H_∞ OUTPUT-FEEDBACK OF DISCRETE-TIME SYSTEMS WITH STATE-MULTIPLICATIVE NOISE

Eli Gershon [*] **Uri Shaked** [**]

[*] *Dept. of Electrical Eng. Holon Academic Institute of Technology, Holon, Israel, e-mail:gershon@eng.tau.ac.il*
[**] *Dept. of Electrical Eng. - Systems, Tel Aviv University, Tel Aviv 69978, Israel, e-mail:gershon@eng.tau.ac.il*

Abstract: Linear discrete-time systems with stochastic uncertainties in their state-space model are considered. The problem of dynamic output-feedback control is solved via a new approach, for both the finite horizon and the stationary cases. In both cases, a cost function is defined which is the expected value of the standard H_∞ performance index with respect to the uncertain parameters. An example is given which demonstrates the applicability of the theory. *Copyright©2005 IFAC.*

Keywords: Stochastic H_∞ output-feedback, multiplicative noise, nonlinear systems, robust control.

1. INTRODUCTION

In the present paper we address the problem of H_∞ output-feedback control of discrete-time state-multiplicative linear systems via a new approach, in both finite and infinite time settings. Systems whose parameter uncertainties are modelled as white noise processes in a linear setting have been treated in (Dragan *et al*., 1997a ; Dragan *et al*., 1997b ; Hinriechsen *et al*., 1998) and (Gershon *et al*., 2001c) for the continuous-time case and in (Dragan *et al*., 1998 ; Bouhtouri *et al*., 1999) and (Gershon *et al*., 2001) for the discrete-time case. Recently, the general problem of discrete-time output-feedback with stochastic uncertainties has been treated by (Dragan *et al*., 1998 ; Bouhtouri *et al*., 1999) and (Gershon *et al*., 2001). The solution in (Dragan *et al*., 1998) includes the finite and the infinite time horizon problems without transients. One drawback of (Dragan *et al*., 1998) is the fact that in the infinite-time horizon case, an infinite number of Linear Matrix Inequality (LMI) sets should be solved. Moreover, the fact that in (Dragan *et al*., 1998) the measurement coupling matrix has no uncertainty is a practical handicap, for example in cases where the measurements include state derivatives (e.g. acceleration control of an aircraft or missile). The treatment of (Bouhtouri *et al*., 1999) includes the derivation of the stochastic Bounded Real Lemma (BRL) and concerns only the stationary case where two coupled nonlinear inequalities were obtained. An attempt to solve the output-feedback problem using the adjoint system has been made in (Gershon *et al*., 2001). A modified-

Riccati recursion is obtained there which guarantees a given H_∞ estimation level, while minimizing an upper-bound on the covariance of the estimation error. The theoretical justification for using an adjoint system in stochastic systems, particularly in the H_∞ control field, is somewhat debatable. In the new approach that is adopted here, the latter obstacle is removed by avoiding the use of the adjoint system.

In the present paper we address the problem via two approaches: In the finite-horizon case we apply the Difference LMI (DLMI) method for the solution of the Riccati inequality obtained and in the stationary case we apply a special Lyapunov function which leads to an LMI derived tractable solution.

Notation:

We denote by $L^2(\Omega, \mathcal{R}^n)$ the space of square-integrable $\mathcal{R}^n-$ valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_k)_{k\in\mathcal{N}}$ we denote an increasing family of σ-algebras which is generated by $v_j, \eta_j,\ 0 \le j \le k$. $\mathcal{F}_k \subset \mathcal{F}$, where $\mathcal{N}$ is the set of natural numbers. We also denote by $\tilde{l}^2(\mathcal{N}; \mathcal{R}^n)$ the space of n-dimensional nonanticipative stochastic processes $\{f_k\}_{k\in\mathcal{N}}$ with respect to $(\mathcal{F}_k)_{k\in\mathcal{N}}$ where $f_k \in L^2(\Omega, \mathcal{R}^n)$. By $||.||_2^2$ we denote the standard l_2-norm: $||d||_2^2 = (\Sigma_{k=0}^{N-1} d_k^T d_k)$ and by $||f_k||_R^2$ the product $f_k^T R f_k$. We denote by $||\cdot||$ is the standard Euclidean norm and by δ_{ij} the Kronecker delta function.

