AN ELEMENTARY PROOF OF THE GENERAL Q-PARAMETRIZATION OF ALL STABILIZING CONTROLLERS

A. Quadrat *

* *INRIA Sophia Antipolis, CAFE project, 2004 Route des Lucioles BP 93, 06902 Sophia Antipolis Cedex, France. Alban.Quadrat@sophia.inria.fr.*

Abstract: It is becoming to be well-known that an internally stabilizable transfer matrix does not necessarily admit doubly coprime factorizations. The equivalence between these two concepts is still open for important classes of plants. Hence, we may wonder whether or not it is possible to parametrize all stabilizing controllers of an internally stabilizable plant which does not necessarily admit doubly coprime factorizations. The aim of this paper is to give an elementary proof of the existence of such a general parametrization. This parametrization is obtained by solving the general conditions for internal stabilizability developed within the fractional representation approach to synthesis problems. We show how such ideas can be traced back to the pioneering work of G. Zames and B. Francis on H_∞-control. Finally, if the transfer matrix admits a doubly coprime factorization, then we show that the Q-parametrization becomes the Youla-Kučera parametrization.

Keywords: Parametrization of all stabilizing controllers, internal stabilizability, Youla-Kučera parametrization, fractional representation approach.

1. FRACTIONAL REPRESENTATION APPROACH

The *fractional representation approach* was introduced in the eighties by C. Desoer, M. Vidyasagar and their co-authors in order to study in a common mathematical framework analysis and synthesis problems for different classes of systems (e.g., finite-/infinite-dimensional systems, continuous/discrete). For more details, see (Curtain and Zwart, 1991; Desoer *et al.*, 1980; Vidyasagar, 1985).

Within the fractional representation approach, the "universal class of systems" is defined by the set of transfer matrices with entries in the quotient field $Q(A) = \{n/d \,|\, 0 \neq d, n \in A\}$ of a commutative integral domain A of SISO stable plants. For instance, we have the following examples of such integral domains $A = RH_\infty$, $H_\infty(\mathbb{D})$, $H_\infty(\mathbb{C}_+)$, W_+, $\hat{\mathcal{A}}$, $A(\mathbb{D})$, $\mathbb{R}(z_1, \ldots, z_n)_S$. See (Curtain and Zwart, 1991; Desoer *et al.*, 1980; Vidyasagar, 1985) for more details.

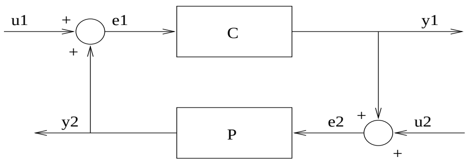
Let us recall a few definitions (Desoer *et al.*, 1980; Vidyasagar, 1985).

Definition 1. Let A be a commutative integral domain of stable SISO plants and $K = Q(A)$.

- We call *fractional representation* of the transfer matrix $P \in K^{q\times r}$ any representation of the form $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ where:

$$(D \ \ -N) \in A^{q\times(q+r)}, \quad \begin{pmatrix} \tilde{N}^T \\ \tilde{D}^T \end{pmatrix} \in A^{(q+r)\times r}.$$

- A transfer matrix $P \in K^{q\times r}$ is said to be *internally stabilizable* if there exists a controller $C \in K^{r\times q}$ such that the transfer matrix defined by
$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P,\,C)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
is A-stable (see Figure 1), i.e., if all the entries of the following matrix belong to A:
$$H(P,\,C) = \begin{pmatrix} I_q & -P \\ -C & I_r \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} (I_q - P\,C)^{-1} & (I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$
$$= \begin{pmatrix} I_q + P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ (I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix}.$$
Then, C is called a *stabilizing controller* of P.



- A transfer matrix $P \in K^{q\times r}$ admits a *left-coprime factorization* if there exist
$$\begin{cases} R = (D \quad -N) \in A^{q\times(q+r)}, \\ S = (X^T \;\; Y^T)^T \in A^{(q+r)\times q} \end{cases}$$
such that $\det D \neq 0$, $P = D^{-1}\,N$ and:
$$R\,S = D\,X - N\,Y = I_q.$$
- A transfer matrix $P \in K^{q\times r}$ admits a *right-coprime factorization* if there exist
$$\begin{cases} \tilde{R} = (\tilde{N}^T \;\; \tilde{D}^T)^T \in A^{(q+r)\times r}, \\ \tilde{S} = (-\tilde{Y} \;\; \tilde{X}) \in A^{r\times(q+r)} \end{cases}$$
such that $\det \tilde{D} \neq 0$, $P = \tilde{N}\,\tilde{D}^{-1}$ and:
$$\tilde{S}\,\tilde{R} = -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r.$$
- A transfer matrix $P \in K^{q\times r}$ admits a *doubly coprime factorization* if P admits a left- and a right-coprime factorization.

