LINEAR-QUADRATIC CONTROL AND QUADRATIC DIFFERENTIAL FORMS

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Abstract: We consider the infinite time linear-quadratic control problem from a behavioral point of view. The performance functional is the integral of a quadratic differential form. A characterization of the stationary trajectories and of the local minima with respect to (left) compact support variations, as well as their relation to stability, are obtained. Finally, several theorems are derived that describe the optimal LQ trajectories with specified initial, and possibly terminal, conditions.

Keywords: Linear-quadratic control, behaviors, stationarity, quadratic differential forms, Hurwitz factorization, state representations, behavioral control.

1. INTRODUCTION

Linear-quadratic (LQ) control, together with Kalman filtering and the theory of the Riccati equation, are the main ingredients that led to the paradigm shift in control theory in the 1960's. Also recent advances, as the double Riccati equation solution of the $\mathcal{H}_\infty$ problem, are in this vein. The starting point of these developments is an input/state/output model for the plant, and a performance functional that is the integral of a memoryless quadratic function of the input and the state. However, often one does not start from such a situation: the model may contain high order derivatives, algebraic equations, and even the input/output structure may be unclear. Furthermore, it is often useful to incorporate, for example, derivatives of the control variables in the cost. The aim of the research that led to the present paper is to approach the LQ problem without assuming any special representation for the plant, and considering a performance functional that is the integral of an arbitrary quadratic expression in the system variables and their derivatives.

The goal of this paper is to present some results on LQ control from a behavioral point of view. We start by explaining the mathematical notions that enter into the formulation: the model of the plant, a linear differential behavior, and the performance functional, a quadratic differential form.

Denote by $\mathfrak{L}^{\mathtt{w}}$ the set of linear differential systems with $\mathtt{w}$ real variables. Thus $\Sigma \in \mathfrak{L}^{\mathtt{w}}$ (or $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$) means that $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is a continuous-time dynamical system with $\mathtt{w}$ real-valued signal variables, whose *behavior* $\mathfrak{B}$ consists of the solutions of a system of linear constant-coefficient differential equations, i.e. there exists $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\mathfrak{B}$ is the set of solutions of

$$R(\frac{d}{dt})w = 0. \qquad (\mathfrak{Ker})$$

The behavior specified by this system of differential equations is defined as

$$\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big) := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\}.$$

The $\mathfrak{C}^\infty$-assumption is made for convenience of exposition. From a modeling point of view, it is often more logical to take solutions in $\mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, and interpret the differential equation ($\mathfrak{Ker}$) in the sense of distributions. We refer to (Willems, 1991) and (Polderman and Willems, 1998) for details.

A *quadratic differential form* (QDF) is a quadratic form in the components of a map $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and its derivatives. Two-variable polynomial matrices lead to a compact notation and a convenient calculus for QDF's. A two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ is a finite sum

$$\Phi(\zeta, \eta) = \sum_{\mathtt{r},\mathtt{s}} \Phi_{\mathtt{r},\mathtt{s}} \zeta^{\mathtt{r}} \eta^{\mathtt{s}},$$

with $\Phi_{\mathtt{r},\mathtt{s}} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ for all $\mathtt{r}, \mathtt{s} \in \mathbb{N}$. $\mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ stands in one-to-one relation with the set of maps

$$\mathsf{L}_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}),$$

defined, for $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$, by

$$\mathsf{L}_\Phi(w_1, w_2) := \sum_{\mathtt{r},\mathtt{s}} (\frac{d^{\mathtt{r}}}{dt^{\mathtt{r}}} w_1)^\top \Phi_{\mathtt{r},\mathtt{s}} (\frac{d^{\mathtt{s}}}{dt^{\mathtt{s}}} w_2).$$

L_Φ is the *bilinear differential form* induced by Φ.

The *dual* of $\Phi(\zeta, \eta) = \sum_{\mathtt{r},\mathtt{s}} \Phi_{\mathtt{r},\mathtt{s}} \zeta^{\mathtt{r}} \eta^{\mathtt{s}}$ is defined by

$$\Phi^\star(\zeta, \eta) := \sum_{\mathtt{r},\mathtt{s}} \Phi^\top_{\mathtt{r},\mathtt{s}} \zeta^{\mathtt{s}} \eta^{\mathtt{r}}.$$

Φ is said to be *symmetric* if $\Phi = \Phi^\star$. Denote the symmetric elements of $\mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ by $\mathbb{R}^{\mathtt{w} \times \mathtt{w}}_{\mathrm{S}}[\zeta, \eta]$. $\mathbb{R}^{\mathtt{w} \times \mathtt{w}}_{\mathrm{S}}[\zeta, \eta]$ stands in one-to-one relation with the quadratic forms in w and its derivatives. Associate with $\Phi(\zeta, \eta) = \sum_{\mathtt{r},\mathtt{s}} \Phi_{\mathtt{r},\mathtt{s}} \zeta^{\mathtt{r}} \eta^{\mathtt{s}} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}_{\mathrm{S}}[\zeta, \eta]$, the *quadratic differential form*

$$\mathsf{Q}_\Phi(w) := \mathsf{L}_\Phi(w, w) = \sum_{\mathtt{r},\mathtt{s}} (\frac{d^{\mathtt{r}}}{dt^{\mathtt{r}}} w)^\top \Phi_{\mathtt{r},\mathtt{s}} (\frac{d^{\mathtt{s}}}{dt^{\mathtt{s}}} w).$$

Note that $\mathsf{Q}_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. We use Q_Φ as standard notation for the QDF induced by $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}_{\mathrm{S}}[\zeta, \eta]$. We refer to (Willems and Trentelman, 1998) for an in-depth study of QDF's.

The (infinite-time) *LQ problem*, the subject of this paper, is to characterize the trajectories w in a given plant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ that are stationary or optimal with respect to the infinite integral of a given QDF $\mathsf{Q}_\Phi(w)$ induced by $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}_{\mathrm{S}}[\zeta, \eta]$. The behavior $\mathfrak{B}$ formalizes the constraints imposed by the plant on the trajectories w. The QDF Q_Φ specifies, through its integral, the performance criterion. This formulation allows, for example, to start with model specifications which are not in state space form and to incorporate (higher) derivatives of the control input in the cost functional. It has also the classical state space formulation as a special case.

We discuss the following problems in this paper. The formulations below are meant to be informal. Precise statements will be given later.

1. Characterize the trajectories in $\mathfrak{B}$ that render $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)\, dt$ stationary with respect to compact support variations in $\mathfrak{B}$.
2. Characterize the stable stationary trajectories in $\mathfrak{B}$, i.e. the stationary trajectories that go to zero as $t \to +\infty$.
3. Characterize the trajectories in $\mathfrak{B}$ that are local minima for $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)\, dt$ with respect to compact support variations in $\mathfrak{B}$.
4. Characterize the trajectories in $\mathfrak{B}$ that are local minima for $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)\, dt$ with respect to left compact support variations in $\mathfrak{B}$.
5. Characterize the infimum of $\int_0^{+\infty} \mathsf{Q}_\Phi(w)\, dt$ over all $w \in \mathfrak{B}$ with constraints on the initial values of the derivatives $\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}} w(0)$. Characterize when this infimum is a minimum and the minimizing trajectories.
6. Characterize the infimum or minimum of $\int_0^{+\infty} \mathsf{Q}_\Phi(w)\, dt$ when, in addition to the initial conditions, there are stability constraints, requiring w and certain of its derivatives to go to zero as $t \to +\infty$.

The basic point of view on which the present article is based was first articulated in (Willems, 1993). It was further developed in (Parlangeli and Valcher, 2004), where an overview of papers following this approach is given. Two related earlier publications are (Ferrante and Zampieri, 2000) and (Weiland and Stoorvogel, 2000). There is also the work of Kučera and his co-workers (see for example (Kucera, 1991), (Hunt, et al. , 1992), and (Hunt, et al. , 1994)), which also starts from polynomial matrix descriptions for solving the LQ problem. We are presently preparing a full paper (Willems and Valcher, 2005) where an extensive discussion of related work will be given.

An important issue is that the behavior $\mathfrak{B}$ admits many representations, the most important ones being kernel, image, state, and latent variable representations. We will meet some of these representations shortly. In principle, therefore, we would like to have algorithms that, starting from any of these representations and Φ as data, return a suitable representation of the stationary or optimal trajectories. Unfortunately, because of space limitations, we have to leave these aspects to (Willems and Valcher, 2005). Another important question which we will not deal with here, is the *synthesis problem*: implement the optimal trajectories as an interconnection of the plant $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and a controller, i.e. find $\mathfrak{C} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathfrak{B} \cap \mathfrak{C}$ consists exactly of the optimal trajectories.