2. PROBLEM FORMULATION

i) **Finite-horizon stochastic H_∞ output-feedback:** We consider the following system:

$$\begin{array}{l} x_{k+1} = (A_k + D_k v_k) x_k + B_{1,k} w_k + B_{2,k} u_k, \\ y_k = (C_{2,k} + F_k \eta_k) x_k + n_k, \\ z_k = C_{1,k} x_k + D_{12,k} u_k \end{array} \quad (1)$$

where $\{v_k\},\ \{\eta_k\}$ are independent white noise sequences that satisfy:

$$E\{\eta_k \eta_j\} = \delta_{kj}, E\{v_k v_j\} = \delta_{kj},\ E\{\eta_k v_j\} = 0 \quad (2)$$

and where $x_k \in R^n$ is the state vector, $w_k \in R^p$ is an exogenous disturbance, $y_k \in R^q$ is the measurement vector, $z_k \in R^m$ is the objective vector and $u_k \in R^l$ is the control input signal. We look for an output-feedback controller that achieves, for a given $\gamma > 0$,

$$\begin{array}{l} J_o \triangleq \underset{v,\eta}{E} \{||z_k||_2^2 - \gamma^2[||w_k||_2^2 + ||n_k||_2^2]\} \\ + \underset{v,\eta}{E} \{x_N^T \bar{Q}_N x_N\} - \gamma^2 x_0^T \bar{Q}_0 x_0 < 0, \end{array} \quad (3)$$

with $\bar{Q}_N \ge 0,\ \bar{Q}_0 \ge 0$ for all nonzero $(\{w_k\}, x_0)$ where $\{w_k\} \in \tilde{l}_2[0, N-1]$ and $x_0 \in \mathcal{R}^n$. Similarly to the standard case (Green and Limebeer, 1995), this problem involves an estimation of an appropriate combination of the states, and the application of the results of the state-feedback with a proper modification.

ii) **Stationary stochastic H_∞ output-feedback:** Given the following system:

$$\begin{array}{l} x_{k+1} = (A + D v_k) x_k + B_1 w_k + B_2 u_k,\ x_0 = 0, \\ y_k = (C_2 + F \eta_k) x_k + n_k, \\ z_k = C_1 x_k + D_{12} u_k \end{array} \quad (4)$$

where the system matrices $A, B_1, B_2, D, C_2, F, C_1$ and D_{12} are all constant, $v_k,\ \eta_k$ are given above. We seek an output-feedback controller that achieves, for a given $\gamma > 0$,

$$J_o \triangleq \underset{v,\eta}{E} \left\{||z_k||_{\tilde{l}_2}^2 - \gamma^2[||w_k||_{\tilde{l}_2}^2 + ||n_k||_{\tilde{l}_2}^2]\right\} < 0 \quad (5)$$

for all nonzero $\{w_k\} \in l^2([0, \infty); \mathcal{R}^p), \{n_k\} \in l^2([0, \infty); \mathcal{R}^q)$.

3. RESULTS

We bring first the known solution the stochastic state-feedback problem which constitutes the first stage in the solution of the dynamic output-feedback problem (Gershon *et al*. 2001). We consider the system of (1), with the measurement equation excluded and we assume, for simplicity, that for $k \in [0\ N]$:

$$[C_{1,k}^T \quad D_{12,k}^T] D_{12,k} = [0 \quad \tilde{R}_k], \quad \tilde{R}_k > 0. \quad (6)$$

Considering, for a given scalar $\gamma > 0$, the performance index of J_o in (3) , we have the following Lemma:

Lemma 1: (Gershon *et al*., 2001) *Consider the system of (1) with the feedback law $u_k = K_k x_k$. Given $\gamma > 0$, a necessary and sufficient condition for J_o of (3) where $||n_k||_2^2 = 0$, to be negative for all nonzero $(\{w_k\}, x_0)$ where $w_k \in \tilde{l}_2[0 \quad N-1]$ and $x_o \in \mathcal{R}^n$ is that there exists a solution $Q_k > 0$ to the following equation:*

$$\begin{array}{l} Q_k = A_k^T \bar{M}_k A_k + D_k^T Q_{k+1} D_k + C_{1,k}^T C_{1,k} \\ -\Delta_{1,k}^T \Phi_k^{-1} \Delta_{1,k}, \quad Q_N = \bar{Q}_N, \end{array} \quad (7)$$

that satisfies $R_k \triangleq \gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k} > 0$ and $Q_0 < \gamma^2 \bar{Q}_0$, where:

$$\begin{array}{l}\bar{M}_k \triangleq Q_{k+1}[I - \gamma^{-2} B_{1,k} B_{1,k}^T Q_{k+1}]^{-1}, \\ \Phi_k \triangleq B_{2,k}^T \bar{M}_k B_{2,k} + \tilde{R}_k, \quad \Delta_{1,k} \triangleq B_{2,k}^T \bar{M}_k A_k.\end{array} \quad (8)$$