2. INTERNAL STABILIZABILITY

We start by giving equivalent necessary and sufficient conditions for internal stabilizability.

Proposition 1. A plant defined by the transfer matrix $P \in K^{q\times r}$ is internally stabilizable iff one of the following equivalent assertions is satisfied:

(1) There exists $L = (U^T \quad V^T)^T \in A^{(q+r)\times q}$ which satisfies $\det U \neq 0$ and:
  (a) $L\,P = \begin{pmatrix} U\,P \\ V\,P \end{pmatrix} \in A^{(q+r)\times r}$,
  (b) $(I_q \quad -P)\,L = U - P\,V = I_q$.
  Then, the controller $C = V\,U^{-1}$ internally stabilizes the plant P and we have:
$$\begin{cases} U = (I_q - P\,C)^{-1}, \\ V = C\,(I_q - P\,C)^{-1}. \end{cases}$$
(2) There exists $\tilde{L} = (-\tilde{V} \;\; \tilde{U}) \in A^{r\times(q+r)}$ which satisfies $\det \tilde{U} \neq 0$ and:
  (a) $P\,\tilde{L} = (-P\,\tilde{V} \;\; P\,\tilde{U}) \in A^{q\times(q+r)}$,
  (b) $\tilde{L}\begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}\,P + \tilde{U} = I_r$.
  Then, the controller $C = \tilde{U}^{-1}\,\tilde{V}$ internally stabilizes the plant P and we have:
$$\begin{cases} \tilde{U} = (I_r - C\,P)^{-1}, \\ \tilde{V} = (I_r - C\,P)^{-1}\,C. \end{cases}$$

We refer the reader to (Quadrat, 2003*a*; Quadrat, 2005) for a proof. We have the following straightforward consequence of Proposition 1.

Corollary 1. P is internally stabilizable iff there exists $V \in A^{r\times q}$ such that:
$$\begin{cases} V\,P \in A^{r\times r}, \\ P\,V \in A^{q\times q}, \\ (P\,V + I_q)\,P = P\,(V\,P + I_r) \in A^{q\times r}. \end{cases}$$
Then, $C = V\,(P\,V + I_q)^{-1} = (V\,P + I_r)^{-1}\,V$ is a stabilizing controller of P and we have:
$$V = C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C.$$

We have recently discovered that Corollary 1 firstly appeared for SISO plants in the pioneering work of G. Zames and B. Francis (Zames and Francis, 1983) on H_∞-control. As they were interested in the case $A = RH_\infty$, they used the important fact that every rational transfer function admitted a coprime factorization over RH_∞ in order to parametrize all stabilizing controllers. They called such a parametrization the *Q-parametrization* and they showed that, up to a stable cancellation, it was the Youla-Kučera parametrization (Desoer *et al.*, 1980; Kučera, 1979; Vidyasagar, 1985; Youla *et al.*, 1976).

Independently, we obtained Proposition 1 within the *fractional ideal/lattice approach* to analysis and synthesis problems developed in (Quadrat, 2003*b*; Quadrat, 2003*a*; Quadrat, 2005) ignoring the first attempt done by G. Zames and B. Francis. As we shall see in Section 3, such a new approach allowed us to solve the general conditions 1.a, 1.b, 2.a and 2.b of Proposition 1, i.e., to obtain the general Q-parametrization of all stabilizing controllers of an internally stabilizable plant (without assuming the existence of doubly coprime factorizations). To our knowledge, it is the first time that the general conditions for internal stabilizability are solved. Similarly to what G. Zames and B. Francis had noticed in the SISO case, if a doubly coprime factorization exists for the transfer matrix, then the general Q-parametrization obtained in (Quadrat, 2003b; Quadrat, 2003a; Quadrat, 2005) becomes the well-known Youla-Kučera parametrization.

It seems that Zames-Francis approach has been forgotten because, in the case of rational transfer matrices, it has been superseded by the fractional representation approach over RH_∞. Indeed, every rational transfer matrix admits doubly coprime factorizations over RH_∞ and explicit algorithms computing them are well-known. However, for infinite-dimensional or multidimensional systems, few transfer matrices admit doubly coprime factorizations. Thus, we need to impose the condition upon the existence of doubly coprime factorizations in order to mimic the results obtained for finite-dimensional systems (e.g., parametrization of all stabilizing controllers, transformation of optimization problems into affine ones).