2. CONTROLLABILITY

One of the properties of behaviors which is very convenient, in particular for LQ problems, is controllability. Recall that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *controllable* if for any $w_1, w_2 \in \mathfrak{B}$ there exists a $T \geq 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geq T$. Denote by $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ the controllable elements of $\mathfrak{L}^{\mathtt{w}}$. It turns out that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ if and only if there exists $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ such that

$$w = M(\frac{d}{dt})\ell \tag{$\mathfrak{Im}$}$$

represents $\mathfrak{B}$. Concretely, ($\mathfrak{Im}$) is called an image representation of $\mathfrak{B}$ if

$$\mathfrak{B} = \text{im}\big(M(\frac{d}{dt})\big) := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}(M)}) \text{ such that } w = M(\frac{d}{dt})\ell\}.$$

The representation ($\mathfrak{Ker}$) is called a *kernel representation*, while the representation ($\mathfrak{Im}$) of a (controllable) element of $\mathfrak{L}^{\mathtt{w}}$ is called an *image representation.* The image representation ($\mathfrak{Im}$) is said to be *observable* if for any $w \in \mathfrak{B}$ there exists a unique $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}(M)})$ such that ($\mathfrak{Im}$) holds. It can be shown that every element $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ actually has an observable image representation. (Observable) image representations are very convenient: the laws governing the system are then expressed through M, while the driving variable ℓ is completely free. We often use the compact support elements of $\mathfrak{B}$. If the image representation is observable, then $w \in \mathfrak{B}$ has compact support if and only if the corresponding $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}(M)})$ has also compact support.

On the other extreme from controllability, we find the autonomous systems. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *autonomous* if

$$[\ w_1, w_2 \in \mathfrak{B} \text{ and } w_1(t) = w_2(t) \text{ for } t < 0\ ] \Rightarrow [\ w_1 = w_2\ ].$$

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is autonomous if and only if it is finite-dimensional, equivalently, if and only if it admits a kernel representation ($\mathfrak{Ker}$) with R square and $\det(R) \neq 0$.

Our notion of stability pertains to autonomous systems. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *stable* if

$$[\, w \in \mathfrak{B}\,] \Rightarrow [\, w(t) \to 0 \text{ for } t \to +\infty\,].$$

Clearly $[\,\mathfrak{B}$ stable$\,] \Rightarrow [\,\mathfrak{B}$ autonomous$\,]$. We will need the 'stable' or 'Hurwitz' part of a behavior. $P \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is said to be *Hurwitz* if $\det(P) \neq 0$ and if the roots of $\det(P)$ are in the open left half of the complex plane: $\{\lambda \in \mathbb{C} \mid \text{Real}(\lambda) < 0\}$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous, and define

$$\mathfrak{B}_{\text{Hurwitz}} := \{w \in \mathfrak{B} \mid w(t) \to 0 \text{ as } t \to +\infty\}.$$

It is easy to see that $\mathfrak{B}_{\text{Hurwitz}} \in \mathfrak{L}^{\mathtt{w}}$. A kernel representation of $\mathfrak{B}_{\text{Hurwitz}}$ can be obtained from a kernel representation ($\mathfrak{Ker}$) of $\mathfrak{B}$ as follows. If $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is square, and $\det(R) \neq 0$, then $R = R' R_{\mathfrak{H}}$ for some $R', R_{\mathfrak{H}} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, with $R_{\mathfrak{H}}$ Hurwitz, and the roots of $\det(R')$ in the closed right half of the complex plane: $\{\lambda \in \mathbb{C} \mid \text{Real}(\lambda) \geq 0\}$. $R_{\mathfrak{H}}$ is called a *Hurwitz factor* of R. A Hurwitz factor is unique up to pre-multiplication by a unimodular matrix $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. It is easy to see that $\mathfrak{B}_{\text{Hurwitz}}$ has $R_{\mathfrak{H}}\big(\frac{d}{dt}\big)\, w = 0$ as kernel representation.

In order to avoid difficulties which are not germane to our aims, we will assume in this paper that the plant is controllable. The controllability assumption has a very effective consequence for LQ problems. Indeed, by using an image representation ($\mathfrak{Im}$) for $\mathfrak{B}$, considering the induced two-variable polynomial matrix Φ' with

$$\Phi'(\zeta, \eta) = M^\top(\zeta)\Phi(\zeta, \eta)M(\eta),$$

and replacing $\mathsf{Q}_\Phi(w)$ by $\mathsf{Q}_{\Phi'}(\ell)$ in the performance functional, we obtain an LQ problem in which the dynamic variable w is replaced by the unconstrained variable ℓ.

Hence, throughout this paper, we will assume that $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. *If the behavior is* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}, \mathfrak{B} \neq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, *one can use the* (*observable*) *image representation* ($\mathfrak{Im}$), *replace* $\Phi(\zeta, \eta)$ *by* $M^\top(\zeta)\Phi(\zeta, \eta)M(\eta)$, *derive conditions on* ℓ, *and transfer these to conditions on* w *using* ($\mathfrak{Im}$).

3. THE STRUCTURE OF QDF'S

In this section, we collect a number of useful notions and results concerning QDF's. We need this background material in order to obtain an uninterrupted development of the LQ problems.

3.1 Factorization of QDF's

A two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}_{\text{S}}[\zeta, \eta]$ can be factored in terms of one-variable polynomial matrices as

$$\Phi(\zeta, \eta) = P^\top(\zeta)P(\eta) - N^\top(\zeta)N(\eta)$$

with $P, N \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, corresponding to an expansion of Q_Φ into sums and differences of squares,

$$\mathsf{Q}_\Phi(w) = ||P\left(\frac{d}{dt}\right)w||^2 - ||N\left(\frac{d}{dt}\right)w||^2.$$

There always exists a factorization (Willems and Trentelman, 1998) with the rows of $\left[\begin{smallmatrix} P \\ N \end{smallmatrix}\right]$ linearly independent over $\mathbb{R}$. We call such a factorization *canonical.* It corresponds to a Sylvester-like expansion of $\mathsf{Q}_\Phi(w)$ into a minimal number of sums and differences of squares. Canonical factorizations play an important role in the sequel. We therefore introduce the notation

$$\begin{aligned} &G_\Phi := \begin{bmatrix} P \\ N \end{bmatrix}, \quad \Sigma_\Phi := \begin{bmatrix} I_{\text{rowdim}(P)} & 0 \\ 0 & -I_{\text{rowdim}(N)} \end{bmatrix}, \\ &\Phi(\zeta, \eta) = G_\Phi(\zeta)^\top \Sigma_\Phi G_\Phi(\eta). \end{aligned} \tag{1}$$

It is easy to see that in a canonical factorization, Σ_Φ is unique, while G_Φ is unique up to pre-multiplication by a non-singular matrix.

Canonical factorizations lead to the *absolute value* of Φ and Q_Φ, denoted as $|\Phi|$ and $\mathsf{Q}_{|\Phi|}$, and defined through a canonical factorization (1) as

$$|\Phi|(\zeta,\eta) := P^\top(\zeta)P(\eta) + N^\top(\zeta)N(\eta), \\ = G_\Phi^\top(\zeta)G_\Phi(\eta)$$

$$\mathsf{Q}_{|\Phi|}(w) =: ||P\left(\frac{d}{dt}\right)w||^2 + ||N\left(\frac{d}{dt}\right)w||^2.$$

The absolute value of Q_Φ is a generalization of the absolute value of a matrix. Note the relation between canonical factorizations of Φ and $|\Phi|$:

$$G_{|\Phi|} = G_\Phi \text{ and } \Sigma_{|\Phi|} = I_{\mathrm{rowdim}(G_\Phi)}. \tag{2}$$

3.2 Positivity of QDF's

Various forms of non-negativity of QDF's play an essential role in this paper. Non-negativity of QDF's is one of the main issues studied in (Willems and Trentelman, 1998), and we collect the main notions here for easy reference.

