

ROBUST STABILITY ANALYSIS FOR STRUCTURED UNCERTAINTIES WITH BOUNDED VARIATION RATES

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Abstract: Robust stability analysis problem is considered for the feedback interconnection of a causal and stable linear time-invariant system with causal and stable linear time-varying block-diagonal structured uncertainties having sub-blocks of bounded norms and variation rates. The problem is studied within the recently developed IQC framework of robust stability analysis, and based on a generalized version of the swapping lemma, a multiplier class capturing the considered uncertainties is obtained and thus a robust stability test is developed in terms of an LMI feasibility problem. *Copyright ©2005 IFAC*

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1. INTRODUCTION

There has been significant amount of research work viewing the controller design problem as a design that is to be performed for a nominal system, which might be affected by some perturbations (uncertainties), and the wide domain of these efforts is by now commonly referred to as robust control, with the robust stability analysis constituting a major part. In this paper, we consider the robust stability analysis problem for structured linear time-varying (LTV) uncertainties with bounded norms and rates-of-variation. Our interest in this problem is intrigued mainly by the results of Poolla and Tikku [1995]; Jönsson and Rantzer [1996], and we are especially interested in developing nonconservative and numerically efficient robust stability analysis tests. For -at least a partial- achievement of this goal, we employ the IQC approach of Megretski and Rantzer [1997] to robust stability analysis and make use of LMI methods. We start with some preliminaries, followed by the description of the considered problem and the relevant literature. Section 4 summarizes the IQC approach to robust stability analysis and Section 5 presents our contributions. We also provide an example, after which we conclude by some final remarks.

2. NOTATION AND PRELIMINARIES

We work in a discrete-time setting with square-summable sequences, $x \in \ell_2^n$, as signals and linear operator maps as systems (see e.g. Willems [1971] for some preliminaries). We treat real-valued matrices, $X \in \mathbb{R}^{m \times n}$; real-rational transfer functions, $H \in \mathcal{RH}_\infty^{m \times n}$; and bounded linear maps, $T \in \mathcal{L}^{m \times n}$, from ℓ_2^n into ℓ_2^m , in a unified setting all as operators. In particular, we refer to the set of symmetric positive-definite matrices by $\mathbb{P}^{m \times m}$; to the set of stable and causal transfer functions by $\mathcal{RH}_\infty^{m \times n}$; and to the set of causal and unity norm-bounded linear operators by $\mathcal{BL}^{m \times n} \triangleq \{T \in \mathcal{L}^{m \times n} : \|T\|_{i2} \leq 1\}$, where $\|\cdot\|_{i2}$ denotes the induced ℓ_2 -norm. We denote the identity operator of size m by I_m ; right shift (delay) operator of size m by $Z_m^{-1} \triangleq z^{-1} \otimes I_m$; and the zero operator of size $m \times n$ by $0_{m \times n}$. For a given linear operator $\Delta \in \mathcal{L}^{m \times n}$, we define its **variation** as $\mathcal{V}_\Delta \triangleq \Delta Z_n^{-1} - Z_m^{-1} \Delta$ and refer to $\|\mathcal{V}_\Delta\|_{i2} (\leq 2\|\Delta\|_{i2})$ as the **rate-of-variation of Δ** . For a compact representation of structures, we denote the block-diagonal operators as $[T_i]_{i=1}^\kappa \triangleq \text{diag}[T_1, \dots, T_\kappa]$ and employ the Kronecker product to describe the commuting structures. We make use of the following Kronecker product commutation identity, which will generalize as the swapping lemma in the sequel:

Lemma 1. (Kronecker Product Commutation Identity) Let $T_1 \in \mathbb{R}^{m_1 \times n_1}$ & $T_2 \in \mathcal{L}^{m_2 \times n_2}$. Then we have

$$(I_{m_2} \otimes T_1)(T_2 \otimes I_{n_1}) = (T_2 \otimes I_{m_1})(I_{n_2} \otimes T_1). \quad (1)$$

The identity is valid also for the case $T_1 \in \mathcal{RH}_\infty^{m_1 \times n_1}$ and $T_2 \in \mathcal{RH}_\infty^{m_2 \times n_2}$.