If there exists such Q_k, then the state-feedback gain is given by:

$$K_k = -\Phi_k^{-1} \Delta_{1,k}. \quad (9)$$

In the case where the system matrices are all constant, $N \to \infty$ and the system $\{A, B_2, C_1, D_{12}\}$ has no invariant zeros on the unit circle and (A, B_2) is stabilizable, the following result is obtained (Gershon *et al.*, 2001):

Lemma 2: (Gershon et al., 2001) *Consider the system of (4) with the measurement equation excluded. Given $\gamma > 0$, a necessary and sufficient condition for J_o of (5),with $||n_k||_{\tilde{l}_2}^2 = 0$, to be negative for all nonzero $\{w_k\}$ where $\{w_k\} \in l^2([0, \infty); \mathcal{R}^p)$ is that there exists a matrix $P = P^T \in \mathcal{R}^{n \times n}$ that satisfies the following LMIs:*

$$\begin{bmatrix} P & PA^T & 0 & PD^T & PC_1^T & 0 \\ AP & \Gamma(2,2) & B_2 \tilde{R}^{-1} & 0 & 0 & B_1 \\ 0 & \tilde{R}^{-1} B_2^T & P & 0 & 0 & 0 \\ DP & 0 & 0 & P & 0 & 0 \\ C_1 P & 0 & 0 & 0 & I_l & 0 \\ 0 & B_1^T & 0 & 0 & 0 & \gamma^2 I_p \end{bmatrix} > 0 \quad (10)$$

and

$$\begin{bmatrix} \gamma^2 I_p & B_1^T \\ B_1 & P \end{bmatrix} > 0, \quad (11)$$

where $\Gamma(2,2) = P + B_2 \tilde{R}^{-1} B_2^T$. In this case the state-feedback gain is $K_s = -[B_2^T \bar{M} B_2 + \tilde{R}]^{-1} B_2^T \bar{M} A$. where $\bar{M} = P^{-1}[I - \gamma^{-2} B_1 B_1^T P^{-1}]^{-1}$.

Proof: See (Gershon *et al.*, 2001) and (Gershon *et al.*, 2004) for the LMI derivation.

4. FINITE-HORIZON STOCHASTIC OUTPUT-FEEDBACK CONTROL

The solution of the output-feedback control problem is obtained below by transforming the problem to one of filtering, to which the result of the discrete-time stochastic state-multiplicative BRL is applied. In order to obtain the equivalent estimation problem, the optimal strategies of both $\{w_k\}$ and $\{u_k\}$ are needed. These are derived below:

Defining $\tilde{J}_k = x_{k+1}^T Q_{k+1} x_{k+1} - x_k^T Q_k x_k$ and substituting (1) in $\tilde{J}_k$ we obtain:

$$\begin{array}{c}\tilde{J}_k = [x_k^T (A_k + D_k v_k)^T + u_k^T B_{2,k}^T] Q_{k+1} [(A_k + D_k v_k) x_k + (B_{2,k}) u_k] \\ +2[x_k^T (A_k + D_k v_k)^T + u_k^T (B_{2,k})^T] Q_{k+1} B_{1,k} w_k \\ + w_k^T B_{1,k}^T Q_{k+1} B_{1,k} w_k - x_k^T Q_k x_k - \gamma^2 w_k^T w_k + \gamma^2 w_k^T w_k + u_k^T \tilde{R}_k u_k \\ + x_k^T C_{1,k}^T C_{1,k} x_k - z_k^T z_k \\ = -w_k^T [\gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}] w_k + 2[x_k^T (A_k + D_k v_k)^T \\ + u_k^T (B_{2,k})^T] Q_{k+1} B_{1,k} w_k + u_k^T [\tilde{R}_k + (B_{2,k})^T Q_{k+1} (B_{2,k})] u_k \\ + 2 x_k^T (A_k + D_k v_k)^T Q_{k+1} (B_{2,k}) u_k + x_k^T [(A_k + D_k v_k)^T Q_{k+1} \\ (A_k + D_k v_k) + C_{1,k}^T C_{1,k} - Q_k] x_k - z_k^T z_k + \gamma^2 w_k^T w_k.\end{array}$$