3. GENERAL PARAMETRIZATION OF ALL STABILIZING CONTROLLERS

Coming back to the roots, in this section, we show how to solve conditions 1.a, 1.b, 2.a and 2.b of Proposition 1 in order to obtain the general Q-parametrization of all stabilizing controllers of an internally stabilizable plant. Moreover, we give elementary proofs. Module-theoretic proofs of the general Q-parametrization were given in (Quadrat, 2003a; Quadrat, 2005). We refer the reader to (Quadrat, 2003b; Quadrat, 2003a; Quadrat, 2005) for the fractional ideal/lattice approach to analysis and synthesis problems which allowed us to independently recover and develop intrinsically the approach of G. Zames-B. Francis.

Proposition 2. Let $P \in K^{q\times r}$ be an internally stabilizable plant, $C_\star \in K^{r\times q}$ a stabilizing controller of P and:

$$\begin{cases} U = (I_q - P\,C_\star)^{-1} \in A^{q\times q},\\ V = C_\star\,(I_q - P\,C_\star)^{-1} \in A^{r\times q},\\ \tilde{U} = (I_r - C_\star\,P)^{-1} \in A^{r\times r},\\ \tilde{V} = (I_r - C_\star\,P)^{-1}\,C_\star \in A^{r\times q}.\end{cases}$$

Then, all stabilizing controllers of P are given by

$$\begin{aligned} C(Q) &= (V+Q)\,(U+P\,Q)^{-1},\\ &= (\tilde{U}+Q\,P)^{-1}\,(\tilde{V}+Q),\end{aligned} \tag{1}$$

where Q is any matrix which belongs to

$$\begin{aligned}\Omega = \{\,Q \in A^{r\times q} \mid Q\,P \in A^{r\times r},\ P\,Q \in A^{q\times q},\\ P\,Q\,P \in A^{q\times r}\,\},\end{aligned} \tag{2}$$

and satisfies:

$$\begin{cases} \det(U+P\,Q) \neq 0,\\ \det(\tilde{U}+Q\,P)\neq 0.\end{cases}$$

PROOF. Let us consider two stabilizing controllers C_1 and $C_2 \in K^{r\times q}$ of $P \in K^{q\times r}$. If we use denote by

$$\begin{cases} U_i = (I_q - P\,C_i)^{-1} \in A^{q\times q},\\ V_i = C_i\,(I_q - P\,C_i)^{-1} \in A^{r\times q},\\ \tilde{U}_i = (I_r - C_i\,P)^{-1} \in A^{r\times r},\\ \tilde{V}_i = (I_r - C_i\,P)^{-1}\,C_i \in A^{r\times q},\end{cases}$$

then, we have $C_i = V_i\,U_i^{-1} = \tilde{U}_i^{-1}\,\tilde{V}_i$ for $i=1,2$. Now, using the well-known identity

$$C_i\,(I_q - P\,C_i)^{-1} = (I_r - C_i\,P)^{-1}\,C_i,$$

we obtain the equality $V_i = \tilde{V}_i$. Then, the matrices

$$\begin{cases} L_i = (U_i^T \;\; V_i^T)^T \in A^{(q+r)\times q},\\ \tilde{L}_i = (-\tilde{V}_i \;\; \tilde{U}_i) \in A^{r\times(q+r)}\end{cases}$$

satisfy 1.a, 1.b, 2.a and 2.b of Proposition 1 or, in other words, for $i=1,2$, we have:

$$\begin{cases} U_i - P\,V_i = I_q, \quad \begin{pmatrix} -U_i\,P\\ -V_i\,P\end{pmatrix} \in A^{(q+r)\times r},\\ \tilde{U}_i - \tilde{V}_i\,P = I_r, \quad (-P\,\tilde{V}_i \quad P\,\tilde{U}_i) \in A^{q\times(q+r)}.\end{cases}$$

Using the two previous equalities, we obtain:

$$\begin{cases} U_2 - U_1 &= P\,V_2 + I_q - P\,V_1 - I_q = P\,(V_2 - V_1),\\ \tilde{U}_2 - \tilde{U}_1 &= \tilde{V}_2\,P + I_r - \tilde{V}_1\,P - I_r = (\tilde{V}_2 - \tilde{V}_1)\,P\\ &= (V_2 - V_1)\,P.\end{cases}$$