We work mainly in an IQC-based robust stability analysis setting, which is established on some basic notions in inner product spaces (see e.g. Jönsson [2001]). With the inner product of $x, y \in \ell_2^n$ defined as $\langle x, y \rangle \triangleq \sum_t x^T(t)y(t)$, the **adjoint** of an operator $T \in \mathcal{L}^{m \times n}$ is defined as the unique operator, $T^* \in \mathcal{L}^{n \times m}$, for which $\langle Tx, y \rangle = \langle x, T^*y \rangle$, $\forall x \in \ell_2^m, \forall y \in \ell_2^n$. A **self-adjoint** operator, $T = T^* \in \mathcal{L}^{m \times m}$, is described as **positive semi-definite** ($T \geq 0$) if it satisfies $\langle Tx, x \rangle \geq 0, \forall x \in \ell_2^m$. Within the environment defined by these notions, we employ in the sequel the following lemma, which is a slightly modified version of a standard inequality in operator theory:

Lemma 2. (Bounding Lemma) Let $T_1, T_2 \in \mathcal{L}^{m \times n}$ and $X \in \mathbb{P}^{m \times m}$. Then we have

$$\mp (T_1^*T_2 + T_2^*T_1) \leq T_1^*XT_1 + T_2^*X^{-1}T_2. \quad (2)$$

3. PROBLEM FORMULATION

The standard setup of stability analysis is given in Figure 1, where $M \in \mathcal{RH}_\infty^{\eta \times \eta}$ denotes a known causal and stable LTI operator (which is referred to as the plant) and $\Delta \in \mathcal{L}^{\eta \times \eta}$ denotes some causal LTV operator (which is referred to as the uncertainty or the perturbation). The feedback interconnection in Figure 1 is **well-posed** (or in short (M, Δ) is **well-posed**), if, for any $u_1, u_2 \in \ell_{2e}^\eta$, there exist $e_1, e_2 \in \ell_{2e}^\eta$, which depend causally on $u_1, u_2 \in \ell_{2e}^\eta$ (see Willems [1971] for a detailed discussion of well-posedness). We say that the system in Figure 1 is ℓ_2 -**stable**, if we have $e_1, e_2 \in \ell_2^\eta$ whenever $u_1, u_2 \in \ell_2^\eta$. We say that M is **robustly stable against Δ** , if the feedback interconnection of M and Δ is stable for all $\Delta \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ denotes the set of uncertainties. In this setting, we consider the following problem:

Problem 3. Consider the feedback system in Figure 1 with a given causal and stable LTI plant $M \in \mathcal{RH}_\infty^{\eta \times \eta}$, together with causal and stable structured LTV uncertainties Δ that come out of the set $\mathbf{\Delta} = \mathbf{\Delta}_\nu^s \subset \mathcal{BL}^{\eta \times \eta}$, which is defined as

$$\mathbf{\Delta}_\nu^s \triangleq \left\{ [I_{l_i} \otimes \Delta_i]_{i=1}^{\kappa} : \Delta_i \in \mathcal{BL}^{m_i \times m_i}, \|\mathcal{V}_{\Delta_i}\|_{i2} \leq \nu_i; i=1:\kappa, \Delta_i^* = \Delta_i; i=1:\kappa_1 \leq \kappa \right\}, \quad (3)$$

where $s = \{l_i, m_i \times m_i\}_{i=1}^{\kappa}$; $\eta = \sum_{i=1}^{\kappa} l_i m_i$ and $\nu = \{\nu_i\}_{i=1}^{\kappa}$; $\nu_i \in [0, 2]$ describe, respectively, the structure and the maximum rate-of-variation of the uncertainty, and κ_1 denotes the number of self-adjoint channels (considered basically with motivations coming from parametric uncertainties). Determine if M is robustly stable against $\mathbf{\Delta}_\nu^s$.

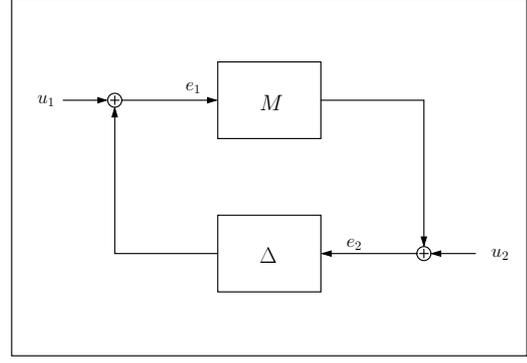


Fig. 1. Standard setup for robust stability analysis.