Taking the expectation with respect to v_k we get:

$$\begin{array}{c}\underset{v}{E}\left\{\tilde{J}_k\right\} = \underset{v}{E}[-w_k^T [\gamma^2 I - B_{1,k}^T Q_{k+1} B_{1,k}] w_k + 2[x_k^T A_k^T Q_{k+1} B_{1,k} \\ + u_k^T B_{2,k}^T Q_{k+1} B_{1,k}] w_k + u_k^T [\tilde{R}_k + B_{2,k}^T Q_{k+1} B_{2,k}] u_k \\ + 2x^T [A_k^T Q_{k+1} B_{2,k}] u_k + x_k^T [A_k^T Q_{k+1} A_k + D_k^T Q_{k+1} D_k \\ + C_{1,k}^T C_{1,k} - Q_k] x_k - z_k^T z_k + \gamma^2 w_k^T w_k] + \underset{v}{E}\{\Psi_k\}\end{array}$$

where

$$\Psi_k = 2 x_k^T D_k^T v_k Q_{k+1} B_{1,k} w_k$$
$$+ 2 x_k^T D_k^T v_k Q_{k+1} B_{2,k} u_k + 2 x_k^T D_k^T v_k Q_{k+1} A_k x_k$$

and where $\underset{v}{E}\{\Psi_k\} = 0, \quad k = 0, 1, ..., N-1$. Completing to squares, first for w_k and then for u_k, we obtain:

$$\begin{array}{l}\underset{v}{E}\left\{\tilde{J}_k\right\} = -||(w_k - w_k^*)||_{R_k}^2 + ||(u_k - u_k^*)||_{\Phi_k}^2 \\ -x_k^T [\Delta_{1,k} \Phi_k^{-1} \Delta_{1,k}^T - A_k^T Q_{k+1} B_{1,k} R_k^{-1} B_{1,k}^T Q_{k+1} A_k] x_k \\ +x_k^T [A_k^T Q_{k+1} A_k + D_k^T Q_{k+1} D_k + C_{1,k}^T C_{1,k} - Q_k] x_k \\ -z_k^T z_k + \gamma^2 w_k^T w_k + \underset{v}{E}\{\Psi_k\}\end{array}$$

where R_k is defined proceeding (7) Φ_k and $\Delta_{1,k}$ are defined in (8),

$$u_k^* \triangleq K_k x_k, \quad w_k^* \triangleq K_{xk} x_k + K_{uk} u_k,$$
$$K_{xk} \triangleq R_k^{-1} B_{1,k}^T Q_{k+1} A_k, \; K_{uk} \triangleq R_k^{-1} B_{1,k}^T Q_{k+1} B_{2,k}$$

and the controller gain K_k is defined in (9).
Rearranging the last equation we obtain:

$$\begin{array}{l}\underset{v}{E}\left\{\tilde{J}_k\right\} = -||(w_k - w_k^*)||_{R_k}^2 + ||(u_k - u_k^*)||_{\Phi_k}^2 \\ +x_k^T \bar{R}(Q_k) x_k - z_k^T z_k + \gamma^2 w_k^T w_k + \underset{v}{E}\{\Psi_k\}\end{array}$$

where $\bar{R}(Q_k) = A_k^T \bar{M}_k A_k + D_k^T Q_{k+1} D_k + C_{1,k}^T C_{1,k} - \Delta_{1,k}^T \Phi_k^{-1} \Delta_{1,k}$. Taking the sum of both sides of $\tilde{J}_k$, from zero to $N-1$, the following is obtained using $\underset{v}{E}\left\{\tilde{J}_k\right\}$:

$$\underset{v}{E} \sum_{k=0}^{N-1} \left\{\tilde{J}_k\right\} = \underset{v}{E}\left\{x_N^T Q_N x_N\right\} - x_0^T Q_0 x_0$$

$$= \sum_{k=0}^{N-1} \underset{v}{E}\left\{-||(w_k - w_k^*)||_{R_k}^2 + ||(u_k - u_k^*)||_{\Phi_k}^2\right\} + \underset{v}{E}\{\Psi_k\}$$

$$+ \sum_{k=0}^{N-1} \underset{v}{E}\left\{x_k^T \bar{R}(Q_k) x_k\right\} + \underset{v}{E}\left\{-||z_k||_2^2 + \gamma^2 ||w_k||_2^2\right\}$$

Hence

$$J_o = \sum_{k=0}^{N-1} \underset{v}{E}\left\{-||(w_k - w_k^*)||^2_{R_k}\right\} + \underset{v}{E}\{\Psi_k\}$$
$$+\sum_{k=0}^{N-1} \underset{v}{E}\left\{||(u_k - u_k^*)||^2_{\Phi_k}\right\} + x_0^T(Q_0 - \gamma^2\bar{Q}_0)x_0 \quad (12)$$
$$+\sum_{k=0}^{N-1} \underset{v}{E}\left\{x_k^T \bar{R}(Q_k)x_k\right\}.$$

Clearly, the optimal strategy for u_k is given by $u_k = u_k^*$ where Q_k that is obtained by

$$Q_k = A_k^T Q_{k+1} A_k - \Delta_{1,k}\Phi_k^{-1}\Delta_{1,k}^T + D_k^T Q_{k+1} D_k$$
$$+C_{1,k}^T C_{1,k} + A_k^T Q_{k+1} B_{1,k} R_k^{-1} B_{1,k}^T Q_{k+1} A_k,$$
$$Q_N = \bar{Q}_N,$$

satisfies $R_k > 0$ and $Q_0 < \gamma^2\bar{Q}_0$.