Definition 1. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The QDF Q_Φ, or simply Φ, is said to be

(i) *non-negative* (denoted $\mathsf{Q}_\Phi \geq 0$, or $\Phi \geq 0$) if $\mathsf{Q}_\Phi(w) \geq 0$ (i.e. $\mathsf{Q}_\Phi(w)(t) \geq 0\ \forall\ t \in \mathbb{R}$) for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$,
(ii) *average non-negative* (denoted $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\, dt \geq 0$) if $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)\, dt \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support, and
(iii) *half-line non-negative* (denoted $\int_{-\infty}^{0} \mathsf{Q}_\Phi\, dt \geq 0$) if $\int_{-\infty}^{0} \mathsf{Q}_\Phi(w)\, dt \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support.

We need also a strict form of non-negativity. We will use one in terms of the absolute value of a QDF.

Definition 2. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The QDF Q_Φ, or simply Φ, is said to be

(i) *strictly positive* (denoted $\mathsf{Q}_\Phi \gg 0$, or $\Phi \gg 0$) if $\exists\ \varepsilon > 0$ such that $Q_{\Phi-\varepsilon|\Phi|} \geq 0$,
(ii) *strictly average positive* (denoted $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\, dt \gg 0$) if $\exists\ \varepsilon > 0$ such that $\int_{-\infty}^{+\infty} Q_{\Phi-\varepsilon|\Phi|}\, dt \geq 0$, and
(iii) *strictly half-line positive* (denoted $\int_{-\infty}^{0} \mathsf{Q}_\Phi\, dt \gg 0$) if $\exists\ \varepsilon > 0$ such that $\int_{-\infty}^{0} Q_{\Phi-\varepsilon|\Phi|}\, dt \geq 0$.

Obviously, half-line non-negativity or half-line strict positivity implies average non-negativity or strict positivity. Obtaining verifiable conditions on Φ for these various forms of non-negativity of QDF's is one of the main issues studied in (Willems and Trentelman, 1998). In particular, $\Phi \geq 0$ is equivalent to non-negative definiteness of the matrix formed by the coefficient matrices expressing $\Phi(\zeta,\eta)$ in powers of ζ and η. Further,

$$[\,\Phi \geq 0\,] \Leftrightarrow [\,\Phi \text{ can be factored as } \\ \Phi(\zeta,\eta) = P^\top(\zeta)P(\eta) \text{ with } P \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]\,].$$

This corresponds to writing $\mathsf{Q}_\Phi(w)$ as a sum of squares $||P\left(\frac{d}{dt}\right)w||^2$. Also, in proposition 5.2 of (Willems and Trentelman, 1998), it is proven that

$$[\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\, dt \geq 0] \Leftrightarrow [\,\Phi(-i\omega, i\omega) \geq 0\ \forall\ \omega \in \mathbb{R}],$$

while in theorem 9.3 a condition for verifying strict half-line positivity in terms of the *Pick matrix* is given.

3.3 The factorization equation (FE) and the dissipation inequality (DinE)

In this section we discuss some results about one- and two-variable polynomial matrices. We limit ourselves to the main points. Details and proofs can be found in (Willems and Trentelman, 1998).

Consider the maps

$$\begin{array}{llll} & ^\bullet : \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta] & \rightarrow & \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta], \\ & ^\bullet : \Phi(\zeta,\eta) & \mapsto & \mathsf{Q}_{\overset{\bullet}{\Phi}} := (\zeta+\eta)\Phi(\zeta,\eta), \\ \text{and} & * : \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\xi] & \rightarrow & \mathbb{R}^{\mathtt{w}_2\times\mathtt{w}_1}[\xi], \\ & * : P(\xi) & \mapsto & P^*(\xi) := P^\top(-\xi). \end{array}$$

P^* is called the *para-hermitian conjugate* of P. If $P = P^*$, we call P *para-hermitian.* $P(i\omega)$, $\omega \in \mathbb{R}$, is then hermitian.

These operators have natural interpretations and entanglements in terms of QDF's. For instance,

$$\frac{d}{dt}\mathsf{Q}_\Phi(w) = \mathsf{Q}_{\overset{\bullet}{\Phi}}(w).$$

Consider now the *factorization equation*

$$Y = F^*F, \text{ i.e. } Y(\xi) = F^\top(-\xi)F(\xi) \tag{FE}$$

viewed as an equation in the unknown $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ with $Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ given, and the *dissipation inequality*

$$\overset{\bullet}{\Psi} \leq \Phi, \tag{DinE}$$

viewed as an equation in the unknown $\Psi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ with $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ given.

These equations are central equations in the field, and they have natural interpretations in terms of QDF's. In fact, (DinE) is equivalent to

$$\frac{d}{dt}\mathsf{Q}_\Psi(w) \leq \mathsf{Q}_\Phi(w)$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. (DinE) is called the *dissipation inequality* because in the theory of dissipative systems $\mathsf{Q}_\Phi(w)$ corresponds to the *supply rate*, $\mathsf{Q}_\Psi(w)$ to the *storage function*, and $\mathsf{Q}_{(\Phi-\overset{\bullet}{\Psi})}(w)$

to the (non-negative) *dissipation rate*. It is well-known that (FE) has a solution $F \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ for a given $Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ if and only if $Y = Y^*$ and $Y(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. However, in the sequel, we need this equation under a somewhat stronger positivity requirement. The following factorization result is well-known.

Proposition 3. Let $Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and consider the factorization equation (FE). The following are equivalent.

(1) $Y = Y^*$ and $Y(i\omega) > 0$ for all $\omega \in \mathbb{R}$,
(2) there exists a Hurwitz polynomial matrix $H \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $Y(\xi) = H^\top(-\xi)H(\xi)$ (this is called a *Hurwitz factorization* of Y).

Note that we use the term 'Hurwitz factor' as meaning something different from 'Hurwitz factorization'. This is only slightly confusing since if $H^\top(-\xi)H(\xi)$ is a Hurwitz factorization of $Y(\xi)$, then $H(\xi)$ is a Hurwitz factor of $Y(\xi)$.

For (DinE) we record the following result.

Proposition 4. (Trentelman and Willems, 1997) Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and consider the dissipation inequality (DinE). The following are equivalent.

(1) $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi \geq 0$,
(2) $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$,
(3) $\Phi(-\xi,\xi)$ can be factored as $\Phi(-\xi,\xi) = F^\top(-\xi)F(\xi)$ for some $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$,
(4) there exists $\Psi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that (DinE) is satisfied.

There is a close relationship between the factorization equation (FE) and the dissipation inequality (DinE). In fact, every solution F of (FE) with $Y(\xi) = \Phi(-\xi,\xi)$ leads to a solution of (DinE), through

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - F^\top(\zeta)F(\eta)}{\zeta+\eta}.$$

The right-hand side of this equation is a (two-variable) polynomial matrix, since

$$\Phi(-\xi,\xi) - F^\top(-\xi)F(\xi) = 0.$$

Conversely, if (DinE) holds, then

$$\Phi(\zeta,\eta) - (\zeta+\eta)\Psi(\zeta,\eta) \geq 0.$$

It can therefore be factored as $F^\top(\zeta)F(\eta)$, leading to a solution of (FE) with $Y(\xi) = \Phi(-\xi,\xi)$.

In particular, assume that

$$\Phi(-i\omega, i\omega) > 0 \text{ for all } \omega \in \mathbb{R}. \tag{3}$$

Define H_Φ by the Hurwitz factorization

$$\Phi(-\xi,\xi) = H_\Phi^\top(-\xi)H_\Phi(\xi), \tag{4}$$

and Ψ_Φ^- by

$$\Psi_\Phi^-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H_\Phi^\top(\zeta)H_\Phi(\eta)}{\zeta+\eta}. \tag{5}$$

We used the notation Ψ_Φ^- because it can be shown that every other solution of (DinE) satisfies $\Psi_\Phi^- \leq \Psi$, i.e. Ψ_Φ^- is the minimum of the solutions of (DinE). This yields the following important relation:

$$\frac{d}{dt}\mathsf{Q}_{\Psi_\Phi^-}(w) = \mathsf{Q}_\Phi(w) - ||H_\Phi(\frac{d}{dt})w||^2 \tag{6}$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

3.4 Observability of QDF's

Much of the rich structure of linear behaviors also applies to QDF's, for example, observability, and image and state representations. The QDF $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ generates the (nonlinear) behavior

$$\mathfrak{B}_\Phi := \operatorname{im}(\mathsf{Q}_\Phi) = \{v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid \exists\, w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \text{ such that } v = \mathsf{Q}_\Phi(w)\}.$$

Consider the image representation

$$z = G_\Phi(\frac{d}{dt})w \tag{7}$$

corresponding to a canonical factorization (1). Note that (7), combined with

$$v = z^\top \Sigma_\Phi z \tag{8}$$

can be viewed as an image representation of $\mathfrak{B}_\Phi$, consisting of a linear system followed by a memoryless quadratic map. Q_Φ, or Φ, is *observable* if the image representation (7) is observable. In section 7 of (Willems and Trentelman, 1998) a number of equivalent conditions for observability of $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ are given. By replacing G_Φ, if need be, by a G'_Φ such that $z = G'_\Phi(\frac{d}{dt})f$ is an observable image representation of $\operatorname{im}\big(G_\Phi(\frac{d}{dt})\big)$, we can always obtain an observable image representation of $\mathfrak{B}_\Phi$. In this sense, therefore, assuming Q_Φ observable, often entails no real loss of generality.