Therefore, using the well-known identity

$$P\,(I_r - C_i\,P)^{-1} = (I_q - P\,C_i)^{-1}\,P,$$

we finally obtain:

$$\begin{cases} V_2 - V_1 &= \tilde{V}_2 - \tilde{V}_1 \in A^{r\times q},\\ (V_2 - V_1)\,P &= \tilde{U}_2 - \tilde{U}_1 \in A^{r\times r},\\ P\,(V_2 - V_1) &= U_2 - U_1 \in A^{q\times q},\\ P\,(V_2 - V_1)\,P &= P\,(\tilde{U}_2 - \tilde{U}_1)\\ &= (U_2 - U_1)\,P \in A^{q\times r},\end{cases}$$

$$\Rightarrow V_2 - V_1 = \tilde{V}_2 - \tilde{V}_1 \in \Omega.$$

If we denote by $Q = V_2 - V_1 = \tilde{V}_2 - \tilde{V}_1 \in \Omega$, then we have

$$\begin{cases} V_2 = V_1 + Q,\\ \tilde{V}_2 = \tilde{V}_1 + Q,\\ U_2 = U_1 + P\,Q,\\ \tilde{U}_2 = \tilde{U}_1 + Q\,P,\end{cases}$$

and, if $\det(U_1 + P\,Q) \neq 0$ and $\det(\tilde{U}_1 + Q\,P) \neq 0$, we obtain:

$$\begin{cases} C_2 = V_2\,U_2^{-1} = (V_1 + Q)\,(U_1 + P\,Q)^{-1},\\ C_2 = \tilde{U}_2^{-1}\,\tilde{V}_2 = (\tilde{U}_1 + Q\,P)^{-1}\,(\tilde{V}_1 + Q).\end{cases}$$

Therefore, if we use the notations

$$U = U_1, \quad V = V_1, \quad \tilde{U} = \tilde{U}_1, \quad \tilde{V} = \tilde{V}_1,$$

then we finally obtain $C_2 = C(Q)$, where $C(Q)$ is defined by (1), for a certain $Q \in \Omega$ which satisfies $\det(U + P\,Q) \neq 0$ and $\det(\tilde{U} + Q\,P) \neq 0$.

Finally, let us prove that, for every $Q \in \Omega$ which satisfies $\det(U + P\,Q) \neq 0$ and $\det(\tilde{U} + Q\,P) \neq 0$, the controller

$$C(Q) = (V{+}Q)\,(U{+}P\,Q)^{-1} = (\tilde{U}{+}Q\,P)^{-1}\,(\tilde{V}{+}Q)$$

internally stabilizes P. We denote by:

$$\begin{cases} L(Q) = ((U + P\,Q)^T \;\; (V + Q)^T)^T, \\ \tilde{L}(Q) = (-(\tilde{V} + Q) \;\; (\tilde{U} + Q\,P)). \end{cases}$$

Then, using the fact that $Q \in \Omega$, we obtain

$$\begin{cases} V + Q \in A^{r\times q}, \quad U + P\,Q \in A^{q\times q}, \\ L(Q)\,P = \begin{pmatrix} U\,P + P\,Q\,P \\ V\,P + Q\,P \end{pmatrix} \in A^{(q+r)\times r}, \\ (I_q \quad -P)\,L(Q) = U - P\,V = I_q, \end{cases}$$

which shows that $C(Q) = (V + Q)\,(U + P\,Q)^{-1}$ internally stabilizes P by 1 of Proposition 1. Moreover, we have

$$\begin{cases} \tilde{V} + Q \in A^{r\times q}, \quad \tilde{U} + Q\,P \in A^{r\times r}, \\ P\,\tilde{L}(Q) = (-(P\,\tilde{V} + P\,Q) \\ \qquad\qquad (P\,\tilde{U} + P\,Q\,P)) \in A^{q\times(q+r)}, \\ \tilde{L}(Q)\begin{pmatrix} P \\ I_r \end{pmatrix} = -\tilde{V}\,P + \tilde{U} = I_r, \end{cases}$$

showing that $C(Q) = (\tilde{U} + Q\,P)^{-1}\,(\tilde{V} + Q)$ internally stabilizes P by 2 of of Proposition 1.