There are various results available in the literature as partial solutions to Problem 3. It is well-established by now that the structured singular value analysis provides an exact test (under some minor technical assumptions) for robust stability analysis in the linear time-invariant (LTI) perturbations case, i.e. $\mathbf{\Delta} = \mathbf{\Delta}_0^s$, which is of natural interest as a limiting case within the description of Problem 3. Due to the computational complexity of the structured singular value calculation, analysis is based on upper bound calculation using D-scales, and $\exists H, H^{-1} \in \mathcal{RH}_\infty^{\eta \times \eta}$ s.t. $H\Delta = \Delta H, \forall \Delta \in \mathbf{\Delta}$ & $\|HMH^{-1}\|_\infty < 1$, is provided as a sufficient test for robust stability (see Packard and Doyle [1993] for details/extensions). Although D scaling test is known to be conservative for certain LTI perturbations, it has been shown by Shamma [1994] in ℓ_2 framework that the test with constant D-scales, $H \in \mathbb{R}^{\eta \times \eta}$, is exact (under some technical assumptions) in the case of arbitrarily time-varying block diagonal uncertainties with full sub-blocks, i.e. $\mathbf{\Delta} = \mathbf{\Delta}_2^s$ (see also Paganini and Doyle [1994]; Meinsma et al. [2000] for some generalizations). The case of slowly time-varying perturbations was considered by Poolla and Tikku [1995] for uncertainties with full sub-blocks, the rates-of-variation of which are all bounded by the same value and it was shown that frequency-dependent D-scales test is necessary and sufficient in the case of strictly proper plants perturbed with arbitrarily slowly time-varying uncertainties. In particular, a conservative bound of the maximum allowable rate-of-variation was provided in terms of the D-scales. Yet, let alone exactness, an efficient calculation of this bound by some sort of search over D-scales seems to be difficult if at all possible. The problem was considered with concerns about numerical efficiency by Jönsson and Rantzer [1994, 1996] for scalar repeated uncertainties and an IQC-based robust stability test was developed in a continuous-time setting. A slight improvement to this was reported by Helmersson [1999]. There are also some relevant works by Tchernychev and Sideris [1999, 2000], who also study the problem for repeated scalar uncertainties with a slightly different approach.

4. IQC-BASED ROBUST STABILITY ANALYSIS

Among the various others, the IQC approach has recently emerged as a unifying framework of robust stability analysis based on operator methods. We summarize the central result by Megretski and Rantzer [1997] in the following theorem, with adaptations from Jönsson [2001]:

Theorem 4. (IQC Stability, Megretski and Rantzer [1997]) Let the set $\mathbf{\Delta}_\Pi$ be defined as

$$\mathbf{\Delta}_\Pi \triangleq \left\{ \Delta \in \mathcal{L}^{\eta \times \eta} : \begin{bmatrix} I_\eta \\ \Delta \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} I_\eta \\ \Delta \end{bmatrix} \geq 0 \right\}, \quad (4)$$

where the IQC multiplier $\Pi \in \mathcal{RL}_\infty^{2\eta \times 2\eta}$ is a self-adjoint transfer function formed by the sub-blocks $\Pi_{11} = \Pi_{11}^*$, $\Pi_{22} = \Pi_{22}^*$, $\Pi_{12} \in \mathcal{RL}_\infty^{\eta \times \eta}$. The feedback system of Figure 1 is robustly stable against $\mathbf{\Delta}$ if the following conditions are all satisfied:

- (i) $(M, \tau\Delta)$ is well-posed $\forall \tau \in [0, 1]$ and $\forall \Delta \in \mathbf{\Delta}$.
- (ii) $\tau\Delta \in \mathbf{\Delta}_\Pi$, $\forall \tau \in [0, 1]$ and $\forall \Delta \in \mathbf{\Delta}$.
- (iii) There exists an $\epsilon \in \mathbb{R}_+$ such that, $\forall \omega \in [-\pi, \pi]$,

$$\begin{bmatrix} M(e^{j\omega}) \\ I_\eta \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(e^{j\omega}) & \Pi_{12}(e^{j\omega}) \\ \Pi_{12}^*(e^{j\omega}) & \Pi_{22}(e^{j\omega}) \end{bmatrix} \begin{bmatrix} M(e^{j\omega}) \\ I_\eta \end{bmatrix} \leq -\epsilon I_\eta. \quad (5)$$