3.5 State representations of QDF's

In the theory of behavioral systems, much attention has been paid to algorithms for state construction. In particular, in (Rapisarda and Willems, 1997) algorithms are given to construct, starting from R in the kernel representation ($\mathfrak{Ker}$) or from M in the image representation ($\mathfrak{Im}$), a polynomial matrix $X_R^{\ker} \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ or $X_M^{\mathrm{im}} \in \mathbb{R}^{\bullet\times \operatorname{coldim}(M)}[\xi]$ such that

$$R(\frac{d}{dt})w = 0, \quad x = X_R^{\ker}(\frac{d}{dt})w \tag{9}$$

and

$$w = M(\frac{d}{dt})\ell, \quad x = X_M^{\mathrm{im}}(\frac{d}{dt})\ell \tag{10}$$

are minimal state representations of the behavior of ($\mathfrak{Ker}$) or ($\mathfrak{Im}$), respectively. The resulting differential operators $X_R^{\ker}(\frac{d}{dt})$ and $X_M^{\mathrm{im}}(\frac{d}{dt})$ are called *state maps.* A state representation has the (defining) property that if (w_1, x_1) and (w_2, x_2) satisfy the state differential equations (9) or (10), and if $x_1(0) = x_2(0)$, then their concatenation at $t = 0$, $(w_1, x_1) \wedge_0 (w_2, x_2)$, belongs to the $\mathcal{L}^{\mathrm{loc}}$-closure of the behavior, consisting of the (w, x)'s respectively satisfying (9), or that are generated by some ℓ through (10). See (Rapisarda and Willems, 1997) for details, for example, for the notion of minimality and its relation to observability.

There is an essential difference between the minimal state maps $X_R^{\mathrm{ker}}(\frac{d}{dt})$ and $X_M^{\mathrm{im}}(\frac{d}{dt})$, as far as uniqueness is concerned. The polynomial matrix X_M^{im} is unique up to pre-multiplication by a non-singular matrix. The polynomial matrix X_R^{ker}, on the other hand, is much further from being unique. Indeed, a minimal state map X_R^{ker} for $(\mathfrak{Ker})$ is unique up to the operation

$$X_R^{\mathrm{ker}} \mapsto S X_R^{\mathrm{ker}} + VR,$$

with $S \in \mathbb{R}^{\mathrm{rowdim}(X_R^{\mathrm{ker}}) \times \mathrm{rowdim}(X_R^{\mathrm{ker}})}$ non-singular, and $V \in \mathbb{R}^{\mathrm{rowdim}(X_R^{\mathrm{ker}}) \times \mathrm{rowdim}(R)}[\xi]$: i.e. there is uniqueness up to a pre-multiplication by a non-singular matrix, and addition to each of the rows of an arbitrary element of the $\mathbb{R}[\xi]$-module generated by the rows of R. This also leads to state representations of the behavior $\mathfrak{B}_\Phi$. Indeed, by representing the image representation (7) in state form, we obtain a state representation of $\mathfrak{B}_\Phi$. So, if $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})$ is a state map for (7), we obtain

$$v = z^\top \Sigma_\Phi z, \;\; z = G_\Phi(\frac{d}{dt})w, \qquad x = X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w. \quad (11)$$

This representation of $\mathfrak{B}_\Phi$ has the property that if $v_1, v_2 \in \mathfrak{B}_\Phi$ result from w_1, w_2 such that $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w_1(0) = X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w_2(0)$, then $v_1 \wedge_0 v_2$ belongs to the $\mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$ – closure of $\mathfrak{B}_\Phi$. Hence $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w$ acts as a state for the nonlinear behavior $\mathfrak{B}_\Phi$.

3.6 Relations among state spaces

We are interested in the relation between the state spaces of $\mathfrak{B}_\Phi$ (see (11)) and of the (autonomous) system with kernel representation $H_\Phi(\frac{d}{dt})w = 0$, with the polynomial matrix H_Φ defined by (4).

Proposition 5. Assume that $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ satisfies (3). Define H_Φ and $\mathsf{Q}_{\Psi_\Phi^-}$ by (4) and (5).

(1) Let $X_{G_\Phi}^{\mathrm{im}}$ induce a minimal state map for the image representation (7) associated with a canonical factorization of Φ. Then $\Psi_\Phi^- \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ can be factored as

$$\Psi_\Phi^-(\zeta, \eta) = \big(X_{G_\Phi}^{\mathrm{im}}(\zeta)\big)^\top K \big(X_{G_\Phi}^{\mathrm{im}}(\eta)\big)$$

with $K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$ a constant matrix.

(2) There exists a polynomial matrix $X_{H_\Phi}^{\mathrm{ker}} \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ whose rows are in the $\mathbb{R}$-span of the rows of $X_{G_\Phi}^{\mathrm{im}}$, which induces a state map for

$$H_\Phi(\frac{d}{dt})w = 0, \quad (12)$$

i.e. such that

$$H_\Phi(\frac{d}{dt})w = 0, \;\; x = X_{H_\Phi}^{\mathrm{ker}}(\frac{d}{dt})w \quad (13)$$

is a state representation.

(3) If, in addition,

$$\mathrm{degree}\big(\det\big(\Phi(-\xi, \xi)\big)\big) = 2\ \mathrm{rowdim}(X_{G_\Phi}^{\mathrm{im}}) \quad (14)$$

(this is the case if e.g. $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi \, dt \gg 0$), then there exists a state representation (13) with $X_{H_\Phi}^{\mathrm{ker}} = X_{G_\Phi}^{\mathrm{im}}$.

Proof. (1) In (Trentelman and Willems, 1997) it is proven that every storage function, hence in particular Ψ_Φ^-, is a memoryless state function, i.e. the dissipation inequality (DinE) is of the form $\frac{d}{dt}|x|_K^2 \leq |v|_{\Sigma_\Phi}^2$, with $K = K^\top$ a constant matrix, and x the state of the image representation (7). Part (1) follows.
(2) It follows from theorem 6.2 of (Trentelman and Willems, 1997) that in the system

$$v = G_\Phi(\frac{d}{dt})w \;\; h = H_\Phi(\frac{d}{dt})w$$

h is a memoryless function of $(X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w, w)$. Hence the state of

$$v = G_\Phi(\frac{d}{dt})w \;\; 0 = H_\Phi(\frac{d}{dt})w,$$

which equals the state of (12), is a sub-vector of the state $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w$.
(3) Note that (4) implies

$$\mathrm{degree}\big(\det\big(\Phi(-\xi, \xi)\big)\big) = 2\ \mathrm{degree}\big(\det\big(H_\Psi(\xi)\big)\big).$$

Hence (14) implies

$$\mathrm{degree}\big(\det(H_\Psi)\big) = \mathrm{rowdim}(X_{G_\Phi}^{\mathrm{im}}).$$

Since the dimension of the state space of (12) equals $\mathrm{degree}\big(\det\big(H_\Psi\big)\big)$, (14) implies that the dimension of this state space equals $\mathrm{rowdim}(X_{G_\Phi}^{\mathrm{im}})$. The result follows. ■

In words, this proposition states that the storage function $\Psi_\Phi^- \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is a memoryless function of the state of $\mathfrak{B}_\Phi$. Further, that the state of (12) is a sub-vector of the state of $\mathfrak{B}_\Phi$, and finally that under condition (14), there corresponds exactly one initial state of (12) (and consequently exactly one solution) to each initial state of $\mathfrak{B}_\Phi$.

