

**A SIMPLE TUBE CONTROLLER FOR
EFFICIENT ROBUST MODEL PREDICTIVE
CONTROL OF CONSTRAINED LINEAR
DISCRETE TIME SYSTEMS SUBJECT TO
BOUNDED DISTURBANCES ***

S. V. Raković^{*,1} D. Q. Mayne*

** Imperial College London, London SW7 2BT, United
Kingdom, Tel: +44-(0)20-7594-6295/87/81. Fax:
+44-(0)20-7594-6282, E-mail: sasa.rakovic@imperial.ac.uk
and d.mayne@imperial.ac.uk*

Abstract: We present a simple tube controller for efficient robust model predictive control of constrained linear, discrete-time systems in the presence of bounded disturbances. The proposed control policy ensures that controlled trajectories are confined to a given tube despite uncertainty. The robust optimal control problem that is solved on-line is a standard quadratic programming problem of marginally increased complexity compared with that required for model predictive control in the deterministic case. We show how to optimize the tube cross section, how to construct an adequate tube terminal set and we establish robust exponential stability of a suitable robust control invariant set (the ‘origin’ for uncertain system) with enlarged domain of attraction. *Copyright*© 2005 IFAC.

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1. INTRODUCTION

Research on robust model predictive control has recognized the advantages of *feedback control* when uncertainty is present (Mayne *et al.*, 2000; Findeisen *et al.*, 2003). The decision variable in feedback model predictive control is a control policy π that is a sequence $\{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ of control laws. Determination of a feedback control policy is usually prohibitively difficult and various simplifying approximations have been proposed in the literature (Kouvaritakis *et al.*, 2000; Chisci *et al.*, 2001; Mayne and Langson, 2001; Löfberg, 2003; van Hessem and Bosgra, 2003; Kerrigan and Alamo, 2004; Smith, 2004). Both *open-*

loop and *feedback* model predictive controller generate a tube of trajectories when uncertainty is present. Feedback model predictive control reduces the spread of predicted trajectories resulting from uncertainty. In this note we present a simple, computationally efficient, controller that forces controlled trajectories to remain within a suitably designed tube (*sequence of sets of states*), ensuring robust constraint satisfaction and appropriate stability properties. Tubes have been extensively studied, see for instance (Quincampoix and Veliov, 2002; Calafiore and El Ghaoui, 2004; Kurzhanski, 2004) and references therein. Application of tubes to robust model predictive control has been reported in (Langson *et al.*, 2004; Mayne *et al.*, 2005; Raković and Mayne, 2004).

Our main focus is the choice of *tube cross-section*, *tube terminal set* and cost function in order to

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¹ Corresponding Author.

simplify the resultant robust optimal control problem. Section 2 is concerned with the preliminaries. Section 3 discusses an appropriate choice for tube cross-section and tube terminal set. Section 4 presents a simple formulation of the resultant robust optimal control problem and discusses stability properties. Section 5 presents an illustrative numerical example. Finally, Section 6 presents conclusions.

NOTATION: Let $\mathbb{N} \triangleq \{0, 1, 2, \dots\}$, $\mathbb{N}_+ \triangleq \{1, 2, \dots\}$ and $\mathbb{N}_q \triangleq \{0, 1, \dots, q\}$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is the closed and bounded polyhedron. Given two sets \mathcal{U} and \mathcal{V} , such that $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, the Minkowski (vector) sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u + v \mid u \in \mathcal{U}, v \in \mathcal{V}\}$, the Pontryagin (geometric) set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{x \mid x \oplus \mathcal{V} \subseteq \mathcal{U}\}$. Given the sequence of sets $\{\mathcal{U}_i \subset \mathbb{R}^n\}_{i=a}^b$, we denote $\bigoplus_{i=a}^b \mathcal{U}_i \triangleq \mathcal{U}_a \oplus \dots \oplus \mathcal{U}_b$. We denote $d(z, X) \triangleq \inf\{|z - x| \mid x \in X\}$.

