FINITE-DIMENSIONAL CONTROLLER DESIGN FOR NONLINEAR EVOLUTION EQUATIONS USING INERTIAL FORMS

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Abstract: This article deals with the problem of designing finite dimensional controllers for a class of nonlinear infinite dimensional systems for which inertial manifolds are known to exist. The synthesis of the controller is based on a finite-dimensional, inertial form approximation of the plant. The controller guarantees exponential convergence of the system state to zero.

Keywords: Partial differential equations, Distributed-parameter systems, Nonlinear Systems, Exponentially stable

1. INTRODUCTION

The problem of stabilising the dynamics of infinite dimensional systems is relevant to many practical engineering systems including chemical reactors (Christofides, 2000), flexible robot arms and space structures, fluid flow and smart material structures (Banks *et al.*,1996).

For obvious reasons only controllers that are finite dimensional can be implemented in practice. The finite dimensional controller can be obtained as a discretisation of an infinite dimensional controller or directly by designing a controller based on a finite dimensional approximation of the infinite dimensional plant.

The direct approach is particularly appealing because many classes of infinite dimensional systems exhibit low-dimensional dynamical behaviour. This behaviour has been related to the existence of a finite dimensional global attractor for the infinite dimensional flow. The global attractor however is in principle very complex and difficult to compute.

A more practical concept, originally introduced in (Foias, *et al.*, 1988a), is that of inertial manifold. The inertial manifold is a smooth, low-dimensional invariant manifold which contains the global attractor and attracts exponentially all the trajectories. Furthermore, the dynamics of the system, when restricted on the inertial manifold, is described by a system of ordinary differential equations, which are known as the inertial form of the corresponding infinite dimensional system.
The existence of an inertial manifold has been proven for certain dissipative evolutionary equations (Foias, *et al.*, 1988a).

The main advantage of the inertial form is that it can be used to carry out detailed simulations, stability and bifurcation analysis for the infinite dimensional system at much lower computational cost than that which is usually associated with standard approximating methods (Foias, *et al.*, 1988b). Many simulation, analysis and control problems related to infinite dimensional systems, which are prohibitively expensive from a computational point of view, can become tractable within the IM framework.

For control applications, the inertial form offers in many ways the optimum finite-dimensional approximation for controller synthesis. In particular, it leads to reduced order controllers and also simplifies stability analysis of the close loop infinite dimensional system, an issue which otherwise is difficult to resolve, especially when the system is nonlinear (Balas,1984,1991).

A number of authors have exploited the inertial manifold theory to address the global stabilisation problem for nonlinear infinite dimensional systems. A good review of existing results in this area can be found in the article by Rosa (2003). The same article also presents the most general approach to date, which uses the concept of inertial manifold.

The approach presented here also applies for a quite large class of semi-linear dissipative evolution problems including well known reaction diffusion equations.

Unlike in the paper by Rosa (2003), where the control law was computed as the fixed point of a contraction mapping, which in practice it is expected to be less robust, this article proposes a nonlinear state feedback controller which does not assume exact cancellation of the nonlinear term. The proposed approach does not require the explicit computation of an approximate inertial manifold used by Christofides and Daoutidis (1997) to synthesize a finite dimensional output feedback controller to exponentially stabilise the closed loop PDE system.

In this work, it is assumed that the inertial form can be identified directly from experimental observations of the process or from data generated by numerical simulations of large scale approximations of the equations.

2. THE EVOLUTION EQUATION

The nonlinear infinite dimensional plant is assumed to be described by the following equations

$$\begin{cases} \dfrac{du}{dt} + Au + F(u) = Bv \\ y = Cu \\ u(0) = u_0 \in H \end{cases} \quad (1)$$

where H is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. In equation (1) Au is a linear unbounded self-adjoint operator on H with domain $D(A)$ dense in H. Furthermore, A is positive and A^{-1} is compact. It follows that A has an orthonormal basis of eigenvectors $\{w_j\}$ and a corresponding set of eigenvalues $\{\lambda_j\}$ that satisfy

$$0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_j \to \infty \text{ as } j \to \infty \quad (2)$$

It should be noted that there is no loss of generality in assuming that A is positive (see Sell and You, 2002: Section 4.7). In practice, this means that the results will apply for the case when A has negative (unstable) eigenvalues.