Note that in the notation used in the above proposition, (6) may be written as

$$\frac{d}{dt}||X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w||_K^2 = \mathsf{Q}_\Phi(w) - ||H_\Phi(\frac{d}{dt})w||^2 \quad (15)$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

3.7 A state convergence lemma

In the proof of theorem 14, we need the following technical result about non-negative QDF's. It states that if Q_Φ is nonnegative, then the state of $\mathfrak{B}_\Phi$ goes to zero whenever $\int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt < +\infty$.

Lemma 6. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Assume that $\Phi \geq 0$. Let $\Phi(\zeta,\eta) = P^\top(\zeta)P(\eta)$ be a canonical factorization of Φ, and assume that $X_P^{\mathrm{im}} \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ induces a minimal state map for the system with image representation $v = P(\frac{d}{dt})w$. Then

$$[\ \int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt < +\infty\ ] \\ \Rightarrow [\ X_P^{\mathrm{im}}(\frac{d}{dt})w(t) \to 0 \text{ as } t \to +\infty\ ].$$

Proof. In an input/output partition of v for the image representation $v = P(\frac{d}{dt})w$, the statement of the lemma is equivalent to the claim that for the controllable and observable state system

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad v = \begin{bmatrix} u \\ y \end{bmatrix}$$

$$[\int_0^{+\infty} (||u||^2 + ||y||^2)\,dt < +\infty\,] \\ \Rightarrow [\,x(t) \to +\infty \text{ as } t \to +\infty\,].$$

This is proven in section 23, theorem 2 of (Brockett, 1970), for the case $D = 0$, but the proof applies to the general case as well. ∎

We now have the required background material, and return to the main story line: LQ problems.

4. STATIONARITY

Consider, for a given $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ and $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, the map

$$\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \\ \mapsto \big(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\big) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}).$$

Obviously,

$$\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w) = 2\,\mathsf{L}_\Phi(\Delta,w) + \mathsf{Q}_\Phi(\Delta).$$

Hence this functional is the sum of a linear and a quadratic term in Δ.

Definition 7. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. The trajectory $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is said to be *stationary* with respect to $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\,dt$, if the linear term in Δ in the integral

$$\int_{-\infty}^{+\infty} \big(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\big)\,dt$$

vanishes for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support:

$$\int_{-\infty}^{+\infty} \mathsf{L}_\Phi(\Delta,w)\,dt = 0.$$

Equivalently,

$$\int_{-\infty}^{+\infty} \big(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\big)\,dt \\ = \int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(\Delta)\,dt$$

Denote the set of stationary trajectories with respect to Φ by $\mathfrak{S}_\Phi$.

Theorem 8. **(Stationarity).** Consider $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The stationary behavior $\mathfrak{S}_\Phi$ consists of all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ that satisfy

$$\Phi(-\frac{d}{dt},\frac{d}{dt})w = 0. \qquad (\mathfrak{S}_\Phi)$$

Proof. Consider $w,\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, with Δ of compact support. Integration by parts yields

$$\int_{-\infty}^{+\infty} \mathsf{L}_\Phi(\Delta,w)\,dt \\ = \int_{-\infty}^{+\infty} \Delta^\top\big(\Phi(-\frac{d}{dt},\frac{d}{dt})w\big)\,dt.$$

For w to be stationary, this integral needs to be zero for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support. Hence $(\mathfrak{S}_\Phi)$ must hold. ∎

Note that in the case $\mathtt{w} = 1$, $\Phi(-\xi,\xi)$ is an even polynomial. The stationary behavior $\Phi(-\frac{d}{dt},\frac{d}{dt})w = 0$ is hence time-reversible (not surprising, since the definition of stationarity does not involve a time direction). Also, the dynamic order of $\mathfrak{S}_\Phi$ is typically twice the order of the largest derivative appearing in Q_Φ. This is characteristic of what happens in the variational principles of mechanics. The consideration of (two-sided) compact support variations makes this situation, as we shall see, very different from the one encountered in optimal control, where stability is one of the main issues.

Example. Consider $\int_{-\infty}^{+\infty}\big(w^2 \pm (\frac{d}{dt}w)^2\big)\,dt$, i.e. $\Phi(\zeta,\eta) = 1 \pm \zeta\eta$. Hence $\Phi(-\xi,\xi) = 1 \mp \xi^2$. The stationary behavior is given by the hyperbolic flow $w - \frac{d^2}{dt^2}w = 0$, or the harmonic oscillator $w + \frac{d^2}{dt^2}w = 0$.

We now define what we mean with stability. We will deal with convergence to zero of w (later we also consider convergence to zero of the state trajectory). In the behavioiral theory it is common to consider convergence to zero of some latent variables, but we will not deal with this here.

Definition 9. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. The stationary trajectory $w \in \mathfrak{S}_\Phi$ is said to be *stable* if $w(t) \to 0$ as $t \to +\infty$. Denote the set of stationary trajectories with respect to Φ that are stable by $\mathfrak{S}_\Phi^{\mathrm{stable}}$.

It is not difficult to specify what $\mathfrak{S}_\Phi^{\mathrm{stable}}$ is.

Theorem 10. **(Stable stationary trajectories).** Consider $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Assume $\det\left(\Phi(-\xi,\xi)\right) \neq 0$. Let $H \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be a Hurwitz factor of $\Phi(-\xi,\xi)$. Then $\mathfrak{S}_\Phi^{\text{stable}}$, the set of $w \in \mathfrak{S}_\Phi$ that satisfy $w(t) \to 0$ as $t \to +\infty$, consists of those $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ that satisfy

$$H(\frac{d}{dt})w = 0. \qquad (\mathfrak{S}_\Phi^{\text{stable}})$$

Proof. Obvious. ∎

The question of how to compute H from Φ will be taken up in (Willems and Valcher, 2005).

In theorem 14 we will see that these stable stationary trajectories also emerge as local minima with respect to variations with left compact support.

5. LOCAL MINIMA

We examine when and in what sense (stable) stationary trajectories are local minima.

5.1 Compact support variations

Definition 11. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. The trajectory $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is said to be *a local minimum for* $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\,dt$ *with respect to compact support variations* if

$$\int_{-\infty}^{+\infty} \left(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\right)dt \geq 0$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support. Denote the set of local minima for $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi$ with respect to compact support variations by $\mathfrak{S}_\Phi^{\min}$.

The question which trajectories are local minima with respect to compact support variations is easy to deal with.

Theorem 12. **(Local minima with respect to compact support variations).** Consider $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\mathfrak{S}_\Phi^{\min}$ is either empty or equal to $\mathfrak{S}_\Phi$. $\mathfrak{S}_\Phi^{\min} = \mathfrak{S}_\Phi$ if and only if Q_Φ is average non-negative, equivalently

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}. \qquad (16)$$

Proof. Let $w, \Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, with Δ of compact support. There holds

$$\int_{-\infty}^{+\infty} \left(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\right)dt = \int_{-\infty}^{+\infty} 2\mathsf{L}_\Phi(\Delta, w)\,dt + \int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(\Delta)\,dt.$$

For this to be non-negative for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support, the first term on the right hand side needs to be zero, and the second term needs to be non-negative for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support. Hence $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is a local minimum if an only if $w \in \mathfrak{S}_\Phi$ and $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(\Delta)\,dt \geq 0$ for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support, i.e. $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\,dt \geq 0$. Equivalently, by proposition 4, if and only if (16) holds. ∎

From the expression of the stationary trajectories or the local minima given in theorem 8 it is apparent that stationarity of w has no bearing on whether $w(t) \to 0$ as $t \to +\infty$. Theorem 12 gives an explicit condition under which stationary trajectories are local minima. This theorem also makes clear that variational principles do not deal with local minima, but merely with stationary trajectories. In (Willems and Valcher, 2005) we will point to ways in which minimality (with respect to a more restricted class of variations) may nevertheless be recovered. It turns out that under a somewhat stronger condition than the one given in theorem 12, we are able to prove (see theorem 14) that the stable stationary trajectories are the local minima with respect to left compact support variations.

5.2 Left compact support variations

We discuss the local optimality with respect to variations that have left compact support.

Definition 13. Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be given. The trajectory $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is said to be *a local minimum for* $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi\,dt$ *with respect to left compact support variations* if

$$\int_{-\infty}^{+\infty} \left(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\right)dt \geq 0 \qquad (17)$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ with left compact support, i.e. for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ for which there exists $T_0 \in \mathbb{R}$ such that Δ has support on the half-line $[T_0, +\infty)$. Denote the set of local minima for $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi$ with respect to left compact support variations by $\mathfrak{S}_\Phi^{\text{onesided}}$.