2. PRELIMINARIES

We consider the following discrete-time linear time-invariant (DLTI) system:

$$x^+ = Ax + Bu + w, \quad (2.1)$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown disturbance and $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. The disturbance w is persistent, but contained in a convex and compact set $W \subset \mathbb{R}^n$ that contains the origin. We make the standing assumption that the couple (A, B) is controllable. We also define the corresponding nominal system:

$$z^+ = Az + Bv, \quad (2.2)$$

where $z \in \mathbb{R}^n$ is the current state, $v \in \mathbb{R}^m$ is the current control action z^+ is the successor state of the nominal system. The system (2.1) is subject to the following set of hard state and control constraints:

$$(x, u) \in \mathbb{X} \times \mathbb{U} \quad (2.3)$$

where $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are polyhedral and polytopic sets respectively and both contain the origin as an interior point. Let $\mathcal{W} \triangleq W^{\mathbb{N}}$ denote the class of admissible disturbance sequences $\mathbf{w} \triangleq \{w(i) \mid i \in \mathbb{N}_{N-1}\}$. Let $\phi(i; x, \pi, \mathbf{w})$ denotes the solution at time i of (2.1) when the control policy is $\pi \triangleq \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$, where $\mu_i(\cdot)$ is the control law (mapping state to control) at time i , the disturbance sequence is \mathbf{w} and the initial state is x at time 0. If the initial state of nominal model is z at time 0 then we denote by $\bar{\phi}(k; z, v)$ the solution to (2.2) at time instant k , given the control sequence $\mathbf{v} \triangleq \{v_0, v_1 \dots v_{N-1}\}$.

Robust model predictive control is defined, as usual, by specifying a finite-horizon robust optimal control problem that is solved on-line. In this note, the robust optimal control problem is the determination of a *simple tube*, defined as a sequence $\mathbf{X} \triangleq \{X_0, X_1, \dots, X_N\}$ of *sets of states*, and an associated control policy π that minimize an appropriately chosen cost function and satisfy the following set of constraints, for a given initial condition $x \in \mathbb{X}$:

$$x \in X_0, X_i \subseteq \mathbb{X}, X_N \subseteq X_f \subseteq \mathbb{X}, \quad (2.4)$$

$$\mu_i(y) \in \mathbb{U}, \forall y \in X_i, \quad (2.5)$$

$$Ay + B\mu_i(y) \oplus W \subseteq X_{i+1}, \forall y \in X_i \quad (2.6)$$

for all $i \in \mathbb{N}_{N-1}$; X_f is the terminal constraint set. In order to exploit linearity and convexity of the considered problem, we recall and generalize some preliminary results.

Definition 1. A set $\Omega \subset \mathbb{R}^n$ is a *robust positively invariant* (RPI) set for system $x^+ = Ax + w$ and constraint set (\mathbb{X}, W) if $\Omega \subseteq \mathbb{X}$ and $Ax + w \in \Omega, \forall w \in W, \forall x \in \Omega$.

Definition 2. A set $\Omega \subset \mathbb{R}^n$ is a *robust control invariant* (RCI) set for system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, W)$ if $\Omega \subseteq \mathbb{X}$ and for every $x \in \Omega$ there exists a $u \in \mathbb{U}$ such that $Ax + Bu + w \in \Omega, \forall w \in W$.

The definition of a *positively invariant* (PI) set for $x^+ = Ax$ and constraint set \mathbb{X} is given by Definition 1 with $W = \{0\}$. Similarly, the definition of a *control invariant* (CI) set for $x^+ = Ax + Bu$ and constraint set (\mathbb{X}, \mathbb{U}) is given by Definition 2 with $W = \{0\}$. First we generalize Proposition 1 of (Mayne and Langson, 2001):

Proposition 1. Let Ω be a RCI set for (2.1) and constraint set $(\mathbb{X}, \mathbb{U}, W)$, and let $\nu : \Omega \rightarrow \mathbb{U}$ be a control law such that Ω is a RPI set for system $x^+ = Ax + B\nu(x) + w$ and constraint set (\mathbb{X}_ν, W) , where $\mathbb{X}_\nu \triangleq \mathbb{X} \cap \{x \mid \nu(x) \in \mathbb{U}\}$. Let also $x \in z \oplus \Omega$ and $u = v + \nu(x - z)$. Then for all $v \in \mathbb{R}^m$, $x^+ \in z^+ \oplus \Omega$ where $x^+ \triangleq Ax + Bu + w$, $w \in W$ and $z^+ \triangleq Az + Bv$.

