynomial matrices $diag(d_Q)$ with all diagonal elements equal to polynomial d_Q, and, at the same time, $\tilde{N}_Q = N_Q$.

Remark 2. In this setup, we only consider the "full-order" closed-loop plant, i.e., the original plant without any pole/zero cancellation. Therefore, the zeros of the denominator polynomial dQ are equivalent to all poles of the closed-loop system.

Hence, the problem of designing a fixed order controller that achieves equalized performance μ for the closed-loop plant (11) over the first m inputs and l

outputs can be recasted a linear program. More specifically, we need to put three kinds of linear constrains: The first constraint set guarantees that the required equalized performance level is achieved. This is done by first ensuring superstability of closed-loop system (11), i.e., $e_0 - \sum_{i=1}^{n} |e_i| > 0$, where $e_0 > 0$. Moreover, equalized performance less than or equal to μ is achieved if $\max_{i=1,\ldots,l} \left(\sum_{m'=1}^{m} \sum_{n'=0}^{n} |f_{im',n'}| \right) - \mu(e_0 - \sum_{i=1}^{n} |e_i|) \leq 0$.

The second set of linear constraints addresses the constraint on the controller order. By controller order we mean the highest power of the controller matrices N_C D_C, $\tilde{N}_C$ and $\tilde{D}_C$. Given maximum controller order n_C, the order of the Youla parameter Q should be at least the order of U plus n_C. Since this might result in controllers of order larger than n_C, additional constraints should be used to ensure that coefficients of the controller corresponding to powers higher then n_C are zero. Again, this corresponds to a set of linear constraints on the coefficients of d_Q and N_Q.

Finally the controller structure depicted in Figure 2 is achieved by restricting the structure of the right and left factorizations of the controller C. More precisely, as D_C and $\tilde{D}_C$ are invertible, having a controller of the form

$$C(\lambda) = \begin{bmatrix} 0_{m\times l} & 0_{m\times q} \\ 0_{p\times l} & C_1(\lambda)_{p\times q} \end{bmatrix}.$$

is equivalent to having numerators of the factorizations of the form

$$N_C = \begin{bmatrix} 0_{(m+p)\times l} & N'_C \end{bmatrix}; \quad \tilde{N}_C = \begin{bmatrix} 0_{m\times(q+l)} \\ \tilde{N}'_C \end{bmatrix}.$$

Hence, one obtains the following result:

Theorem 4. Consider the setup depicted in Figure 2, there exists a controller of order less than or equal to n_C achieving closed-loop equalized performance less than or equal to μ if and only if there exists Q of the form (10) satisfying the linear constraints

$$\forall n' = 1,\ldots,n, \quad l' = 1,\ldots,l, \quad m' = 1,\ldots,m,$$
$$s = 0,1,\ldots, \quad k = n_C+1, n_C+2,\ldots$$

$$e_0 > 0; \quad e_0 - \sum_{i=1}^{n} \alpha_i > 0. \tag{12}$$

$$\begin{cases} \alpha_{n'} \geq 0; & \beta_{l'm',n'} \geq 0; \\ \alpha_{n'} - e_{n'}(Q) \geq 0; & \alpha_{n'} + e_{n'}(Q) \geq 0; \\ \beta_{l'm',n'} - f_{l'm',n'}(Q) \geq 0; & \beta_{l'm',n'} + f_{l'm',n'}(Q) \geq 0; \\ \mu e_0 - \mu \sum_{n'=1}^{n} \alpha_{n'} - \sum_{m'=1}^{m} \sum_{n'=0}^{n} \beta_{l'm',n'} \geq 0. \end{cases} \tag{13}$$

$$\begin{cases} coef[N_C(1:m, 1:(l+q))]_s = 0; \\ coef[\tilde{N}_C(1:(m+p), 1:l)]_s = 0; \\ coef[N_C((m+1):(m+p), 1:(l+q))]_k = 0; \\ coef[\tilde{N}_C(1:(m+p), (l+1):(l+q))]_k = 0; \\ coef[D_C]_k = 0; \\ coef[\tilde{D}_C]_k = 0. \end{cases} \tag{14}$$

where $C = N_C D_C^{-1} = \tilde{D}_C^{-1} \tilde{N}_C$ are defined in equation (3), $coef(W)_k$ denotes the coefficient of k_{th} order delay in function W, and $F(i_1 : i_2, j_1 : j_2)$ denotes the elements from row i_1 to row i_2 from column j_1 to j_2 in matrix F.