Up to now, we only considered infinite integrals with integrands of compact support. No convergence issues occurred. However the integrand of the integral (17) has support on a half-line $[T_0, +\infty)$. A priori this infinite integral could be finite or infinite, but it could also not exist as an infinite integral. In the specification of $\mathfrak{S}_\Phi^{\text{onesided}}$ in definition 13 and the characterization obtained in theorem 14, this integral exists and must be ≥ 0 or $+\infty$. In stronger versions of theorem 14 (with weaker assumptions on Φ), one could have to resort to a third possibility, requiring, instead of (17), that

$$\liminf_{T\to+\infty} \int_{-\infty}^{T} \left(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\right)dt \geq 0.$$

It is clear that if w is a local minimum with respect to left compact support variations, then it is a local minimum with respect to compact support variations. In particular, for the set of local minima for left compact support variations to be non-empty we need that $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi \geq 0$. But, as can be expected, we need a stronger non-negativity of the QDF in the left compact support case.

It is easy to prove that $\mathfrak{S}_\Phi^{\text{onesided}}$ is empty if Φ is not half-line non-negative. In the next theorem, the *pièce de résistance* of this paper, we show that strict half-line positivity implies that a local minimum with respect to compact support variations is a local minimum with respect to left compact support variations if and only if it is also stable.

Theorem 14. **(Local minima with respect to left compact support variations).** Consider $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. If Q_Φ is not half-line non-negative, then $\mathfrak{S}_\Phi^{\text{onesided}}$ is empty. If Q_Φ is strictly half-line positive, then $\mathfrak{S}_\Phi^{\text{onesided}} = \mathfrak{S}_\Phi^{\text{stable}}$.

If Q_Φ is strictly half-line positive and observable, $\mathfrak{S}_\Phi^{\text{onesided}}$ can be computed as follows. Since (3) holds, Φ admits a Hurwitz factorization as (4). Then $\mathfrak{S}_\Phi^{\text{onesided}}$, the set of $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ that are local minima with respect to left compact support variations for Φ, is given by (12).

Proof. (17) shows that $w \in \mathfrak{S}_\Phi^{\text{onesided}}$ if and only if

$$\int_{-\infty}^{+\infty} \big(2\mathsf{L}_\Phi(\Delta, w) + \mathsf{Q}_\Phi(\Delta)\big)\, dt \geq 0$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ with left compact support. Linearity and time-invariance of $\mathfrak{S}_\Phi^{\text{onesided}}$ are immediately clear from this expression. Hence $\mathfrak{S}_\Phi^{\text{onesided}}$ is empty if and only if $0 \notin \mathfrak{S}_\Phi^{\text{onesided}}$, whence if and only if Q_Φ is not half-line non-negative.

Assume now that Q_Φ is strictly half-line positive and observable (generalization to the non-observable case is straightforward). Then (3) holds. Hence $\mathfrak{S}_\Phi^{\text{onesided}} \subseteq \mathfrak{S}_\Phi^{\min} = \mathfrak{S}_\Phi$. $\mathfrak{S}_\Phi$ is then given by the autonomous system $\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0$. $\mathfrak{S}_\Phi$ is hence finite-dimensional, without oscillatory modes (which would correspond to roots of $\det\big(\Phi(-\xi,\xi)\big)$ on the imaginary axis). Consequently, $\mathfrak{S}_\Phi^{\text{onesided}}$ is a linear shift-invariant subspace of the finite-dimensional behavior $\mathfrak{S}_\Phi$. Our aim is to show that it consists exactly of the elements $w \in \mathfrak{S}_\Phi$ such that $w(t) \to 0$ as $t \to +\infty$.

Let $\exp_\lambda$ denote the *exponential* with parameter $\lambda \in \mathbb{R}$, i.e. $\exp_\lambda : t \in \mathbb{R} \mapsto e^{\lambda t} \in \mathbb{R}$. We will prove that $\exp_\lambda a$, with $0 \neq a \in \mathbb{R}^{\mathtt{w}}$ and $\lambda > 0$ does not belong to $\mathfrak{S}_\Phi^{\text{onesided}}$. Subsequently, we will prove that $\exp_\lambda a \in \mathfrak{S}_\Phi$ with $\lambda < 0$ does belong to $\mathfrak{S}_\Phi^{\text{onesided}}$. In other words, we prove that increasing exponentials do not, and decreasing exponentials in $\mathfrak{S}_\Phi$ do belong to $\mathfrak{S}_\Phi^{\text{onesided}}$. For simplicity, we consider only real exponentials — the complex case is analogous, but requires more complicated notation.

To prove that $\exp_\lambda a$ with $0 \neq a \in \mathbb{R}^{\mathtt{w}}$, $\lambda > 0$, cannot belong to $\mathfrak{S}_\Phi^{\text{onesided}}$, consider the integral

$$\int_{-\infty}^{T} \big(\mathsf{Q}_\Phi(\exp_\lambda a + \exp_{\lambda'} b) - \mathsf{Q}_\Phi(\exp_\lambda a)\big)\, dt$$

with $b \in \mathbb{R}^{\mathtt{w}}$ and $\lambda > \lambda' > 0$. This integral equals

$$2b^\top \Phi(\lambda',\lambda) a \frac{e^{(\lambda+\lambda')T}}{\lambda+\lambda'} + b^\top \Phi(\lambda',\lambda') b \frac{e^{2\lambda' T}}{2\lambda'}. \quad (18)$$

Observability of Φ implies (see corollary 3, item 3 of (Trentelman and Willems, 1997)) that $\Phi(\lambda',\lambda)a$ cannot be identically zero for all λ' with $\lambda > \lambda' > 0$. Take in (18) $\Phi(\lambda',\lambda)a \neq 0$, and choose $b = \alpha a$, with $\alpha \in \mathbb{R}$. It is easy to see that, for a suitable α, (18) approaches $-\infty$ as $T \to +\infty$. We have hence shown that for $w = \exp_\lambda a$ with $0 \neq a \in \mathbb{R}^{\mathtt{w}}$ and $\lambda > 0$, a suitable $\Delta = \exp_{\lambda'} b$ with $b \in \mathbb{R}^{\mathtt{w}}$ and $\lambda' > 0$, yields

$$\int_{-\infty}^{+\infty} \big(\mathsf{Q}_\Phi(w+\Delta) - \mathsf{Q}_\Phi(w)\big)\, dt = -\infty.$$

This Δ does not have left compact support, but both $w(t)$ and $\Delta(t) \to 0$ as $t \to -\infty$. By suitably approximating Δ by a Δ' with left compact support, we obtain

$$\int_{-\infty}^{+\infty} \big(\mathsf{Q}_\Phi(w+\Delta') - \mathsf{Q}_\Phi(w)\big)\, dt = -\infty.$$

This shows that $\mathfrak{S}_\Phi^{\text{onesided}}$ does not contain increasing exponentials.

Next, we set out to show that decreasing exponentials in $\mathfrak{S}_\Phi^{\min}$ belong to $\mathfrak{S}_\Phi^{\text{onesided}}$. Assume therefore that $\exp_\lambda a, a \neq 0$, satisfies $H_\Phi(\frac{d}{dt})\exp_\lambda a = 0$. Since H_Φ is Hurwitz, $\lambda < 0$. Let $\Delta \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ have left compact support. We need to show that

$$\int_{-\infty}^{+\infty} \big(\mathsf{Q}_\Phi(\exp_\lambda a + \Delta) - \mathsf{Q}_\Phi(\exp_\lambda a)\big)\, dt \geq 0. \quad (19)$$

Note that we may as well assume that Δ has support on $[0,+\infty)$. Now replace $\exp_\lambda a$ by a trajectory $\hat{w} \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of left compact support (hence $\hat{w}(t) = 0$ for t sufficiently small) that coincides with $\exp_\lambda a$ for $t \geq 0$, i.e. $\hat{w}(t) = (\exp_\lambda a)(t)$ for $t \geq 0$. Since this does not change the integrand of (19), we need to show that

$$\int_{-\infty}^{+\infty} \big(\mathsf{Q}_\Phi(\hat{w}+\Delta) - \mathsf{Q}_\Phi(\hat{w})\big)\, dt \geq 0. \quad (20)$$