PROOF. See Megretski and Rantzer [1997]. \square

Remark 5. The well-posedness of the feedback loop will be assured if M is strictly proper. If the IQC multiplier satisfies $\Pi_{11} \geq 0$ & $\Pi_{12} = 0$ or $\Pi_{11} \geq 0$ & $\Pi_{22} \leq 0$, we have $\tau\Delta \in \mathbf{\Delta}_\Pi$ for $\tau \in [0, 1]$, whenever $\Delta \in \mathbf{\Delta}_\Pi$.

After an appropriate factorization of the multiplier, the frequency domain inequality in (5) can be transformed into an LMI by employing the (discrete-time and strict version of the) celebrated Kalman-Yakubovich-Popov (KYP) lemma:

Lemma 6. (Kalman-Yakubovich-Popov Lemma)

Let $\Phi \in \mathcal{RH}_\infty^{m_\Phi \times n_\Phi}$ be a causal and stable transfer function that has a realization of the form $\Phi = \begin{bmatrix} A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{bmatrix} \triangleq C_\Phi(Z_{k_\Phi} - A_\Phi)^{-1}B_\Phi + D_\Phi$, where $A_\Phi \in \mathbb{R}^{k_\Phi \times k_\Phi}$, $B_\Phi \in \mathbb{R}^{k_\Phi \times n_\Phi}$, $C_\Phi \in \mathbb{R}^{m_\Phi \times k_\Phi}$, $D_\Phi \in \mathbb{R}^{m_\Phi \times n_\Phi}$, with A_Φ having all its eigenvalues strictly inside the unit disk, and let $Q = Q^T \in \mathbb{R}^{m_\Phi \times m_\Phi}$ be a symmetric matrix. Then the following conditions are equivalent:

- (i) There exists an $\epsilon \in \mathbb{R}_+$ such that $\Phi^*(e^{j\omega})Q\Phi(e^{j\omega}) \leq -\epsilon I_{n_\Phi}$, $\forall \omega \in [-\pi, \pi]$. (6)
- (ii) There exists a symmetric matrix $P = P^T \in \mathbb{R}^{k_\Phi \times k_\Phi}$ and an $\epsilon \in \mathbb{R}_+$ such that

$$\begin{bmatrix} I_{k_\Phi} & 0 \\ A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{bmatrix}^T \begin{bmatrix} -P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} I_{k_\Phi} & 0 \\ A_\Phi & B_\Phi \\ C_\Phi & D_\Phi \end{bmatrix} \leq -\epsilon I_{n_\Phi}. \quad (7)$$

PROOF. See e.g. Rantzer [1996]. \square

5. AN IQC-BASED ROBUST STABILITY TEST FOR STRUCTURED LTV UNCERTAINTIES

In this section, we present the main contributions of the paper, which basically generalizes the robust stability analysis method developed by Jönsson and Rantzer [1994] using the so-called swapping lemma (see Chou and Tits [1995] for a discrete-time version). We first generalize the swapping lemma, which we then employ together with the bounding lemma to prove our main result. As a direct corollary, we finally provide a robust stability test in terms of an LMI feasibility problem.

5.1 Generalized Swapping Lemma

Lemma 7. (Swapping Lemma) Let $\Delta \in \mathcal{L}^{m_\Delta \times n_\Delta}$ be a causal and bounded linear operator and let $H \in \mathcal{RH}_\infty^{m_H \times n_H}$ be a causal and stable transfer function that admits a realization of the form $H = \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix}$, where $A_H \in \mathbb{R}^{k_H \times k_H}$, $B_H \in \mathbb{R}^{k_H \times n_H}$, $C_H \in \mathbb{R}^{m_H \times k_H}$, $D_H \in \mathbb{R}^{m_H \times n_H}$, with A_H having all its eigenvalues strictly inside the unit disk. With $H_B \in \mathcal{RH}_\infty^{k_H \times n_H}$ and $H_C \in \mathcal{RH}_\infty^{m_H \times k_H}$ defined respectively as