Proposition 3. Let $P \in K^{q\times r}$ be a stabilizable plant, $C_\star$ a stabilizing controller of P and:

$$\begin{cases} L = \begin{pmatrix} (I_q - P\,C_\star)^{-1} \\ C_\star\,(I_q - P\,C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{L} = (-(I_r - C_\star\,P)^{-1}\,C_\star \\ \qquad\qquad (I_r - C_\star\,P)^{-1}) \in A^{r\times(q+r)}. \end{cases}$$

Then, the A-module Ω defined by (2) satisfies

$$\Omega = \tilde{L}\,A^{(q+r)\times(q+r)}\,L. \tag{3}$$

Equivalently, Ω is generated over A by the $(q{+}r)^2$ matrices $\tilde{L}_i\,L^j$, where $\tilde{L}_i$ denotes the i^{th} column of $\tilde{L}$ and L^j the j^{th} row of L, i.e., we have:

$$\Omega = \sum_{i,j=1}^{q+r} A\,(\tilde{L}_i\,L^j). \tag{4}$$

PROOF. Let $Q \in \Omega$, i.e., the matrix $Q \in A^{r\times q}$ satisfies $Q\,P \in A^{r\times r}$, $P\,Q \in A^{q\times q}$, $P\,Q\,P \in A^{q\times r}$. Then, using 2.b of Proposition 1, we obtain:

$$\begin{cases} Q = \tilde{L}\begin{pmatrix} P \\ I_r \end{pmatrix} Q = \tilde{L}\begin{pmatrix} P\,Q \\ Q \end{pmatrix}, \\ \begin{pmatrix} P\,Q \\ Q \end{pmatrix} \in A^{(q+r)\times q}. \end{cases} \tag{5}$$

Moreover, using 1.b of Proposition 1, we obtain:

$$\begin{cases} Q = Q\,(I_q \quad -P)\,L = (Q \quad -Q\,P)\,L, \\ (Q \quad -Q\,P) \in A^{r\times(q+r)}, \end{cases}$$

$$\Rightarrow \begin{cases} P\,Q = (P\,Q \quad -P\,Q\,P)\,L, \\ (P\,Q \quad -P\,Q\,P) \in A^{q\times(q+r)}. \end{cases}$$

Therefore, we have:

$$\begin{cases} \begin{pmatrix} P\,Q \\ Q \end{pmatrix} = \begin{pmatrix} P\,Q & -P\,Q\,P \\ Q & -Q\,P \end{pmatrix} L, \\ \Lambda = \begin{pmatrix} P\,Q & -P\,Q\,P \\ Q & -Q\,P \end{pmatrix} \in A^{(q+r)\times(q+r)}. \end{cases}$$

Then, using the equation of (5), we obtain that Q has the form $Q = \tilde{L}\,\Lambda\,L$ where $\Lambda \in A^{(q+r)\times(q+r)}$, i.e., $Q \in \tilde{L}\,A^{(q+r)\times(q+r)}\,L$.

If $Q \in \tilde{L}\,A^{(q+r)\times(q+r)}\,L$, then there exists a matrix $\Lambda \in A^{(q+r)\times(q+r)}$ such that $Q = \tilde{L}\,\Lambda\,L$, where L and $\tilde{L}$ satisfy 1 and 2 of Proposition 1. Then, using 1.a and 2.a of Proposition 1, we finally obtain

$$\begin{cases} Q \in A^{r\times q}, \\ Q\,P = \tilde{L}\,\Lambda\,(L\,P) \in A^{r\times r}, \\ P\,Q = (P\,\tilde{L})\,\Lambda\,L \in A^{q\times q}, \\ P\,Q\,P = (P\,\tilde{L})\,\Lambda\,(L\,P) \in A^{q\times r}, \end{cases}$$

showing that $Q \in \Omega$ and proving (3).

Finally, (4) follows from the fact that $A^{(q+r)\times(q+r)}$ is a free A-module of rank $(q + r)^2$ with a basis defined by $\{E_{ij}\}_{i,j=1,\ldots,q+r}$, where E_{ij} denotes the matrix defined by 1 in the i^{th} row and the j^{th} and 0 elsewhere. Indeed, $\Lambda \in A^{(q+r)\times(q+r)}$ can be uniquely written as $\Lambda = \sum_{i,j=1}^{q+r} \lambda_{ij}\,E_{ij}$ where $\lambda_{ij} \in A$, and thus, every $Q \in \Omega$ can be written (non necessarily uniquely) as:

$$Q = \sum_{i,j=1}^{q+r} \lambda_{ij}\,(\tilde{L}\,E_{ij}\,L).$$

Therefore, $\{\tilde{L}\,E_{ij}\,L\}_{i,j=1,\ldots,q+r}$ is a family of generators of Ω and $\tilde{L}\,E_{ij}\,L$ is the product of the i^{th} column $\tilde{L}_i$ of $\tilde{L}$ by the j^{th} row L^j of L.