This result allows us to exploit a simple parameterization of the tube-policy pair (\mathbf{X}, π) as follows. The state tube $\mathbf{X} = \{X_0, X_1, \dots, X_N\}$ is parametrized by $\{z_i\}$ and \mathcal{R} as follows:

$$X_i \triangleq z_i \oplus \mathcal{R}, \quad i \in \mathbb{N}_N \quad (2.7)$$

where z_i is the *tube cross-section center* at time i and \mathcal{R} is a set. The control laws $\mu_i(\cdot)$ defining the control policy $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ are parametrized by $\{z_i\}$ and $\{v_i\}$ as follows:

$$\mu_i(y) \triangleq v_i + \nu(y - z_i), \quad y \in X_i, \quad (2.8)$$

for all $i \in \mathbb{N}_{N-1}$, where v_i is the feedforward component of the control law and $\nu(y - z_i)$ is feedback component of the control law $\mu_i(\cdot)$. A suitable choice for the *tube cross-section* \mathcal{R} is any RCI set with a $\nu : \mathcal{R} \rightarrow \mathbb{U}$ such that \mathcal{R} is RPI for system $Ax + B\nu(x) + w$ and constraint set (\mathbb{X}_ν, W) , where $\mathbb{X}_\nu \triangleq \mathbb{X} \cap \{x \mid \nu(x) \in \mathbb{U}\}$. The sequence $\{z_i\}$ is *the sequence of tube centers* and is required to satisfy (2.2), subject to tighter constraints than those in (2.2), as discussed in the sequel.

3. CONSTRUCTION OF A SIMPLE TUBE

3.1 Optimized tube cross-section

To reduce conservativeness we minimize an appropriate norm of the set \mathcal{R} by exploiting a relevant result recently established in (Raković, 2005). Let $M_i \in \mathbb{R}^{m \times n}$, $i \in \mathbb{N}$ and for each $k \in \mathbb{N}$ let $\mathbf{M}_k \triangleq (M_0, M_1, \dots, M_{k-2}, M_{k-1})$. An appropriate characterization of a family of RCI sets for (2.1) for *unconstrained case*, for constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$, is given by the following sets for $k \geq n$:

$$R_k(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} D_i(\mathbf{M}_k)W \quad (3.1)$$

where the matrices $D_i(\mathbf{M}_k)$, $i \in \mathbb{N}_k$, $k \geq n$ are defined by:

$$\begin{aligned} D_0(\mathbf{M}_k) &\triangleq I, \\ D_i(\mathbf{M}_k) &\triangleq A^i + \sum_{j=0}^{i-1} A^{i-1-j} B M_j, \quad i \geq 1 \end{aligned} \quad (3.2)$$

where \mathbf{M}_k satisfies:

$$D_k(\mathbf{M}_k) = \mathbf{0} \quad (3.3)$$

Since the couple (A, B) is assumed to be controllable, such a choice exists for all $k \geq n$. Let \mathbb{M}_k denote the set of all matrices \mathbf{M}_k satisfying condition (3.3):

$$\mathbb{M}_k \triangleq \{\mathbf{M}_k \mid D_k(\mathbf{M}_k) = \mathbf{0}\} \quad (3.4)$$

Theorem 1. (Raković, 2005) Given any $\mathbf{M}_k \in \mathbb{M}_k$ and the corresponding set $R_k(\mathbf{M}_k)$ there exists a control law $\nu : R_k(\mathbf{M}_k) \rightarrow \mathbb{R}^m$ such that $Ax + B\nu(x) \oplus W \subseteq R_k(\mathbf{M}_k)$, $\forall x \in R_k(\mathbf{M}_k)$, i.e. the set $R_k(\mathbf{M}_k)$ is RCI for the system (2.1) and constraint set $(\mathbb{R}^n, \mathbb{R}^m, W)$.

The feedback control law $\nu : R_k(\mathbf{M}_k) \rightarrow \mathbb{R}^m$ in Theorem 1 is a selection from the set valued map:

$$\mathcal{U}(x) \triangleq \mathbf{M}_k \mathbf{W}(x) \quad (3.5)$$

where $\mathbf{M}_k \in \mathbb{M}_k$ and the set of *disturbance sequences* $\mathbf{W}(x)$ is defined for each $x \in R_k(\mathbf{M}_k)$ by:

$$\mathbf{W}(x) \triangleq \{\mathbf{w} \mid \mathbf{w} \in \mathbf{W}^k, D\mathbf{w} = x\}, \quad (3.6)$$

where $\mathbf{W}^k \triangleq W \times W \times \dots \times W$ and $D = [D_{k-1}(\mathbf{M}_k) \dots D_0(\mathbf{M}_k)]$. A $\nu(\cdot)$ satisfying Theorem 1 can be defined, for instance, as follows:

$$\nu(x) \triangleq \mathbf{M}_k \mathbf{w}^0(x) \quad (3.7a)$$

$$\mathbf{w}^0(x) \triangleq \arg \min_{\mathbf{w}} \{|\mathbf{w}|^2 \mid \mathbf{w} \in \mathbf{W}(x)\} \quad (3.7b)$$