Furthermore, the fractional powers A^{α} of A and the fractional power spaces $V^{2\alpha} := D(A^{\alpha})$ are defined for all $\alpha \in$ IR. V^{α} is a Hilbert space endowed with the scalar product $(u,v)_{\alpha}=(A^{\alpha/2}u,\ A^{\alpha/2}v)$ and norm $|u|_{\alpha}=(u,u)_{\alpha}^{1/2}$ where $u,v \in D(A^{\alpha/2})$.

The control and observation operators are assumed to be the spectral projectors of the linear part of the equation that is $P=B=C=P_N$ where $P_N: H \to P_N H$ is the orthogonal projection on the finite dimensional subspace $P_N H$ spanned by the first N eigenvectors $w_1,\dots,w_N$ of A. Let $Q=Q_N$ be the orthogonal complement of P_N , i.e. $Q_N=I- P_N$.

In practice, the infinite dimensional system will be controlled using a finite-dimensional feedback law $v=v_N= Pv(y_N)$ obtained as a function of the finite dimensional output $y_N= P u$.

Here it is assumed that the function $v_N(y_N)$ is chosen such that the nonlinear term

$$F_v(u)=F(u)-v_N(Pu)$$

is bounded

$$|F_v(u)|<c_0 \text{ for all } u \in V^{\alpha} \quad (3a)$$

globally Lipschitz continuous

$$|F_v(u_1)-F_v(u_2)| \le c|u_1- u_2|_{\alpha} \text{ for all } u_1, u_2 \in V^{\alpha} \quad (3b)$$

for some $0 \le \alpha < 2$, and $|F_v(0)|=0$.

In practice, if F_v is locally Lipschitz, the results remain valid locally, for solutions starting in a region around the origin.

For every solution $u(t)$ of (1) it is possible to define $p(t)=Pu(t)$ and $q(t)=Qu(t)$. It follows that p and q are solutions of the following system of equations

$$\begin{cases} \dfrac{dp}{dt} + Ap + PF(p+q) = v_N(y_N) \\ \dfrac{dq}{dt} + Aq + QF(p+q) = 0 \\ y_N = Pu = p \end{cases} \quad (4)$$

An inertial manifold for (1) is a finite dimensional Lipschitz continuous manifold in $V^{\alpha} \subset H$, which is positively invariant and exponentially attracting (Foias, *et al.*,1988, Sell and You, 2002) .

An inertial manifold for (1) has been shown to exist (Foias, *et al.*,1988, Sell and You, 2002) providing that the gap between λ_N and λ_{N+1} is sufficiently large. More precisely

$$\lambda_{N+1} - \lambda_N \le 2c(\lambda_{N+1}^{\alpha/2} + \lambda_N^{\alpha/2}) \quad (5)$$

The inertial manifold is constructed as a graph $\mathcal{M}$=Graph Φ of a Lipschitz function $\Phi(p): PH \cap V^{\alpha} \to QH \cap V^{\alpha}$.

The function $\Phi(p,v_N)$, which depends on the choice of function v_N, is commonly obtained as the fixed point of a Lyapunov-Perron integral operator

$$[T\Phi](p(0)) = -\int_{-\infty}^{0} e^{AQ\tau} QF(p(\tau) + \Phi(p(\tau), v_N))d\tau \quad (6)$$

where $p(t)=p(\Phi,p_0,t)$ denotes the solution (backward in time) of

$$\frac{dp}{dt}+Ap+PF(p+\Phi(p,v_N))=v_N(p), \quad (7)$$
$$p(0)=p_0\in PH\cap V^{\alpha}$$