PROOF. The closed-loop system (11) is superstable with the equalized performance less than or equal to μ if and only if

$$\xi = e_0 - \sum_{n'=1}^{n} |e_{n'}| > 0 \tag{15}$$

$$\mu\, \xi + \|F\|_\infty \leq 0 \tag{16}$$

where

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_l \end{bmatrix}, \quad F_{l'} = \sum_{m'=1}^{m} \sum_{n'=0}^{n} |f_{im',n'}|, \quad l' = 1,\ldots,l.$$

Now, note that the two inequalities (15) and (16) are satisfied if and only if there exist α_i and $\beta_{pg,j}$ satisfying the inequalities

$$\begin{aligned} & e_0 - \sum_{n'=1}^{n} \alpha_{n'} > 0; \\ & \alpha_{n'} \geq 0, \quad n' = 1,\ldots,n,; \\ & \beta_{l'm',n'} \geq 0, \quad l' = 1,\ldots,l,\ m' = 1,\ldots,m,\ n' = 0,\ldots,n; \\ & -\alpha_{n'} \leq e_{n'} \leq \alpha_{n'}, \quad n' = 1,\ldots,n; \\ & -\beta_{l'm',n'} \leq f_{l'm',n'} \leq \beta_{l'm',n'}, \quad l' = 1,\ldots,l,\ m' = 1,\ldots,m,\ n' = 0,\ldots,n; \\ & \mu e_0 - \mu \sum_{n'=1}^{n} \alpha_{n'} - \sum_{m'=1}^{m} \sum_{n'=0}^{n} \beta_{l'm',n'} \geq 0, \quad l' = 1,\ldots,l. \end{aligned}$$

In other words, the system achieves equalized performance μ if and only if the inequalities (12) and (13) are satisfied. In addition, as D_C and $\tilde{D}_C$ are invertible, the following equivalent relation exists,

$$C(\lambda) = \begin{bmatrix} 0_{m\times l} & 0_{m\times q} \\ 0_{p\times l} & C_1(\lambda)_{p\times q} \end{bmatrix}$$
$$\Updownarrow$$
$$N_C = \begin{bmatrix} 0_{m\times(l+q)} \\ N'_{p\times(l+q)} \end{bmatrix} \text{ and } \tilde{N}_C = \begin{bmatrix} 0_{(m+p)\times l} & \tilde{N}'_{(m+p)\times q} \end{bmatrix},$$

which is given by $coef[\tilde{N}_C(1:(m+p), 1:l)]_s = 0$ and $coef[N_C(1:m, 1:(l+q))]_s = 0$, where $s = 0,1,\ldots$. And $coef(N_C$ and $D_C)_k = 0$ for all integer $k > n_C$ puts extra constrains on the order of the controller.

Given the results above, the problem of finding a controller of order less than or equal n_C that achieves optimal closed-loop equalized performance can be formulated as the generalized eigenvalue problem

$$\begin{aligned} & \min \quad \mu \\ & \text{subject to } (12), (13) \text{ and } (14) \end{aligned}$$

which can be solved using available numerical tools.