The function $\mathbf{w}^0(\cdot)$ is piecewise affine, being the solution of a *parametric* quadratic programme; thus, the feedback control law $\nu : R_k(\mathbf{M}_k) \rightarrow \mathbb{R}^m$ is piecewise affine (being a linear map of a piecewise affine function). The set $R_k(\mathbf{M}_k)$ and the feedback control law $\nu(\cdot)$ are parametrized by the matrix \mathbf{M}_k . It is shown in (Raković, 2005) that a suitable \mathbf{M}_k can be obtained by solving an optimization problem. The optimization problem is:

$$\begin{aligned} \bar{\mathbb{P}}_k : (\mathbf{M}_k^0, \alpha^0, \beta^0, \delta^0) = \\ \arg \min_{\mathbf{M}_k, \alpha, \beta, \delta} \{\delta \mid (\mathbf{M}_k, \alpha, \beta, \delta) \in \bar{\Omega}\} \end{aligned} \quad (3.8)$$

where the constraint set $\bar{\Omega}$ is defined by:

$$\begin{aligned} \bar{\Omega} \triangleq \{(\mathbf{M}_k, \alpha, \beta, \delta) \mid \mathbf{M}_k \in \mathbb{M}_k, R_k(\mathbf{M}_k) \subseteq \alpha X, \\ U(\mathbf{M}_k) \subseteq \beta U, \\ (\alpha, \beta) \in [0, 1] \times [0, 1], \\ q_\alpha \alpha + q_\beta \beta \leq \delta\} \end{aligned} \quad (3.9)$$

with $R_k(\mathbf{M}_k)$ defined by (3.1), $U(\mathbf{M}_k)$ defined by:

$$U(\mathbf{M}_k) \triangleq \bigoplus_{i=0}^{k-1} M_i W \quad (3.10)$$

with weights q_α and q_β reflecting a desired contraction of state and control constraint sets. The solution \mathbf{M}_k^0 to problem $\bar{\mathbb{P}}_k$ (which exists if $\bar{\Omega} \neq \emptyset$) yields a set $R_k^0 \triangleq R_k(\mathbf{M}_k^0)$ and feedback control law $\nu^0(x) = \mathbf{M}_k^0 \mathbf{w}^0(x)$ satisfying:

$$R_k^0 \subseteq \alpha^0 X, \nu^0(x) \in U(\mathbf{M}_k) \subseteq \beta^0 U, \quad (3.11)$$

for all $x \in R_k^0$. It follows from Theorem 1 and the discussion above that the set R_k^0 , if it exists, is RPI for system $x^+ = Ax + B\nu^0(x) + w$ and constraint set (\mathbb{X}_{ν^0}, W) , where $\mathbb{X}_{\nu^0} \triangleq \alpha^0 X \cap \{x \mid \nu^0(x) \in \beta^0 U\}$.

3.2 Tube Terminal Set

The parametrization for the state tube \mathbf{X} motivates the introduction of a *set of sets* of the form $\Phi \triangleq \{z \oplus \mathcal{R} \mid z \in Z_f\}$ (Φ is a *set of sets*, each of the form $z \oplus \mathcal{R}$ where \mathcal{R} is a set) that is *set robust control invariant*.

Definition 3. A set of sets Φ is *set robust control invariant* (SRCI) for system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, W)$ if for any set $X \in \Phi$: (i) $X \subseteq \mathbb{X}$ and, (ii) there exists a set $Y \in \Phi$ such

that for all $x \in X$, there exists a $u \in \mathbb{U}$ such that $Ax + Bu \oplus W \subseteq Y$.

We assume that:

A1: (i) The set \mathcal{R} is a RCI set for system (2.1) and constraint set $(\alpha\mathbb{X}, \beta\mathbb{U}, W)$ where $(\alpha, \beta) \in [0, 1) \times [0, 1)$, (ii) The control law $\nu : \mathcal{R} \rightarrow \beta\mathbb{U}$ is such that \mathcal{R} is RPI for system $x^+ = Ax + B\nu(x) + w$ and constraint set (\mathbb{X}_ν, W) , where $\mathbb{X}_\nu \triangleq \alpha\mathbb{X} \cap X_\nu$ where $X_\nu \triangleq \{x \mid \nu(x) \in \beta\mathbb{U}\}$ ($\nu(\cdot)$ exists by **A1** (i)).