Furthermore $(p(t),q(t))$ is a solution of (4) and $u(t)=p(t)+q(t)$ is a solution of (1) where

$$q(t)=\Phi_v(p)=\Phi(p,v_N(p)).$$

The function $\Phi_v:PH\rightarrow QH$ satisfies

$$|\Phi_v(p)|_{\alpha}\le b, \text{ for all } p\in PD(A^{\alpha/2}) \quad (8a)$$

$$|\Phi_v(p_1)-\Phi_v(p_2)|_{\alpha}\le l|p_1-p_2|_{\alpha}, \text{ for all } p_1, p_2\in PD(A^{\alpha/2}) \quad (8b)$$

The existence of an inertial manifold implies that the long term dynamics of the system (1) is described by the inertial form (7).

3. COMPUTING THE INERTIAL MANIFOLD

The theoretical approach for the construction of the inertial manifold does not provide an explicit form of $\Phi(p)$. Moreover since $\Phi(p)$ has infinite dimensional range, in practice, it has to be approximated by a finite dimensional function $\Phi_M(p)$, which, for example, can be computed as in (Foias *et al.,* 1988b) as the fixed point of a finite dimensional contraction mapping or as in (Christofides and Daoutidis, 1997) using a power series expansion. Other numerical approximation schemes can be found in (Demengel and Ghidaglia,1991) and (Robinson, 2002).

Specifically, following the scheme proposed in (Foias, 1988b), for example, the finite dimensional approximation $\Phi_M(p)$ of $\Phi(p)$ could be computed as

$$\Phi_M(p)=-\tau(I+\tau AQ)^{-1}QF(p)$$

where τ is a constant comparable to $1/\lambda_{N+1}$.

A possible solution to capture implicitly the exact mapping $\Phi_v(p)$ would be by direct identification of the nonlinear term from (experimental) input-output data v_N and y_N.

A similar identification method to that proposed by Coca and Billings (2003), which is based on finite element approximations, could be used to estimate equation (7) based on the spectral projection of the solution $y_N=Pu$ for a given forcing function $v_N\in PH\cap V^{\alpha}$.

Here it assumed that the conditions (2),(3a,b) and (4) are valid uniformly with respect to the forcing function v_N, which ensures the existence of an inertial manifold $\mathcal{M}(v_N)$ and of a fixed point Φ_v of (6) that depends continuously on v_N (Foias, *et al.*,1988).

The identification procedure could be used to estimate

$$\frac{dp}{dt}+Ap+P\hat{F}(\Phi_v)(p)=v_N(t) \quad (9)$$

where

$$p(x,t)=\sum_{i=1}^{N}p_i(t)w_i(x)$$
$$v_N(x,t)=\sum_{i=1}^{N}v_i(t)w_i(x)$$

The advantage of this approach is that (9), in principle, would represent the exact inertial form (7) where $\Phi_v(p)$ is implicitly modelled by

$$P\hat{F}(\Phi_v)(p)\overset{def}{=}PF(p+\Phi(p,v(p)))$$

Another advantage is that this method could be used when an accurate PDE model of the system is not available.

The system identification approach can also be used to construct an arbitrarily accurate approximation of the inertial form using data obtained by numerical simulation of a large scale approximation of the original PDE.

Because of the reduced computational costs involved (there is no need to compute an approximation of $\Phi_v(p)$ at every time step), the identification approach would be better suited for real-time implementation than the other alternative, which involves the explicit computation of Approximate Inertial Manifolds based on the original equations.

4. THE FINITE DIMENSIONAL CONTROLLER

The aim is to design, using the inertial form (7), a feedback control law such that the origin of the infinite dimensional system is globally exponentially stable.