4.1 *Example*

Consider the open-loop plant as presented in Figure 2 with $l = m = 2$ and $p = q = 1$

$$P = \begin{bmatrix} 0 & \frac{-1}{\lambda+1.1} & \frac{-1}{\lambda+1.1} \\ \frac{1}{\lambda-0.18} & \frac{1}{\lambda+0.19} & \frac{0.37}{\lambda^2+0.01\lambda-0.0342} \\ \frac{1}{\lambda-0.18} & \frac{-0.91}{\lambda^2+1.29\lambda+0.209} & \frac{4\lambda^3+3.84\lambda^2+0.331\lambda-0.04156}{\lambda^4+1.28\lambda^3+0.1655\lambda^2-0.04156\lambda-0.006395} \end{bmatrix} \tag{17}$$

By considering controller with order up until four and Youla parameters up until eight, which is the summation of the order of U and the order of the controller, the procedure described in the previous sections was used and the following results were obtained: The Youla parameter is

$$Q = \frac{\begin{bmatrix} \begin{array}{c} -0.082\lambda^5 - 0.427\lambda^4 \\ -0.011\lambda^3 + 0.033\lambda^2 \\ -1.082\lambda - 17.27 \end{array} & \begin{array}{c} -0.002\lambda^5 + 0.020\lambda^4 \\ -0.008\lambda^3 + 0.212\lambda^2 \\ +4.392\lambda + 3.621 \end{array} & \begin{array}{c} 0.0004\lambda^5 - 0.007\lambda^4 \\ -0.036\lambda^3 - 0.219\lambda^2 \\ -0.93\lambda - 0.676 \end{array} \\ \begin{array}{c} 0.101\lambda^5 + 0.535\lambda^4 \\ +0.011\lambda^3 - 0.1\lambda^2 \\ -1.306\lambda + 5.832 \end{array} & \begin{array}{c} 0.005\lambda^5 - 0.005\lambda^4 \\ +0.023\lambda^3 + 0.281\lambda^2 \\ -1.825\lambda - 1.802 \end{array} & \begin{array}{c} 0.001\lambda^5 - 0.0005\lambda^4 \\ -0.0016\lambda^3 + 0.025\lambda^2 \\ +0.394\lambda + 0.335 \end{array} \\ \begin{array}{c} 0.328\lambda^4 + 1.471\lambda^3 \\ -1.309\lambda^2 + 1.053\lambda \\ -2.333 \end{array} & \begin{array}{c} 0.015\lambda^4 - 0.004\lambda^3 \\ +0.041\lambda^2 + 0.267\lambda \\ -2.46 \end{array} & \begin{array}{c} -0.004\lambda^4 + 0.0004\lambda^3 \\ -0.003\lambda^2 + 0.036\lambda \\ +0.449 \end{array} \end{bmatrix}}{-1.125\lambda^5 - 6.329\lambda^4 - 0.031\lambda^3 - 0.233\lambda^2 + 0.166\lambda + 49.3}$$

and the corresponding controller is $C = N_C D_C^{-1} = \tilde{D}_C^{-1} \tilde{N}_C$ where

$$N_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \begin{array}{c} 0.071\lambda^3 + 1.99\lambda^2 \\ -2.65\lambda - 58.88 \end{array} & \begin{array}{c} -0.014\lambda^4 - 0.414\lambda^3 + 0.169\lambda^2 \\ +12.32\lambda + 10.76 \end{array} & \begin{array}{c} -0.016\lambda^3 - 0.540\lambda^2 \\ -2.60\lambda - 1.93 \end{array} \end{bmatrix},$$

$$D_c = \begin{bmatrix} \begin{array}{c} -1.13\lambda^4 - 5.37\lambda^3 + \\ 3.04\lambda^2 - 0.584\lambda + 54.22 \end{array} & \begin{array}{c} 0.013\lambda^4 + 0.364\lambda^3 \\ -0.485\lambda^2 - 10.76\lambda \end{array} & \begin{array}{c} 0.014\lambda^3 + 0.478\lambda^2 \\ +1.931\lambda \end{array} \\ -0.764\lambda^3 - 21.79\lambda^2 & \begin{array}{c} 0.154\lambda^4 + 4.52\lambda^3 \\ +3.982\lambda^2 \end{array} & \begin{array}{c} -0.954\lambda^3 - 0.764\lambda^2 \\ +0.011\lambda - 1.686 \end{array} \\ \begin{array}{c} 6.053\lambda^3 - 0.202\lambda^2 \\ -177.5\lambda - 45.84 \end{array} & \begin{array}{c} -1.22\lambda^4 - 1.07\lambda^3 + \\ 35.72\lambda^2 + 41.66\lambda + 8.38 \end{array} & \begin{array}{c} -1.356\lambda^3 - 7.97\lambda^2 \\ -7.779\lambda - 1.504 \end{array} \end{bmatrix},$$