Let $U_\nu, \mathbb{Z}, \mathbb{V}$ be defined as follows:

$$U_\nu \triangleq \{\nu(x) \mid x \in \mathcal{R}\}, \quad \mathbb{Z} \triangleq \mathbb{X} \ominus \mathcal{R}, \quad \mathbb{V} \triangleq \mathbb{U} \ominus U_\nu. \quad (3.12)$$

We also assume:

A2: (i) The set Z_f is a CI set for the nominal system (2.2) and constraint set (\mathbb{Z}, \mathbb{V}) , (ii) The control law $\varphi : Z_f \rightarrow \mathbb{V}$ is such that Z_f is PI for system $z^+ = Az + B\varphi(z)$ and constraint set \mathbb{Z}_φ , where $\mathbb{Z}_\varphi \triangleq \mathbb{Z} \cap \{z \mid \varphi(z) \in \mathbb{V}\}$. ($\varphi(\cdot)$ exists by **A2** (i)).

We can now establish the following relevant result:

Theorem 2. Suppose that **A1** and **A2** are satisfied. Then $\Phi \triangleq \{z \oplus \mathcal{R} \mid z \in Z_f\}$ is a *set robust control invariant* for system $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, W)$.

PROOF. Let $X \in \Phi$, then $X = z \oplus \mathcal{R}$ for some $z \in Z_f$. For every $x \in X$ we have $x = z + y$, where $y \triangleq x - z \in \mathcal{R}$. Let the control law $\theta : Z_f \oplus \mathcal{R} \rightarrow \mathbb{U}$ be defined by $\theta(x) \triangleq \varphi(z) + \nu(y)$ and let $u = \theta(x)$. Then $x^+ = Ax + B\theta(x) + w = A(z + y) + B(\varphi(z) + \nu(y)) + w = Az + B\varphi(z) + Ay + B\nu(y) + w$. It follows from **A2** that $Az + B\varphi(z) \in Z_f$ and, by **A1**, we have $Ay + B\nu(y) + w \in \mathcal{R}$, $\forall (y, w) \in \mathcal{R} \times W$. Hence we conclude that $Ax + B\theta(x) + w \in Y$, $\forall (x, w) \in X \times W$ where $Y \triangleq z^+ \oplus \mathcal{R} \in \Phi$. The fact that $X \subseteq \mathbb{X}$ follows from **A1** and **A2** because $Z_f \oplus \mathcal{R} \subseteq \mathbb{X}$. The fact that $u = \theta(x) \in \mathbb{U}$ for all $x \in X$ and every $X \in \Phi$ follows from **A1** and **A2** since $\varphi(z) \in \mathbb{U} \ominus U_\nu \subseteq (1 - \beta)\mathbb{U}$, $\forall z \in Z_f$ and $\nu(y) \in \beta\mathbb{U}$, $\forall y \in \mathcal{R}$ implying that $u = \theta(x) \in \mathbb{U}$, $\forall x \in X$ and every $X \in \Phi$.

The *terminal set* X_f is defined by:

$$X_f \triangleq Z_f \oplus \mathcal{R} \quad (3.13)$$

where the sets \mathcal{R} and Z_f satisfy **A1** and **A2**. With this choice for *terminal set* the domain of attraction is enlarged (compared to the case when $X_f = \mathcal{R}$).

4. SIMPLE TUBE CONTROLLER

We are now ready to propose a robust optimal control problem, whose solution yields the tube and the corresponding control policy satisfying the set of constraints specified in (2.4) – (2.6) *providing that there exists RCI set* $\mathcal{R} = R_k^0 = R_k(\mathbf{M}_k^0)$, $k \in \mathbb{N}$ (for the system $x^+ = Ax + Bu + w$ and constraint set $(\alpha\mathbb{X}, \beta\mathbb{U}, W)$, with $(\alpha, \beta) \in [0, 1) \times [0, 1)$) defined in (3.1) with the corresponding feedback control law (3.6) – (3.7). We define:

$$\mathbb{Z} \triangleq \mathbb{X} \ominus \mathcal{R}, \quad \mathbb{V} \triangleq \mathbb{U} \ominus U(\mathbf{M}_k^0) \quad (4.1)$$

where $\mathcal{R} \triangleq R_k^0 = R_k(\mathbf{M}_k^0)$, $R_k(\mathbf{M}_k^0)$ and $U(\mathbf{M}_k^0)$ are defined in (3.1) and (3.10), respectively. In order to insure satisfaction of (2.4) – (2.6) and use of the simple tube–policy parametrization (2.7) – (2.8) we require that the trajectory of the nominal model (*the sequence of tube centers*) satisfies the tighter constraints (4.1). We assume in the sequel that **A1** and **A2** hold so that the terminal set Z_f for the nominal model can be any CI set satisfying **A2**. Let the set $\mathcal{V}_N(x)$ of admissible control–states pairs for nominal system at state x be defined as follows:

$$\mathcal{V}_N(x) \triangleq \{(\mathbf{v}, z) \mid (\bar{\phi}(k; z, \mathbf{v}), v_k) \in \mathbb{Z} \times \mathbb{V}, \\ k \in \mathbb{N}_{N-1}, \bar{\phi}(N; z, \mathbf{v}) \in Z_f, x \in z \oplus \mathcal{R}\} \quad (4.2)$$