We will show that (20) is either $+\infty$, or finite and ≥ 0. This integral is the difference of

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(\hat{w}+\Delta)\, dt$$

and the finite integral

$$\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(\hat{w})\, dt.$$

Since Q_Φ is strictly half-line positive, there exists $\varepsilon > 0$ such that

$$\int_{-\infty}^{T} \mathsf{Q}_\Phi(\hat{w}+\Delta)\,dt \geq \varepsilon \int_{-\infty}^{T} \mathsf{Q}_{|\Phi|}(\hat{w}+\Delta)\,dt$$

for all $T \in \mathbb{R}$. There are 2 possibilities: either

$$\int_{-\infty}^{+\infty} \mathsf{Q}_{|\Phi|}(\hat{w}+\Delta)\,dt = +\infty,$$

in which case (20) is also $+\infty$, or

$$\int_{-\infty}^{+\infty} \mathsf{Q}_{|\Phi|}(\hat{w}+\Delta)\,dt < +\infty. \qquad (21)$$

We now consider the case that (21) holds. This implies, by lemma 6, that $X^{\mathrm{im}}_{G_{|\Phi|}}(\hat{w}+\Delta)(t) \to 0$ as $t \to +\infty$. Hence $X^{\mathrm{im}}_{G_{|\Phi|}}(\exp_\lambda a + \Delta)(t) \to 0$ as $t \to +\infty$. But, by (2), $X^{\mathrm{im}}_{G_\Phi} = X^{\mathrm{im}}_{G_{|\Phi|}}$. Therefore $X^{\mathrm{im}}_{G_\Phi}(\exp_\lambda a + \Delta)(t) \to 0$ as $t \to +\infty$. Now integrate (15) twice from 0 to T, once with $w = \exp_\lambda a + \Delta$ and once with $w = \exp_\lambda a$, to obtain

$$\begin{aligned}\int_0^T \big(\mathsf{Q}_\Phi(\exp_\lambda a+\Delta) - \mathsf{Q}_\Phi(\exp_\lambda a)\big)\,dt \\ = ||\big(X^{\mathrm{im}}_{G_\Phi}(\frac{d}{dt})(\exp_\lambda a+\Delta)\big)(T)||_K^2 \\ - ||X^{\mathrm{im}}_{G_\Phi}(\frac{d}{dt})(\exp_\lambda a)\big)(T)||_K^2 \\ + \int_0^T \big(||H_\Phi(\frac{d}{dt})\Delta||^2\big)\,dt.\end{aligned}$$

Let $T \to +\infty$, and conclude

$$\begin{aligned}\int_0^{+\infty} \big(\mathsf{Q}_\Phi(\exp_\lambda a+\Delta) - \mathsf{Q}_\Phi(\exp_\lambda a)\big)\,dt \\ = \int_0^{+\infty} \big(||H_\Phi(\frac{d}{dt})\Delta||^2\big)\,dt \geq 0.\end{aligned}$$

The proof is complete. ∎

Recapitulating, we have shown that the functional $\int_{-\infty}^{+\infty} \mathsf{Q}_\Phi(w)\,dt$ has the solutions of $\Phi(-\frac{d}{dt},\frac{d}{dt})w = 0$ as its stationary points with respect to compact support variations. These stationary points are local minima with respect to compact support variations if and only if Q_Φ is average non-negative. Roughly speaking, there are local minima with respect to left compact support variations if and only if Q_Φ is (strictly) half-line non-negative. If this is the case, then the stable stationary trajectories are the local minima with respect to left compact support variations.

Variational principles are about stationary points. In optimal control, on the other hand, stability is crucial. Stability can be 'artificially' imposed, by looking for the stable local minima with respect to compact support variations. But, as we have seen in theorem 14, stability can be made to emerge 'naturally' through the local minima with respect to *left* compact support variations.

We end this section with a remark about the dimensions of the behaviors. If $\det(\Phi(-\xi,\xi)) = 0$, then $\dim(\mathfrak{S}_\Phi) = \infty$, and therefore, if, in addition, (14) holds, $\dim(\mathfrak{S}_\Phi^{\min}) = \infty$. In the case $\det\big(\Phi(-\xi,\xi)\big) \neq 0$,

$$\begin{aligned}\dim(\mathfrak{S}_\Phi) = \mathrm{degree}\big(\det\big(\Phi(-\xi,\xi)\big)\big) \\ \leq 2\,\mathrm{rowdim}(X^{\mathrm{im}}_{G_\Phi})\end{aligned}$$

Using the results of theorem 5, it can furthermore be shown that, if (3) holds, and

$$\mathrm{degree}\big(\det\big(\Phi(-\xi,\xi)\big)\big) = 2\ \mathrm{rowdim}(X^{\mathrm{im}}_{G_\Phi})$$

then there passes, through every non-zero initial state $X^{\mathrm{im}}_{G_\Phi}(\frac{d}{dt})w(0)$, exactly one element of $\mathfrak{S}_\Phi^{\min}$ that is stable ($w(t) \to 0$ as $t \to +\infty$) and exactly one that is anti-stable ($w(t) \to 0$ as $t \to -\infty$). If

$$\mathrm{degree}\big(\det\big(\Phi(-\xi,\xi)\big)\big) < 2\ \mathrm{rowdim}(X^{\mathrm{im}}_{G_\Phi}),$$

then replace 'exactly one' by 'at most one'.

6. LQ OPTIMAL TRAJECTORIES

In this section, we consider LQ trajectory optimization with initial and terminal conditions. Guided by controllability and image representations, we assume in this section again that $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is free. The question which we consider is to minimize (or find the infimum of)

$$\int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt \qquad (22)$$

under certain conditions on w and its derivatives at $t = 0$, and on their limits as $t \to +\infty$.

The following is a general way of specifying initial conditions. Let $I \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and $a \in \mathbb{R}^{\mathrm{rowdim}(I)}$ be given, and consider the initial conditions

$$I(\frac{d}{dt})w(0) = a. \qquad (23)$$

Effectively, this constrains the initial values of w and its derivatives to belong to an affine subspace with finite co-dimension. Note that without loss of generality, we can assume that the rows of I are linearly independent over $\mathbb{R}$.

Consider now the problem of finding the minimum or infimum of (22) subject to the initial conditions (23) and possibly conditions on the limits of w and some of its derivatives as $t \to +\infty$. It is clear that a necessary condition for the infimum to be $> -\infty$ is that $\mathfrak{S}_\Phi^{\min}$ is non-empty, i.e. (see theorem 12) that (16) holds. We will assume however that the slightly stronger condition (3) holds. In this case $\Phi(-\xi,\xi)$ admits the Hurwitz factorization (4), leading to H_Φ and, through (5), to Ψ_Φ^-.

Our dynamic LQ minimization problem leads to the following static LQ problem. Consider the quadratic functional $\mathsf{Q}_{\Psi_\Phi^-}(w)(0)$ subject to the constraints (23), with $w(0), \frac{d}{dt}w(0), \frac{d^2}{dt^2}w(0), \ldots$ viewed as independent variables. The infimum of this functional may be $-\infty$. If it is $> -\infty$, e.g. if $\mathsf{Q}_{\Psi_\Phi^-} \geq 0$, then the minimum exists. It is easy to see that the set of minimizing $w(0), \frac{d}{dt}w(0), \frac{d^2}{dt^2}w(0), \ldots$ is then also a linear variety. Denote the infimum and minimum by $\inf(\Psi_\Phi^-, I, a)$ and $\min(\Psi_\Phi^-, I, a)$, respectively. The set of minimizing $w(0), \frac{d}{dt}w(0), \frac{d^2}{dt^2}w(0), \ldots$ is an affine sub-variety of (23). Hence it is given by an affine subspace, which can be expressed by an matrix/vector equation requiring equality of a matrix acting on the initial conditions and a vector. Of course, this matrix and vector depend on Ψ_Φ^-, I, a. We denote this equation by

$$I^\star_{(\Psi_\Phi^-, I, a)}(\frac{d}{dt})w(0) = a^\star_{(\Psi_\Phi^-, I, a)}. \quad (24)$$

Assume, again without loss of generality, that also the rows of $I^\star_{(\Psi_\Phi^-, I, a)} \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ are linearly independent. Note that if e.g. $I = X^{\rm im}_{G_\Phi}$, then this static minimization problem simply yields (23) for (24), since $X^{\rm im}_{G_\Phi}(\frac{d}{dt})w(0)$ determines, by proposition 5, part (1) $\mathsf{Q}_{\Psi_\Phi^-}(w)(0)$.