Unlike in (Rosa, 2003) where the control law was computed as the fixed point of a contraction mapping, which owing to parameter uncertainty and computational errors means that in practice the controller may not be very robust, a nonlinear state feedback controller is proposed here. It is shown that the nonlinear feedback control

$$v_N(y_N)=v_N(Pu)=-Kp-G(p) \quad (10)$$

where $K: PH\cap V^{\alpha}\rightarrow PH\cap V^{\alpha}$, $G(p)\in PH\cap V^{\alpha}$, results in a infinite dimensional closed loop system which is globally exponentially stable in the origin. The role of the linear feedback term K is to assign the eigenvalues to the linear part so that it dominates the resulting nonlinear term

$$P\hat{F}_G(\Phi_v)(p) \overset{def}{=} PF(p+\Phi_v(p))+G(p)$$

whilst ensuring that the spectral gap condition is satisfied.

The main assumptions are as follows:

[A1] $\langle P\hat{F}_G(\Phi_v)(p_1) - P\hat{F}_G(\Phi_v)(p_2), p_1 - p_2\rangle_\alpha + c_2|p_1 - p_2|^2_\alpha \geq 0$, $\forall\, p_1, p_2 \in PH \cap V^\alpha$

[A2] $|F_G(u_1) - F_G(u_2)|_\alpha \leq (c-\lambda_N)|u_1 - u_2|_\alpha$, $\forall u_1$, $u_2 \in H \cap V^\alpha$

[A3] $P\hat{F}_v(\Phi_v)(0) = 0$

[A4] $\langle (A+K)p, p\rangle_\alpha \geq c_3|p|^2_\alpha$, with $c_2 \leq c_3$,$\forall p \in PD(A^{\alpha/2})$

[A5] $\max_{j=1,N} (\lambda'_j) \leq \lambda_N$ where λ'_j represent the eigenvalues of $A+K$ and λ_N is the Nth eigenvalue of A.

Assumptions [A1-A5] are achieved by design, i.e. by appropriately choosing K and G. Assumption [A1] ensures that the nonlinear term satisfies a global sector condition. Assumptions [A2-A5] guarantee existence of an inertial manifold for the closed-loop infinite dimensional system. The fact that exact cancellation of the nonlinear term is not required makes this design procedure more robust.

The control input calculated from (10) can be expressed in terms of the first N eigenvectors of A. In order to implement the control however, v_N has to be represented as $v_N=\Sigma b_i(x)v'_i(t)$ where b_i describe how the action of the ith actuator is distributed in the spatial domain. That is, the actual actuator actions need to be determined subsequently from v_N. The number and distribution of actuators could be optimised based on the number and characteristics of the eigenfunctions w_j and properties of b_i.

The following theorem provides a stability result for the infinite dimensional closed-loop system

$$\begin{cases} \dfrac{du}{dt} + Au + F(u) = -KPu - G(Pu) \\ u(0) = u_0 \in H \cap V^\alpha \end{cases} \qquad (11)$$

where K: $PH \cap V^\alpha \rightarrow PH \cap V^\alpha$, $G \in PH \cap V^\alpha$.

Theorem 1: *For the closed-loop infinite dimensional system of* Eq. (11)*, resulted from applying the control law in* Eq (10), *for which assumptions* [A1-A5] *hold, to the original PDE system in* Eq. (1), *the origin is globally exponentially stable.*

Proof: The closed-loop system (10) admits an inertial manifold as the graph of $\Phi_v(p(t))$ which is the unique fixed point of the operator

$$T[\Phi_v](p(0)) = -\int_{-\infty}^{0} e^{AQ\tau} QF(p(\tau)+\Phi_v(p(\tau)))d\tau$$

where $p(t)$ is the unique solution ($t\rightarrow-\infty$) of

$$\begin{cases} \dfrac{dp}{dt} + Ap + F(p+\Phi_v(p)) = -Kp - G(p) \\ p(0) = p_0 \in PH \cap V^\alpha \end{cases}$$

This follows from [A5], which states that the eigenvalue gap for $(A+K)$ is the same as for A. This also implies that $|KPu|_\alpha < \lambda_N |Pu|_\alpha$ so that using [A2] we have $F_G+KPu=F(u)+G(Pu)+KPu$ is globally Lipschitz with constant c. It follows that the spectral gap condition (5) holds.