It is clear that the set $\mathcal{V}_N(x)$ is a polyhedral set providing that Z_f is a polyhedral set. An appropriate cost function can be defined as follows:

$$V_N(\mathbf{v}, z) \triangleq \sum_{i=0}^{N-1} \ell(z_i, v_i) + V_f(z_N), \quad (4.3)$$

where for all i , $z_i \triangleq \bar{\phi}(i; z, \mathbf{v})$ and $\ell(\cdot)$ is the stage cost and $V_f(\cdot)$ is the terminal cost, that can be chosen to be :

$$\ell(x, u) \triangleq \|Qx\|_p + \|Ru\|_p, \quad p = 1, 2, \infty \quad (4.4a)$$

$$V_f(x) \triangleq \|Px\|_p, \quad p = 1, 2, \infty \quad (4.4b)$$

where P , Q and R are matrices of suitable dimensions. We assume additionally, as is standard (Mayne *et al.*, 2000), that:

A3: The terminal cost satisfies $V_f(Az + B\varphi(z)) + \ell(z, \varphi(z)) \leq V_f(z)$ for all $z \in Z_f$.

If $p = 2$ and $Q = Q' > 0$, $P = P' > 0$ and $R = R' > 0$ the resultant optimal control problem is a quadratic programme (Mayne *et al.*, 2005), since the set $\mathcal{V}_N(x)$ is polyhedral, defined by :

$$\mathbb{P}_N(x) : \quad V_N^0(x) \triangleq \min_{\mathbf{v}, z} \{V_N(\mathbf{v}, z) \mid (\mathbf{v}, z) \in \mathcal{V}_N(x)\} \quad (4.5)$$

and its unique minimizer is:

$$(\mathbf{v}^0(x), z^0(x)) \triangleq \arg \min_{\mathbf{v}, z} \{V_N(\mathbf{v}, z) \mid (\mathbf{v}, z) \in \mathcal{V}_N(x)\} \quad (4.6)$$

The domain of the value function $V_N^0(\cdot)$, the controllability set, is:

$$\mathcal{X}_N \triangleq \{x \mid \mathcal{V}_N(x) \neq \emptyset\} \quad (4.7)$$

For each i let $\mathcal{V}_i(x)$ and \mathcal{X}_i be defined, respectively, by (4.2) and (4.7) with i replacing N . The sequence $\{\mathcal{X}_i\}$ is a monotonically non-decreasing set sequence, i.e. $\mathcal{X}_i \subseteq \mathcal{X}_{i+1}$ for all i . Given any $x \in \mathcal{X}_N$ the solution to $\mathbb{P}_N(x)$ defines the corresponding optimal simple tube:

$$\mathbf{X}^0(x) = \{X_i^0(x)\}, \quad X_i^0(x) = z_i^0(x) \oplus \mathcal{R}, \quad (4.8)$$

for $i \in \mathbb{N}_N$, and the corresponding control policy $\pi^0(x) = \{\mu_i^0(\cdot) \mid i \in \mathbb{N}_{N-1}\}$ with

$$\mu_i^0(y; x) = v_i^0(x) + \nu(y - z_i^0(x)), \quad y \in X_i^0(x) \quad (4.9)$$

where, for each i , $z_i^0(x) = \bar{\phi}(i; z^0(x), \mathbf{v}^0(x))$. The following result (Langson *et al.*, 2004) follows easily from the construction of a simple tube:

Proposition 2. Let $x \in \mathcal{X}_N$ and let $(\mathbf{v}^0(x), z^0(x))$ be defined by (4.6). Let the control policy $\pi^0 \triangleq \{\mu_i^0(\cdot) \mid i \in \mathbb{N}_{N-1}\}$, where each $\mu_i^0 : X_i \rightarrow \mathbb{U}$ is defined by (4.9), be applied to actual system (2.1). Then $\phi(i; x, \pi, \mathbf{w}) \in \bar{\phi}(i; z^0(x), \mathbf{v}^0(x)) \oplus \mathcal{R}$ for all $i \in \mathbb{N}_N$ and all $\mathbf{w} \in \mathcal{W}$. Moreover, $(\phi(i; x, \pi^0, \mathbf{w}), \mu_i^0(\phi(i; x, \pi^0, \mathbf{w}); x)) \in \mathbb{X} \times \mathbb{U}$, $i \in \mathbb{N}_{N-1}$ and $\phi(N; x, \pi^0, \mathbf{w}) \in X_f$.