$$H_B \triangleq Z_{k_H}(Z_{k_H} - A_H)^{-1}B_H, \quad (8)$$

$$H_C \triangleq C_H(Z_{k_H} - A_H)^{-1}Z_{k_H}, \quad (9)$$

and with $\mathcal{V}_\Delta \triangleq \Delta Z_{n_\Delta}^{-1} - Z_{m_\Delta}^{-1}\Delta$ denoting the variation of Δ , we have

$$\begin{aligned} & (I_{m_H} \otimes \Delta)(H \otimes I_{n_\Delta}) - (H \otimes I_{m_\Delta})(I_{n_H} \otimes \Delta) \\ & = (H_C \otimes I_{m_\Delta})(I_{k_H} \otimes \mathcal{V}_\Delta)(H_B \otimes I_{n_\Delta}). \end{aligned} \quad (10)$$

PROOF. Let us consider the identity

$$\begin{aligned} & [(I_{k_H} - Z_{k_H}^{-1}A_H) \otimes I_{m_\Delta}][I_{k_H} \otimes (\Delta Z_{n_\Delta}^{-1})] \\ & = (I_{k_H} \otimes \mathcal{V}_\Delta) \\ & + [I_{k_H} \otimes (Z_{m_\Delta}^{-1}\Delta)][(I_{k_H} - Z_{k_H}^{-1}A_H) \otimes I_{n_\Delta}], \end{aligned}$$

which can be verified by adding/subtracting $[I_{k_H} \otimes (Z_{m_\Delta}^{-1}\Delta)]$ to/from the left hand side and rearranging the terms after employing the identity $(A_H \otimes I_{m_\Delta})(I_{k_H} \otimes \Delta) = (I_{k_H} \otimes \Delta)(A_H \otimes I_{n_\Delta})$. Moving the last term to the left hand side, left/right multiplying both sides of the equation, respectively, by $H_C \otimes I_{m_\Delta}$ and $H_B \otimes I_{n_\Delta}$, and finally employing the commutation identities $(I_{k_H} \otimes \Delta)(B_H \otimes I_{n_\Delta}) = (B_H \otimes I_{m_\Delta})(I_{n_H} \otimes \Delta)$, $(C_H \otimes I_{m_\Delta})(I_{k_H} \otimes \Delta) = (I_{m_H} \otimes \Delta)(C_H \otimes I_{n_\Delta})$, we obtain

$$\begin{aligned} & (I_{m_H} \otimes \Delta) \left([C_H(Z_{k_H} - A_H)^{-1}B_H] \otimes I_{n_\Delta} \right) \\ & - \left([C_H(Z_{k_H} - A_H)^{-1}B_H] \otimes I_{m_\Delta} \right) (I_{n_H} \otimes \Delta) \\ & = (H_C \otimes I_{m_\Delta})(I_{k_H} \otimes \mathcal{V}_\Delta)(H_B \otimes I_{n_\Delta}). \end{aligned}$$

Adding/subtracting $(I_{m_H} \otimes \Delta)(D_H \otimes I_{n_\Delta}) = (D_H \otimes I_{m_\Delta})(I_{n_H} \otimes \Delta)$ to/from the left hand side, we end up with (10). \square

Remark 8. The form of the swapping lemma for scalar repeated uncertainties, as it appears in Jönsson and Rantzer [1996]; Chou and Tits [1995], can be recovered from (10) as a special case corresponding to $m_\Delta = n_\Delta = 1$. Another special case deserving interest is the case of full-block uncertainty considered with a repeated scalar LTI system, which is basically the case corresponding to $m_H = n_H = 1$.

Remark 9. When a block-diagonal structured uncertainty, $\Delta = [I_{l_i} \otimes \Delta_i]_{i=1}^{\kappa}$; $\Delta_i \in \mathcal{L}^{m_i \times n_i}$, is considered together with a transfer function that has a structure of the form $H = [H_i \otimes I_{m_i}]_{i=1}^{\kappa}$; $H_i \in \mathcal{RH}_\infty^{q_i \times l_i}$, application of (10) for each block leads to