We introduce a bit of more notation. Let $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. Define the span over $\mathbb{R}$ of the rows of F, and the $\mathbb{R}[\xi]$-module generated by the rows of F by respectively

$$\begin{aligned}
&\mathrm{rowspan}_{\mathbb{R}}(F) := \{f \in \mathbb{R}^{1\times\mathtt{w}}[\xi] \mid \\
&\quad \exists\, \alpha \in \mathbb{R}^{\mathrm{rowdim}(F)} \text{ such that } f = \alpha^\top F\},\\
&\mathrm{rowmodule}_{\mathbb{R}[\xi]}(F) := \{f \in \mathbb{R}^{1\times\mathtt{w}}[\xi] \mid \\
&\quad \exists\, \alpha \in \mathbb{R}^{\mathrm{rowdim}(F)}[\xi] \text{ such that } f = \alpha^\top F\}.
\end{aligned}$$

Theorem 15. **(LQ optimal trajectories with initial and terminal constraints).** Assume that $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ satisfies (3), and define H_Φ and $\mathsf{Q}_{\Psi_\Phi^-}$ by (4) and (5). Let $X^{\rm im}_{G_\Phi}$ induce a minimal state map for the image representation (7) associated with a canonical factorization of Φ.

1. The infimum of $\int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt$ over all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, subject to the initial conditions (23) and the stability conditions $X^{\rm im}_{G_\Phi}(\frac{d}{dt})w(t) \to 0$ as $t \to +\infty$, equals $\inf(\Psi_\Phi^-, I, a)$.

2. It is a minimum if and only if $\inf(\Psi_\Phi^-, I, a) > -\infty$ and

$$\begin{aligned}
&[\,\alpha^\top I^\star_{(\Psi_\Phi^-, I, a)} \in \mathrm{rowmodule}_{\mathbb{R}[\xi]}(H_\Phi),\\
&\qquad \alpha \in \mathbb{R}^{\mathrm{rowdim}(I^\star_{(\Psi_\Phi^-, I, a)})}\,]\\
&\qquad\qquad \Rightarrow [\,\alpha^\top a^\star_{(\Psi_\Phi^-, I, a)} = 0\,]. \quad (25)
\end{aligned}$$

3. The minimizing trajectory is unique iff
(i) $\inf(\Psi_\Phi^-, I, a) > -\infty$,
(ii) (25) holds, and
(iii) $\mathrm{rowspan}_{\mathbb{R}}(I^\star_{(\Psi_\Phi^-, I, a)}) + \mathrm{rowmodule}_{\mathbb{R}[\xi]}(H_\Phi) = \mathbb{R}^{1\times\mathtt{w}}[\xi]. \quad (26)$

Proof. Integrating (15) from $t = 0$ to $t = +\infty$, yields, since $X^{\rm im}_{G_\Phi}(\frac{d}{dt})w(t) \to 0$ as $t \to +\infty$,

$$\int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt = -\mathsf{Q}_{\Psi_\Phi^-}(w)(0) + \int_0^{+\infty} ||H_\Phi(\frac{d}{dt})w||^2\,dt.$$

Part 1 of the theorem follows immediately from this expression and the following lemma.

Lemma 16. Let $H \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be Hurwitz. Assume that the rows of I are linearly independent over $\mathbb{R}$. Then the infimum of $\int_0^{+\infty} ||H(\frac{d}{dt})w||^2\,dt$ over all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, and subject to initial conditions as (23), equals 0.

Proof of lemma 16. This result follows from proposition 11 (v) of (Willems, e.a., 1986).

We now turn to parts 2 and 3 of theorem 15. This requires imposing conditions such that $H_\Phi(\frac{d}{dt})w = 0$ subject to (24) has a (unique) solution. This leads to the question under what conditions there exists a (unique) solution to a system of differential equations as $R(\frac{d}{dt})w = 0$, which satisfies initial conditions like (23)? This is covered in the following lemma.

Lemma 17. There exists a solution to ($\mathfrak{Ker}$) that satisfies (23) if and only if

$$[\,\alpha^\top I \in \mathrm{rowmodule}_{\mathbb{R}[\xi]}(R), \alpha \in \mathbb{R}^{\mathrm{rowdim}(I)}\,] \Rightarrow [\,\alpha^\top a = 0\,].$$

This solution is unique if and only if, in addition, the behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ described by ($\mathfrak{Ker}$) is autonomous, and

$$\mathrm{rowspan}_{\mathbb{R}}(I) + \mathrm{rowmodule}_{\mathbb{R}[\xi]}(R) = \mathbb{R}^{1\times\mathtt{w}}[\xi].$$

Proof of lemma 17. The proof is straightforward. It will be given in (Willems and Valcher, 2005). ■

The proof of part 2 and 3 of theorem 15 is an immediate consequence of this lemma. ■

Theorem 18. **(LQ optimal trajectories with initial constraints).** Assume that $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is observable and strictly positive, and define H_Φ and $\mathsf{Q}_{\Psi_\Phi^-}$ by (4) and (5). The infimum of $\int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt$ over all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, subject to the initial conditions (23), equals $\inf(\Psi_\Phi^-, I, a)$. It is a minimum if and only if in addition (25) holds. This minimum is unique if and only if, in addition, (25) and (26) hold.

Proof. It follows from lemmas 6 and 17 that (22) $= +\infty$, unless $X^{\rm im}_{G_\Phi}(\frac{d}{dt})w(t) \to 0$ as $t \to +\infty$. The result then follows from theorem 15. ■

Theorem 18 is rather involved due to the generality of the initial conditions (23) considered. If the initial conditions specify $X^{\rm im}_{G_\Phi}(\frac{d}{dt})w(0) = x_0$, as is the case in the classical LQ theory, then we obtain the following result.

Theorem 19. **(LQ optimal trajectories with initial and terminal state constraints).** Assume that $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ satisfies (3), and define H_Φ and $\mathsf{Q}_{\Psi_\Phi^-}$ by (4) and (5). Let $X_{G_\Phi}^{\mathrm{im}}$ induce a minimal state map for the image representation (7) associated with a canonical factorization of Φ. Assume also that (14) holds.

Then the infimum of $\int_0^{+\infty} \mathsf{Q}_\Phi(w)\,dt$ over all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, subject to the initial conditions $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w(0) = x_0$ and the stability condition $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w(t) \to 0$ as $t \to +\infty$, is equal to $\mathsf{Q}_{\Psi_\Phi^-}w(0)$, which is uniquely specified by x_0. If, in addition, (14) holds, then there exists a unique minimum, given by the unique solution of

$$H_\Phi(\frac{d}{dt})w = 0, \quad X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w(0) = x_0$$

Proof. This is a special case of theorem 18. ∎

Of course, it is again possible to replace the stability condition $X_{G_\Phi}^{\mathrm{im}}(\frac{d}{dt})w(t) \to 0$ as $t \to +\infty$ by strict positivity of Φ. Note that in all the results the stability requirement can always be replaced by $\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}}w(t) \to 0$ as $t \to +\infty$, for all $\mathtt{k} \in \mathbb{N}$.

7. CONCLUSIONS

In this paper we have discussed the LQ problem from a behavioral point of view. We assume throughout that the plant is controllable and given in image representation. The performance functional is the integral of a QDF.

We obtained a differential equation for the stationary trajectories. This immediately yields a specification of the stable stationary behavior. The stationary trajectories are local minima with respect to compact support variations if and only if the QDF is average non-negative. Furthermore, if the QDF is strictly half-line positive, then the stable stationary trajectories are the local minima with respect to left compact support variations.

We also considered the problem of characterizing the optimal LQ trajectories with initial and terminal constraints. This leads to an auxiliary static LQ problem involving the initial condition constraints and the minimal storage function, a QDF. The dynamic LQ problem has a minimum if and only if this static problem has a minimum, and in addition one of the minimizing elements is a possible initial condition for a differential equation involving the Hurwitz factor induced by the QDF.

In a future publication (Willems and Valcher, 2005), we will obtain algorithms that start from a kernel, image, latent variable, or state representation of the plant, and return a specification of the stationary or optimal behavior. We will also discuss the synthesis problem, i.e. achieving the optimal behavior as the intersection of the plant behavior and the behavior of a controller.

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