$$\tilde{N}_c = \begin{bmatrix} 0 & 0 & -0.162\lambda^3 - 5.633\lambda^2 - 29.36\lambda - 22.16 \\ 0 & 0 & -0.0142\lambda^4 - 0.414\lambda^3 + 0.169\lambda^2 + 12.32\lambda + 10.76 \\ 0 & 0 & -0.0182\lambda^3 - 0.506\lambda^2 + 0.887\lambda + 13.81 \end{bmatrix},$$

$$\tilde{D}_c = \begin{bmatrix} \begin{array}{c} -1.125\lambda^4 - 11.68\lambda^3 - \\ 33.6\lambda^2 - 23.79\lambda - 8.873 \end{array} & 0.707\lambda^3 + 20.16\lambda^2 & \begin{array}{c} -13.92\lambda^3 - 89.53\lambda^2 \\ -88.87\lambda - 17.25 \end{array} \\ \begin{array}{c} 0.079\lambda^4 + 2.736\lambda^3 \\ +14.26\lambda^2 + 10.76\lambda \end{array} & \begin{array}{c} 0.062\lambda^4 + 1.42\lambda^3 \\ -9.794\lambda^2 \end{array} & \begin{array}{c} -1.217\lambda^4 - 1.066\lambda^3 + \\ 35.72\lambda^2 + 41.66\lambda + 8.38 \end{array} \\ \begin{array}{c} 0.101\lambda^3 + 3.374\lambda^2 \\ +13.81\lambda \end{array} & \begin{array}{c} -1.046\lambda^3 + 2.327\lambda^2 \\ -13.26\lambda + 10.3 \end{array} & \begin{array}{c} -1.563\lambda^3 + 0.806\lambda^2 \\ +41.84\lambda + 10.76 \end{array} \end{bmatrix},$$

and the respective closed-loop equalized performance is $\mu^* \approx 5.1$. We tested the closed-loop system performance with disturbance inputs uniformly distributed over $[-1, 1]$. The sampling time is 0.01 seconds. The results from $t = 0$ to $t = 100$ s are shown in Figure 3 where one can see that the outputs are indeed bounded below μ^* for these arbitrary persistent disturbances.

5. CONCLUSION

In this paper, we addressed the problem of designing fixed-order controllers for discrete-time systems. We introduced the notion of I/O superstability which extends previous definitions of superstability to the MIMO case. Based on this definition, we provided an algorithm for optimal fixed-order controller design which results in a closed-loop system that minimizes the effects of bounded persistent perturbations. It is shown that the problem of optimal controller design reduces to a generalized eigenvalue program.

Since the notion of equalized performance relies only on the coefficients of the difference equations that describe the system, it is easily extended to the time varying and nonlinear cases. Therefore, effort will be put in the development of algorithms for these cases.

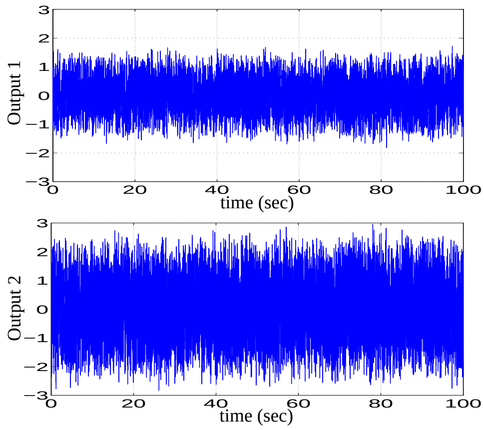

Fig. 3. Simulation Results

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