$$\Delta_1 H_r - H \Delta = H_C \tilde{\Delta} H_B, \quad (11)$$

where H_B and H_C are defined (consistently with (8) and (9)) as $H_B = [H_{iB} \otimes I_{n_i}]_{i=1}^{\kappa}$; $H_{iB} \in \mathcal{RH}_\infty^{k_i \times l_i}$, $H_C = [H_{iC} \otimes I_{m_i}]_{i=1}^{\kappa}$; $H_{iC} \in \mathcal{RH}_\infty^{q_i \times k_i}$ respectively; H_r and Δ_1 are given by $H_r = [H_i \otimes I_{n_i}]_{i=1}^{\kappa}$ and $\Delta_1 = [I_{q_i} \otimes \Delta_i]_{i=1}^{\kappa}$; and $\tilde{\Delta}$ is an operator that is formed with the repeated forms of the variations of the sub-blocks of Δ according to $\tilde{\Delta} = [I_{k_i} \otimes \mathcal{V}_{\Delta_i}]_{i=1}^{\kappa}$. We would like to note that an implicit version of the swapping lemma (based on the commutation properties of the realization matrices of the LTI system with the LTV uncertainty) is given by Helmersson [1999].

5.2 Main Result

Theorem 10. Consider the setup of Problem 3. Assume that $(M, \tau\Delta)$ is well-posed for all $\Delta \in \Delta_\nu^s$ and $\tau \in [0, 1]$. Then, M is robustly stable against Δ_ν^s if (5) is satisfied for some $\Pi \in \Pi_\nu^{s^c}$, with the multiplier set $\Pi_\nu^{s^c}$ being defined as

$$\Pi_\nu^{s^c} \triangleq \{\Pi_R + \Pi_S : \Pi_R \in \Pi_R, \Pi_S \in \Pi_S\}, \quad (12)$$

where

$$\Pi_R \triangleq \left\{ \begin{array}{l} \Pi_R = \begin{bmatrix} \Pi_{11} & 0 \\ 0 & \Pi_{22} \end{bmatrix} : \\ \Pi_{11} = R^* X_{R1} R + R_B^* V_R Y_R V_R R_B \\ \Pi_{22} = R^* X_{R2} X_{R1}^{-1} X_{R2} R - 2R^* X_{R2} R \\ \quad + R^* X_{R2} R_C V_R Y_R^{-1} V_R R_C^* X_{R2} R \\ R = [R_i \otimes I_{m_i}]_{i=1}^{\kappa}; R_i \in \mathcal{RH}_\infty^{q_i \times l_i} \\ X_{R1} = [X_{R1i} \otimes I_{m_i}]_{i=1}^{\kappa}; X_{R1i} \in \mathbb{P}^{q_i \times q_i} \\ X_{R2} = [X_{R2i} \otimes I_{m_i}]_{i=1}^{\kappa}; X_{R2i} \in \mathbb{P}^{q_i \times q_i} \\ Y_R = [Y_{Ri} \otimes I_{m_i}]_{i=1}^{\kappa}; Y_{Ri} \in \mathbb{P}^{k_i \times k_i} \end{array} \right\} \quad (13)$$

$$\Pi_S \triangleq \left\{ \begin{array}{l} \Pi_S = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^T & 0 \end{bmatrix} : \\ \Pi_{11} = E^T S_B^* V_S Y_S V_S S_B E \\ \quad + E^T X_S S_C V_S Y_S^{-1} V_S S_C^* X_S^T E \\ \Pi_{12} = E^T (X_S S - S^* X_S^T) E \\ S = [S_i \otimes I_{m_i}]_{i=1}^{\kappa_1}; S_i \in \mathcal{RH}_\infty^{q_i \times l_i} \\ X_S = [X_{Si} \otimes I_{m_i}]_{i=1}^{\kappa_1}; X_{Si} \in \mathbb{R}^{l_i \times q_i} \\ Y_S = [Y_{Si} \otimes I_{m_i}]_{i=1}^{\kappa_1}; Y_{Si} \in \mathbb{P}^{k_i \times k_i} \end{array} \right\} \quad (14)$$