The above results are used now to derive a solution to the output-feedback problem. Denoting $r_k \stackrel{\Delta}{=} w_k - w_k^*$ and using $u_k = K_k\hat{x}_k$, where $\hat{x}_k$ is yet to be found, we obtain from (1) that

$$x_{k+1} = (A_k + D_k v_k + B_{1,k}K_{xk})x_k + B_{1,k}r_k$$
$$+(B_{1,k}K_{uk} + B_{2,k})K_k\hat{x}_k,$$
$$y_k = (C_{2,k} + F_k\eta_k)x_k + n_k.$$

Substituting in (12) we seek for $\hat{x}_k$ for which

$$J \stackrel{\Delta}{=} \underset{v,\eta}{E}\left\{\Sigma_{k=0}^{N-1}||z_k||^2_{\Phi_k} - ||r_k||^2_{R_k} - \gamma^2||n_k||^2_2 + \Psi_k\right\}$$
$$-x_0^T\tilde{S}x_0$$

is negative for all nonzero $(\{w_k\}, \{n_k\}, x_0)$, where

$$z_k = K_k(x_k - \hat{x}_k) \text{ and } \quad \tilde{S} = \gamma^2\bar{Q}_0 - Q_0. \quad (13)$$

We consider the following state estimator:

$$\hat{x}_{k+1} = (A_k + B_{1,k}K_{xk})\hat{x}_k + (B_{1,k}K_{uk} + B_{2,k})u_k$$
$$+K_{0,k}(y_k - C_{2,k}\hat{x}_k), \quad \hat{x}_0 = 0. \quad (14)$$

Defining $e_k = x_k - \hat{x}_k$ we obtain

$$x_{k+1} = [A_k + B_{1,k}K_{xk} + B_{1,k}K_{uk}K_k + B_{2,k}K_k]x_k$$
$$+D_k x_k v_k + B_{1,k}r_k - (B_{1,k}K_{uk} + B_{2,k})K_k e_k$$

and

$$\hat{x}_{k+1} = [A_k + B_{1,k}K_{xk} + (B_{2,k} + B_{1,k}K_{uk})K_k]\hat{x}_k$$
$$+K_{0,k}[(C_{2,k} + F_k\eta_k)x_k + n_k - C_{2,k}\hat{x}_k].$$

Defining

$$\xi_k = \begin{bmatrix} x_k^T & e_k^T \end{bmatrix}^T \quad \text{and} \quad \tilde{w}_k = \begin{bmatrix} r_k^T & n_k^T \end{bmatrix}^T,$$

we obtain the following augmented system:

$$\xi_{k+1} = \tilde{A}_k\xi_k + \tilde{D}_k\xi_k v_k + \tilde{F}_k\xi_k\eta_k + \tilde{B}_k\tilde{w}_k,$$
$$z_k = \tilde{C}_{1,k}\xi_k, \quad (15)$$

where

$$\tilde{A}_k = \begin{bmatrix} A_{11,k} & -[B_{1,k}K_{uk} + B_{2,k}]K_k \\ 0 & A_k + B_{1,k}K_{xk} - K_{0,k}C_{2,k} \end{bmatrix}$$

$$\tilde{B}_k = \begin{bmatrix} B_{1,k} & 0 \\ B_{1,k} & -K_{0,k} \end{bmatrix}, \tilde{D}_k = \begin{bmatrix} D_k & 0 \\ D_k & 0 \end{bmatrix},$$

$$\tilde{C}_{1,k} = \begin{bmatrix} 0 & K_k \end{bmatrix}, \quad \tilde{F}_k = \begin{bmatrix} 0 & 0 \\ -K_{0,k}F_k & 0 \end{bmatrix},$$

and where $A_{11,k} = A_k + (B_{1,k}K_{uk} + B_{2,k})K_k + B_{1,k}K_{xk}$. Using the above notation we arrive at the following theorem:

Theorem 1: *Consider the system of (1) where $u_k = K_k\hat{x}_k$ and where $\hat{x}_k$ is defined above. Given $\gamma > 0$, there exists a controller that achieves (3) iff there exists a solution $(\hat{P}_k, K_{0,k})$ to the following difference linear matrix inequality (DLMI)(Gershon et al., 2001b):*

$$\begin{bmatrix} \hat{P}_k & \hat{P}_k\tilde{A}_k^T & 0 & \hat{P}_k\tilde{D}_k^T & \hat{P}_k\tilde{F}_k^T & \hat{P}_k\tilde{C}_{1,k}^T \\ \tilde{A}_k\hat{P}_k & \hat{P}_{k+1} & \gamma^{-1}\tilde{B}_{1,k} & 0 & 0 & 0 \\ 0 & \gamma^{-1}\tilde{B}_{1,k}^T & I & 0 & 0 & 0 \\ \tilde{D}_k\hat{P}_k & 0 & 0 & \hat{P}_{k+1} & 0 & 0 \\ \tilde{F}_k\hat{P}_k & 0 & 0 & 0 & \hat{P}_{k+1} & 0 \\ \tilde{C}_{1,k}\hat{P}_k & 0 & 0 & 0 & 0 & I \end{bmatrix} > 0, (16)$$

with a forward iteration, starting from the following initial condition:

$$\hat{P}_0 = \begin{bmatrix} I_n \\ I_n \end{bmatrix} (\gamma^2\bar{Q}_0 - Q_0) \begin{bmatrix} I_n & I_n \end{bmatrix}. \quad (17)$$

Remark 1: We note that the solution of the latter DLMI proceeds the solution of the finite-horizon state-feedback of Lemma 1 starting from Q_N in (7), for a given attenuation level of γ. Once a solution to the latter problem is achieved, the DLMI of (16) is solved for the same γ starting from the above initial condition.

Remark 2: We note that $\bar{P}_0 = E\{x_0x_0^T\}$ where $\bar{P}_0 = (\gamma^2\bar{Q}_0 - Q_0)$. The latter suggests that the initial condition $\hat{P}_0$ of (17) is

$$\hat{P}_0 = E\{\begin{bmatrix} x_0 \\ e_0 \end{bmatrix} \begin{bmatrix} x_0^T & e_0^T \end{bmatrix}\} = \begin{bmatrix} \bar{P}_0 & \bar{P}_0 \\ \bar{P}_0 & \bar{P}_0 \end{bmatrix},$$

since $e_0 = x_0$, hence the structure of (17).

Proof: Applying the result of the discrete-time stochastic BRL (Gershon *et al.*, 2001) to the system (15) the following Riccati-type inequality is obtained:

$$\begin{array}{c} -\hat{Q}_k + \tilde{A}_k^T \hat{Q}_{k+1} \tilde{A}_k + \tilde{A}_k^T \hat{Q}_{k+1} \tilde{B}_k \Theta_k^{-1} \tilde{B}_k^T \hat{Q}_{k+1} \tilde{A}_k \\ + \tilde{D}_k^T \hat{Q}_{k+1} \tilde{D}_k + \tilde{F}_k^T \hat{Q}_{k+1} \tilde{F}_k + \tilde{C}_{1,k}^T \tilde{C}_{1,k} > 0, \\ \Theta_k = \gamma^2 I - \tilde{B}_k^T \hat{Q}_{k+1} \tilde{B}_k,\ \Theta_k > 0. \end{array} \tag{18}$$

By simple manipulations, including the inversion Lemma, on the latter the following inequality is obtained:

$$\begin{array}{c} -\hat{Q}_k + \tilde{A}_k^T [\hat{Q}_{k+1}^{-1} - \gamma^{-2} \tilde{B}_k \tilde{B}_k^T]^{-1} \tilde{A}_k + \tilde{C}_{1,k}^T \tilde{C}_{1,k} \\ + \tilde{D}_k^T \hat{Q}_{k+1} \tilde{D}_k + \tilde{F}_k^T \hat{Q}_{k+1} \tilde{F}_k > 0 \end{array} \tag{19}$$

Denoting $\hat{P}_k = \hat{Q}_k^{-1}$ and using Schur's complement the result of (16) is obtained.

5. STATIONARY STOCHASTIC OUTPUT-FEEDBACK CONTROL

We consider the system of (4). Introducing the following Lyapunov function:

$$V_k = \xi_k^T \tilde{Q} \xi_k,\ \text{with } \tilde{Q} = \begin{bmatrix} Q & \alpha\hat{Q} \\ \alpha\hat{Q} & \hat{Q} \end{bmatrix}, \tag{20}$$

where ξ_k is the state vector of (15), Q and $\hat{Q}$ are $n \times n$ matrices and α ia a scalar. We obtain the following result