A result analogous to Proposition 2 holds for any arbitrary couple $(\mathbf{v}, z) \in \mathcal{V}_N(x)$. The solution of $\mathbb{P}_N(x)$ allows for a variety of controller implementations. Here we follow a useful proposal recently made in (Mayne *et al.*, 2005) and consider the following implicit robust model predictive control law $\kappa_N^0(\cdot)$ yielded by the solution of $\mathbb{P}_N(x)$:

$$\kappa_N^0(x) \triangleq v_0^0(x) + \nu(x - z^0(x)) \quad (4.10)$$

where $\nu(\cdot)$ is defined in (3.6) – (3.7). We establish some relevant properties of the proposed controller $\kappa_N^0(\cdot)$ by exploiting the results reported in (Mayne *et al.*, 2005). First we recall the following definition:

Definition 4. A set \mathcal{R} is robustly exponentially stable (Lyapunov stable and exponentially attractive) for $x^+ = Ax + B\kappa(x) + w$, $w \in W$, with a region of attraction \mathcal{X} if there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that any solution $x(\cdot)$ of $x^+ = Ax + B\kappa(x) + w$ with initial state $x(0) \in \mathcal{X}$, and admissible disturbance sequence $w(\cdot)$ ($w(i) \in W$ for all $i \geq 0$) satisfies $d(x(i), \mathcal{R}) \leq c\gamma^i d(x(0), \mathcal{R})$ for all $i \geq 0$.

Proposition 3. (i) For all $x \in \mathcal{R}$, $V_N^0(x) = 0$, $z^0(x) = 0$, $\mathbf{v}^0(x) = \{0, 0, \dots, 0\}$ and $\kappa_N^0(x) = \nu(x)$. (ii) Let $x \in \mathcal{X}_N$ and let $(\mathbf{v}^0(x), z^0(x))$ be defined by (4.6), then for all $x^+ \in Ax + B\kappa_N^0(x) \oplus W$ there exists $(\mathbf{v}(x^+), z(x^+)) \in \mathcal{V}_N(x^+)$ and

$$V_N^0(x^+) \leq V_N^0(x) - \ell(z^0(x), v_0^0(x)). \quad (4.11)$$

The main stability result follows (see Theorem 1 in (Mayne *et al.*, 2005)):

Theorem 3. Suppose that \mathcal{X}_N is bounded, then the set \mathcal{R} is robustly exponentially stable for controlled uncertain system $x^+ = Ax + B\kappa_N^0(x) + w$, $w \in W$. The region of attraction is \mathcal{X}_N .

The proposed controller $\kappa_N^0(\cdot)$ results in a set sequence $\{X_0^0(x(i))\}$, where:

$$X_0^0(x(i)) = z^0(x(i)) \oplus \mathcal{R}, \quad i \in \mathbb{N} \quad (4.12)$$

and $z^0(x(i)) \rightarrow 0$ exponentially as $i \rightarrow \infty$. The actual trajectory $x(\cdot) \triangleq \{x(i)\}$, where $x(i)$ is the solution of $x^+ = Ax + B\kappa_N^0(x) + w$ at time $i \geq 0$, corresponding to a particular realization of an infinite admissible disturbance sequence $w(\cdot) \triangleq \{w_i\}$, satisfies $x(i) \in X_0^0(x(i))$, $\forall i \in \mathbb{N}$. Theorem 3 implies that $X_0^0(x(i)) \rightarrow \mathcal{R}$ as $i \rightarrow \infty$ exponentially in the Hausdorff metric.