Here; R_B, R_C, S_B, S_C are defined for $R = [R_i \otimes I_{m_i}]_{i=1}^{\kappa}$ and $S = [S_i \otimes I_{m_i}]_{i=1}^{\kappa_1}$ (consistently with (8) and (9)) as $R_B = [R_{iB} \otimes I_{m_i}]_{i=1}^{\kappa}$; $R_{iB} \in \mathcal{RH}_\infty^{k_i \times l_i}$, $R_C = [R_{iC} \otimes I_{m_i}]_{i=1}^{\kappa}$; $R_{iC} \in \mathcal{RH}_\infty^{q_i \times k_i}$, $S_B = [S_{iB} \otimes I_{m_i}]_{i=1}^{\kappa_1}$; $S_{iB} \in \mathcal{RH}_\infty^{q_i \times l_i}$, $S_C = [S_{iC} \otimes I_{m_i}]_{i=1}^{\kappa_1}$; $S_{iC} \in \mathcal{RH}_\infty^{q_i \times k_i}$; V_R and V_S are matrices that contain the information on rate-of-variation bounds, with their definitions being $V_R \triangleq [\sqrt{\nu_i} I_{k_i} \otimes I_{m_i}]_{i=1}^{\kappa}$ & $V_S \triangleq [\sqrt{\nu_i} I_{k_i} \otimes I_{m_i}]_{i=1}^{\kappa_1}$; and E is a matrix that is introduced to assure dimensional compatibility by extending the relevant operators with zeros according to $E \triangleq [I_{\eta_1} \ 0_{\eta_1 \times \eta_2}]$ with $\eta_1 = \sum_{i=1}^{\kappa_1} l_i m_i$ and $\eta_2 = \sum_{i=\kappa_1+1}^{\kappa} l_i m_i = \eta - \eta_1$.

PROOF. The proof is based on an application of the IQC stability theorem and follows somewhat parallel lines to the work of Jönsson and Rantzer [1996]. We prove the theorem by showing that, for all $\Delta \in \Delta_\nu^s$ and all $\tau \in [0, 1]$, we have $\tau\Delta \in \Delta_\Pi$ for any $\Pi \in \Pi_R$ as well as for any $\Pi \in \Pi_S$. It then follows that, for all $\tau \in [0, 1]$, we have $\tau\Delta \in \Delta_\Pi$ for any $\Pi \in \Pi_\nu^{s^c}$, which means according to the IQC stability theorem that the feedback system is robustly stable, provided that the frequency domain inequality of (5) is satisfied for some $\Pi \in \Pi_\nu^{s^c}$.

We can show that, $\Delta \in \Delta_\nu^s$ implies $\Delta \in \Delta_\Pi$ for any $\Pi \in \Pi_R$, by considering the self-adjoint operator $2(H\Delta)^*(H\Delta)$ for some $H = UR$, where $R = [R_i \otimes I_{m_i}]_{i=1}^{\kappa}$; $R_i \in \mathcal{RH}_\infty^{q_i \times l_i}$; $U = [U_i \otimes I_{m_i}]_{i=1}^{\kappa}$; $U_i, U_i^{-1} \in \mathbb{R}^{q_i \times q_i}$ and applying the general form of the swapping lemma given in (11) for structured uncertainties, together with the bounding lemma to write

$$2\Delta^* H^* H \Delta \leq H^* \Delta_1^* X \Delta_1 H + \Delta^* H^* X^{-1} H \Delta \\ + H_B^* \tilde{\Delta}^* Y \tilde{\Delta} H_B + \Delta^* H^* H_C Y^{-1} H_C^* H \Delta,$$

where we restrict the choices of X and Y to the structured matrices of the form $X = K^T K = [K_i^T K_i \otimes I_{m_i}]_{i=1}^{\kappa}$; $K_i, K_i^{-1} \in \mathbb{R}^{q_i \times q_i}$ and $Y = L^T L = [L_i^T L_i \otimes I_{m_i}]_{i=1}^{\kappa}$; $L_i, L_i^{-1} \in \mathbb{R}^{k_i \times k_i}$. Thanks to these specific choices of the structures of X and Y , it can easily be shown (by using the Kronecker product commutation identity and by employing $\|\Delta_i\|_{i2} \leq 1$ and $\|\mathcal{V}_{\Delta_i}\|_{i2} \leq \nu_i$) that $H^* \Delta_1^* X \Delta_1 H \leq H^* X H$ and $H_B^* \tilde{\Delta}^* Y \tilde{\Delta} H_B \leq H_B^* V_R^2 Y V_R^2 H_B$. Inserting $H = UR$; $H_B = R_B$; $H_C = U R_C$, from which it follows that $H^* H = R^* U^T U R = R^* X_{R2} R$; $H^* X H = R^* X_{R1} R$; $H^* H_C = R^* X_{R2} R_C$; $H^* X^{-1} H = R^* X_{R2} X_{R1}^{-1} X_{R2} R$, with the definitions $X_{R1} \triangleq U^T X U$; $X_{R2} \triangleq U^T U$, and setting $Y = V_R^{-1} Y_R V_R^{-1}$ for purposes of annihilating in the channels with $\nu_i = 0$, the terms attributed to the swapping affect (which is clearly nonexistent for the time-invariant channels), we conclude that $\Delta \in \Delta_\Pi$. Since we have $\Pi_{12} = 0$ for $\Pi \in \Pi_R$, we conclude that $\tau\Delta \in \Delta_\Pi$ for all $\Pi \in \Pi_R$ and for all $\tau \in [0, 1]$.