Theorem 2: *Consider the system of (15) where the matrices $A, B_1, B_2, D, C_2, F, C_1$ and D_{12} are all constant, $u_k = K_s \hat{x}_k$ and where $\hat{x}_k$ is defined in (14). Given $\gamma > 0$, there exists a controller that achieves (5) if there exist $Q = Q^T \in \mathcal{R}^{n\times n},\ \hat{Q} = \hat{Q}^T \in \mathcal{R}^{n\times n}$, $Y \in \mathcal{R}^{n\times q}$ and a tuning scalar parameter α that satisfy the following LMIs:*

$$\begin{bmatrix} Q & \alpha\hat{Q} & \Upsilon(1,3) & \Upsilon(1,4) & 0 & 0 & -\alpha D^T Y & -D^T Y & F^T(Q+\alpha\hat{Q}) & F^T\hat{Q}(1+\alpha) & 0 \\ * & \hat{Q} & \Upsilon(2,3) & \tilde{\Upsilon}(2,4) & 0 & 0 & 0 & 0 & 0 & 0 & \hat{C}_1^T \\ * & * & Q & \alpha\hat{Q} & \Upsilon(3,5) & -\gamma^{-1}\alpha Y & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \hat{Q} & \Upsilon(4,5) & -\gamma^{-1} Y & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & I_p & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & I_q & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & Q & \alpha\hat{Q} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \hat{Q} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & Q & \alpha\hat{Q} & 0 \\ * & * & * & * & * & * & * & * & * & \hat{Q} & 0 \\ * & * & * & * & * & * & * & * & * & 0 & I_l \end{bmatrix} > 0,$$

and

$$\begin{bmatrix} \gamma^2 I_{p+q} & \tilde{B}^T \\ \tilde{B} & \tilde{Q} \end{bmatrix} > 0. \tag{21}$$

where

$\Upsilon(1,3) \triangleq [K_{s,x}^T B_1^T + K_s^T (B_2^T + K_{s,u}^T B_1^T) + A^T] Q,$

$\Upsilon(1,4) \triangleq \tilde{\alpha}\hat{Q}\Upsilon(1,3)Q^{-1},\ \Upsilon(2,3) \triangleq -\tilde{\alpha} C_2^T Y^T$
$-K_s^T [B_2^T + K_{s,u}^T B_1^T] Q + \tilde{\alpha}[K_{s,x}^T B_1^T + A^T]\hat{Q},$

$\Upsilon(2,4) \triangleq \tilde{\alpha}[K_{s,x}^T B_1^T + K_s^T (B_2^T + K_{s,u}^T B_1^T) + A^T]\hat{Q}$
$+[K_{s,x}^T B_1^T + A^T]\hat{Q} - C_2^T Y^T,$

$\Upsilon(3,5) \triangleq \gamma^{-1}[Q B_1 + \tilde{\alpha}\hat{Q} B_1],\ \Upsilon(4,5) \triangleq \gamma^{-1}(\tilde{\alpha}+1)\hat{Q} B_1,$
$K_{s,x} = R^{-1} B_1^T P^{-1} A,\ K_{s,u} = R^{-1} B_1^T P^{-1} B_2$

and where P is the solution of (10).

Proof: The proof outline for the above stationary case resembles the one of the finite-horizon case. Considering the system of (4) we first solve the stationary state-feedback problem to obtain the optimal stationary strategies of both $w_{s,k}^*$ and $u_{s,k}^*$ and the stationary controller gain K_s. Thus we obtain:

$$\begin{array}{c} u_{s,k}^* \triangleq K_s x_k,\ w_{s,k}^* \triangleq R^{-1} B_1^T P^{-1} A x_k + R^{-1} B_1^T P^{-1} \\ B_2 u_{s,k},\ \ K_s \triangleq -[B_2^T \bar{M} B_2 + \tilde{R}]^{-1} B_2^T \bar{M} A, \\ \bar{M} \triangleq P^{-1}[I - \gamma^{-2} B_1 B_1^T P^{-1}]^{-1} \end{array}$$

where P is the solution of (10). Using the optimal strategies we transform the problem to an estimation one, thus arriving to the stationary counterpart of the augmented system of (15). Applying the discrete BRL for the stationary case (see Bouhtouri *et al.*, 1999; Gershon *et al.*, 2002) to the latter system the algebraic counterpart of (18) is obtained (see Gershon *et al.*, 2002) which, similarly to the finite horizon case, becomes the stationary version of (16). Multiplying the srtationary version of (16) from the left and the right by $diag\{\hat{P}^{-1},\hat{P}^{-1},I_{p+q},\hat{P}^{-1},\hat{P}^{-1},I_l\}$, denoting $\tilde{Q}=\hat{P}^{-1},\ Y=\hat{Q}K_o$ where K_o is the observer gain and carrying out the various multiplications the LMI of Theorem 2 obtained.