One can check that all conditions in (Foias, *et al.*,1988;Theorem 2.1) are satisfied and hence the long term dynamics of closed-loop system (11) is described by

$$\frac{dp}{dt} + (A+K)p + PF(p+\Phi_v(p)) + G(p) = 0 \qquad (12)$$

where as before $PAu=APu=p$. By taking the scalar product of the second equation with p it follows that

$$\frac{1}{2}\frac{d}{dt}|p|^2 + \langle (A+K)p, p\rangle + \langle PF(p+\Phi_v(p)) + G(p), p\rangle = 0 \qquad (13)$$

Using [A1] with $p_1=p$ and $p_2=0$, [A3] and [A4] it is easy to show that

$$\langle (A+K)p, p\rangle + \langle PF(p+\Phi_v(p)) + G(p), p\rangle \geq (c_3 - c_2)|p|^2$$

where c_3-c_2>0. Consequently one has

$$\frac{1}{2}\frac{d}{dt}|p|^2 \leq -(c_3 - c_2)|p|^2 \qquad (14)$$

which implies that

$$|p(t)| \leq |p(0)|\, e^{-(c_3-c_2)t} \qquad (15)$$

Since we assumed that $F(0)=0$ then $u(t)=0$ is a long term solution (with $t\rightarrow\infty$) of (11). Therefore the pair (p=0,q=0) should belong to the inertial manifold that is $q(t)=\Phi_v(p(t))$ for $q(t)=p(t)=0$. It follows that $\Phi_v(0)=0$.

The asymptotic completeness property of the inertial manifold states that for any trajectory starting off the inertial manifold, the distance from the manifold decays at an exponential rate dist($u(t;u_0)$,$\mathcal{M}$) $\leq k_1(u_0)e^{-\mu t}$ (Foias, 1989). Using the Lipschitz property of Φ_v (8b) it is easy to show that

$$|q(t) - \Phi_v(p(t))| \leq k_2(l)(|q(t) - \Phi_v(p')| + |p - p'|) \qquad (16)$$

for any $p' \in \mathcal{M}$ where k_2 is a positive constant which depends of l. If follows that

$$|q(t) - \Phi_v(p(t))| \leq k_1(u_0)k_2(l)e^{-\mu t} \qquad (17)$$

which implies

$$|q(t)| = |q(t) - \Phi_v(p(t)) + \Phi_v(p(t))| \le |q(t) - \Phi_v(p(t))| + |\Phi_v(p(t))| \le k_3(l,u_0)e^{-\gamma t} \quad (18)$$

where $\gamma=\min(\mu,c_3\text{-}c_2)$. Finally,

$$|u(t)| = |p(t)| + |q(t)| \le k(l,u_0)e^{-\gamma t} \quad (19)$$

which means that $u(t)=0$ is globally exponentially stable.

Remark 1: In many practical cases, F is locally Lipschitz on a domain $\Omega_\rho \subset H$ which contains the origin and possibly the global attractor of (1). In that case, it is possible to prove exponential stability of the system for any initial condition inside any compact subset of Ω_ρ. A typical example is the reaction-diffusion system involving a polynomial nonlinearity, $F(u)=u(1-u)(u-a)$, for $0<a<1$.

Remark 2: The stability result in Theorem 1 will hold for any alternative control strategy which stabilises the inertial form and guarantees existence of the inertial manifold for the closed-loop infinite dimensional system.

5. CONCLUSIONS

This paper has shown that nonlinear infinite dimensional systems for which inertial manifolds are known to exists, are globally/locally exponentially stabilisable using finite dimensional controllers based on the inertial form. The proposed methodology does not require the explicit computation of the inertial manifold, which is often a requirement of other approaches. A numerical study is currently under way.

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