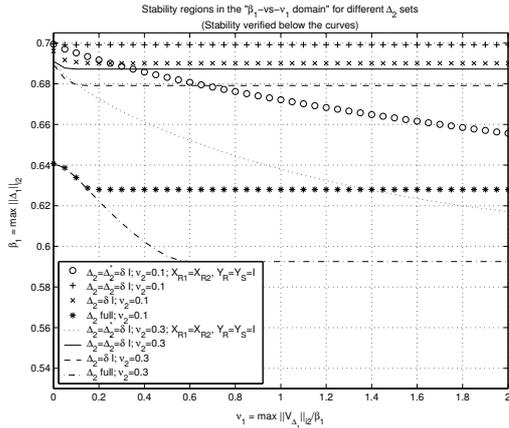


Fig. 2. Robust stability regions in ν_1 - β_1 domain.

6. ILLUSTRATIVE EXAMPLE

We provide in this section some analysis results (obtained via an implementation of the test described by Corollary 12 in MATLAB-LMILAB within a bisection search) for a strictly proper plant M with the realization

$$\begin{bmatrix} A_M & B_M \\ C_M & D_M \end{bmatrix} = \begin{bmatrix} 0.04 & 0.09 & -0.14 & 0.02 & 0.22 & 0.06 \\ 0.09 & 0.25 & 0.05 & -0.24 & -0.08 & 0.64 \\ -0.04 & -0.06 & -0.04 & 0.22 & -0.82 & 0.40 \\ 1.00 & -1.00 & 2.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & 2.00 & -1.00 & 0.00 & 0.00 & 0.00 \\ 1.00 & 1.00 & 1.00 & 0.00 & 0.00 & 0.00 \end{bmatrix},$$

and block-diagonal structured LTV uncertainties with two uncertainty channels, $\Delta_1 \in \mathcal{L}^{1 \times 1}$ and $\Delta_2 \in \mathcal{L}^{2 \times 2}$. With magnitude and normalized rate-of-variation bounds imposed on Δ_i denoted by $\beta_i = \max \|\Delta_i\|_{i2}$ and $\nu_i = \max \|\mathcal{V}_{\Delta_i}\|_{i2} / \beta_i$, we present in Figure 2 the robust stability regions (verified using bases of the form $R_i = S_i = [I \ Z^{-1}]^T$) in the β_1 - ν_1 domain (as ν_1 varies from 0 to 2), for three different structure (self-adjoint scalar repeated, arbitrary scalar repeated and full blocks) and two different normalized maximal rate-of-variation ($\nu_2 = 0.1$ and $\nu_2 = 0.3$) choices of Δ_2 , for all of which $\beta_2 = 0.45$. As expected, the region where the stability is verified (usually) gets larger as the restrictions on the structure and the nature of the second block are relaxed. Moreover, the verified region gets significantly smaller (especially for large ν_1 values) for $X_{R1} = X_{R2}$, $Y_R = I$ and $Y_S = I$, with which we get the multipliers of Jönsson and Rantzer [1996].

7. CONCLUDING REMARKS

We have generalized the stability analysis method developed by Jönsson and Rantzer [1994, 1996] for structured LTV uncertainties with repeated scalar sub-blocks to general structured LTV uncertainties. The results admit generalizations to the case of non-square plants/uncertainties and to continuous-time setting with modest difficulty. A naturally challenging research direction is the investigation of the degree of conservatism involved in the method and in particular the possibility of (or conditions for) exactness.

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