6. EXAMPLE

We consider the system of (4) and the objective function of (5) with the following matrices:

$$A=\begin{bmatrix}0 & 1\\ -0.8 & 1.6\end{bmatrix}, D=\begin{bmatrix}0 & 0\\ 0.08 & 0.16\end{bmatrix},\ B_1=\begin{bmatrix}1\\ -1\end{bmatrix},$$

$$B_2=\begin{bmatrix}0\\ 1\end{bmatrix},\ D_{12}=\begin{bmatrix}0\\ .1\end{bmatrix}, C_1=\begin{bmatrix}-0.5 & 0.4\\ 0 & 0\end{bmatrix}$$

and $C=\begin{bmatrix}0 & 1\end{bmatrix}$ where $F=0$.

We apply the result of Lemma 2 and Theorem 2 where we solve first for (10) and then for the LMIs of (21) and obtain for a near minimum of $\gamma = 7.18$ and $\alpha=0.15$ the following results:

$$Q=\begin{bmatrix}0.5950 & -0.2184\\ -0.2184 & 1.0805\end{bmatrix},\ \hat{Q}=\begin{bmatrix}8.2663 & -3.1948\\ -3.1948 & 2.7360\end{bmatrix},$$

$$K_s=\begin{bmatrix}0.7790 & -0.9454\end{bmatrix},\ Ko^T=\begin{bmatrix}1.2273 & 2.0104\end{bmatrix}.$$

The resulting closed-loop transfer function, from w_k to z_k, is $G_z=(-0.9446Z+0.9646)(z^2-0.5350z-0.3589)^{-1}$. We note that for the state-feedback solution of this problem (see Lemma 2) (where we assume that there is an access to the states of the system) one obtains a near minimum attenuation level of $\gamma=1.02$. We note also that for the deterministic counterpart of this example (where $D=0$) we obtain for the output-feedback case, a near minimum γ of 4.88.

7. CONCLUSIONS

The problem of H_∞-optimal output-feedback control of finite-horizon and stationary discrete-time linear systems with multiplicative stochastic uncertainties is solved. In both problems the solution is carried out along the standard approach where, using the optimal strategy for the state-feedback case, the problem is transformed to an estimation one, to which the stochastic BRL is applied. Unlike the previous works of (Dragan *et al*., 1998), (Bouhtouri et al., 1999) and (Gershon *et al*., 2001), our solution is tractable and involves an LMI based recursion (DLMI) in the finite horizon case and two simple LMIs for the stationary case. We note that in the latter case one has only to search for a single tuning parameter of α, a fact that renders our solution as a simple and easily tractable one. We note that the stationary solution is based on a specific selection of a Lyapunov function which leads to a sufficient solution of the output-feedback problem. In the example the latter solution is easily achieved by solving for a set of two LMIs.

REFERENCES

Bouhtouri, A. El, D. Hinriechsen and A. J. Pritchard (1999). H_∞ type control for discrete-time stochastic systems. *Int. J. of Robust and Nonlinear Control*, **9** 923-948.

Dragan, V., A. Halanay and A stoica (1997a). A small gain theorem for linear stochastic sys tem. *System and control letters,* **30**, 243-251.

Dragan, V. and T. Morozan (1997b). Mixed input-output optimization for time-varying Ito sys tems with state dependent noise. *Dynamics of Continuous, Discrete and Impulsive Systems*, **3**, 317-333.

Dragan, V. and A. Stoica (1998). A γ attenua tion problem for discrete-time-varying stochas tic systems with multiplicative noise. *Reprint series of the Institute of Mathematics of the Romanian Academy,* **10**.

Gershon, E., U. Shaked, I. Yaesh (2001). H_∞ con trol and filtering of discrete-time stochastic systems with multiplicative noise. *Automatica,* **37**, 409-417.

Gershon, E., A. Pila and U. Shaked (2001b). Dif ference LMIs for robust H_∞ control and filter ing. *Proceedings of the European Control Con ference (ECC01),* Porto, Portugal, 3469 - 3474.

Gershon, E., D. J. N. Limebeer, U. Shaked, I. Yaesh (2001c). Robust H_∞ filtering of station ary continuous-time linear systems with stochas tic uncertainties. *IEEE Transactions on Automatic Control*, **46** (11),1788-1793.

Gershon, E., D. J. N. Limebeer, U. Shaked, I. Yaesh (2004). Stochastic H_∞ tracking with preview for state-multiplicative systems. *IEEE Transactions on Automatic Control*, to appear.

Green, M. and D. J. N. Limebeer (1995). *Linear Robust Control,* Prentice Hall.

Hinriechsen, D. and A. J. Pritchard (1998). Stochastic H_∞. *SIAM J. of Contr. Optimiz.*, **36**, 1504-1538.