5. NUMERICAL EXAMPLE

Our illustrative example is a double integrator:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \quad (5.1)$$

with $w \in W \triangleq \{w \in \mathbb{R}^2 \mid |w|_\infty \leq 0.5\}$, $x \in X = \{x \in \mathbb{R}^2 \mid x^1 \leq 1.85, x^2 \leq 2\}$ and $u \in U = \{u \mid |u| \leq 2.4\}$, where x^i is the i^{th} coordinate of a vector x . The cost function is defined by (4.4) with $Q = 100I$, $R = 100$; the terminal cost $V_f(x)$ is the value function $(1/2)x'P_f x$ for the optimal unconstrained problem for the nominal system. The horizon is $N = 21$. The design parameters for the minimization problem \mathbb{P}_k (See (3.10)) defining the components of feedback actions of control policy are $k = 5$, $q_\alpha = q_\beta = 1$. The optimization problem \mathbb{P}_k , which in this case is a *linear program*, yielded the following matrix \mathbf{M}_k^0 :

$$\mathbf{M}_k^0 = \begin{bmatrix} -0.3833 & 0 & 0.15 & 0.233 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}' \quad (5.2)$$

The *tube cross-section* is constructed by using the set $\mathcal{R} = R_k(\mathbf{M}_k^0)$. The sequence of the sets \mathcal{X}_i , $i = 0, 1, \dots, 21$, where \mathcal{X}_i is the domain of $V_i^0(\cdot)$ and the terminal set $X_f = Z_f \oplus \mathcal{R}$ where Z_f satisfies **A2** and is the maximal positively invariant set (Blanchini, 1999) for system $z^+ = (A + BK)z$ under the tighter constraints $\mathbb{Z} = \mathbb{X} \ominus \mathcal{R}$ and $\mathbb{V} = \mathbb{U} \ominus U(\mathbf{M}_k^0)$ where K is unconstrained DLQR controller for (A, B, Q, R) , is shown in Figure 1. A RMPC tube $\{X_0^0(x(i)) = z_0^0(x(i)) \oplus \mathcal{R}\}$ for initial state $x_0 = (0.5, -8.5)'$ is shown in Figure 2 for a sequence of random admissible disturbances. The dash-dot line is the actual trajectory $\{x(i)\}$ due to the disturbance realization while the dotted line is the sequence $\{z_0^0(x(i))\}$ of optimal initial states for corresponding nominal system.

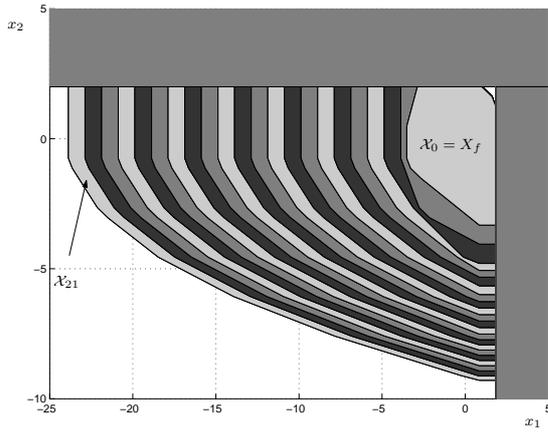


Fig. 1. Controllability Sets \mathcal{X}_i , $i = 0, 1, \dots, 21$

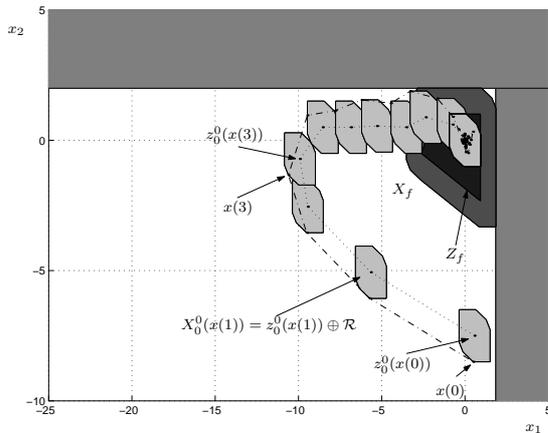


Fig. 2. RMPC Tube Trajectory

6. CONCLUSIONS

The main contribution of this note is a simple tube controller that ensures robust exponential stability of \mathcal{R} , a RCI set – the ‘origin’ for the uncertain system. *The complexity of the corresponding robust optimal control problem is marginally increased compared with that for conventional model predictive control.* A set of necessary ingredients ensuring robust exponential stability has been identified. The proposed scheme is computationally simpler than the schemes proposed in (Löfberg, 2003; van Hessem and Bosgra, 2003; Kerrigan and Alamo, 2004; Langson *et al.*, 2004) and it has an advantage over schemes proposed in (Kouvaritakis *et al.*, 2000; Chisci *et al.*, 2001; Mayne and Langson, 2001; Smith, 2004; Mayne *et al.*, 2005) because the feedback component of control policy, which is piecewise affine, *results in a smaller tube cross-section.*

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