Combining Propositions 2 and 3, we obtain the main result of this paper.

Theorem 1. Let $P \in K^{q\times r}$ be an internally stabilizable plant, $C_\star$ a stabilizing controller of P and:

$$\begin{cases} L = \begin{pmatrix} (I_q - P\,C_\star)^{-1} \\ C_\star\,(I_q - P\,C_\star)^{-1} \end{pmatrix}, \\ \tilde{L} = (-(I_r - C_\star\,P)^{-1}\,C_\star \;\; (I_r - C_\star\,P)^{-1}). \end{cases}$$

Then, all stabilizing controllers of P are of the form

$$\begin{aligned} C(Q) &= (C_\star\,(I_q - P\,C_\star)^{-1} + Q)\,((I_q - P\,C_\star)^{-1} + P\,Q)^{-1} \\ &= ((I_r - C_\star\,P)^{-1} + Q\,P)^{-1}\,((I_r - C_\star\,P)^{-1}\,C_\star + Q), \end{aligned}$$

where Q is any matrix which belongs to

$$\Omega = \sum_{i,j=1}^{q+r} A\,(\tilde{L}_i\,L^j)$$

and satisfies:

$$\begin{cases} \det((I_q - P\,C_\star)^{-1} + P\,Q) \neq 0, \\ \det((I_r - C_\star\,P)^{-1} + Q\,P) \neq 0. \end{cases}$$

Using a fractional representation approach of P, we can prove that the general Q-parametrization given in Theorem 1 has the following equivalent form (see (Quadrat, 2005) for a proof).

Theorem 2. Let $P \in K^{q\times r}$ be an internally stabilizable plant, $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$ a fractional representation of P, $C_\star \in K^{r\times q}$ a stabilizing controller of P and:

$$\begin{cases} S = \begin{pmatrix} (D - N\,C_\star)^{-1} \\ C_\star\,(D - N\,C_\star)^{-1} \end{pmatrix} \in A^{(q+r)\times q}, \\ \tilde{S} = (-(\tilde{D} - C_\star\,\tilde{N})^{-1}\,C_\star \\ \qquad\qquad (\tilde{D} - C_\star\,\tilde{N})^{-1}) \in A^{r\times(q+r)}. \end{cases}$$

Let us denote by $\tilde{S}_i$ the i^{th} column of $\tilde{S}$ and by S^j the j^{th} row of S. Then, all stabilizing controllers of P are of the form

$$\begin{aligned} &C(Q) \\ &= (C_\star\,(D - N\,C_\star)^{-1} + \tilde{D}\,Q)\,((D - N\,C_\star)^{-1} + \tilde{N}\,Q)^{-1} \\ &= ((\tilde{D} - C_\star\,\tilde{N})^{-1} + Q\,N)^{-1}\,((\tilde{D} - C_\star\,\tilde{N})^{-1}\,C_\star + Q\,D), \end{aligned}$$

where Q is any matrix which belongs to

$$\Delta = \tilde{S}\,A^{(q+r)\times(q+r)}\,S = \sum_{i,j=1}^{q+r} A\,(\tilde{S}_i\,S^j)$$

and satisfies:

$$\begin{cases} \det((D - N\,C_\star)^{-1} + \tilde{N}\,Q) \neq 0, \\ \det((\tilde{D} - C_\star\,\tilde{N})^{-1} + Q\,N) \neq 0. \end{cases}$$

We note that V. Sule obtained in (Sule, 1994) a parametrization of all stabilizing controllers for an internally stabilizable plant over *a unique factorization domain* (UFD) A. However, his parametrization has the major inconvenience of not being explicit in terms of the free parameters and the set of free parameters is not determined. Moreover, we prove in (Quadrat, 2005) that no non-trivial Banach algebra (e.g., $H_\infty(\mathbb{C}_+)$, $\hat{\mathcal{A}}$, W_+, $A(\mathbb{D})$) is a UFD. Another parametrization of all stabilizing controllers has been recently developed in (Mori, 2004). However, this new parametrization is less explicit than the parametrization obtained in this paper as, for instance, it has not the explicit form of a linear fractional transformation in the free parameters and the set of free parameters is not completely characterized contrary to the sets Ω and Δ.

4. YOULA-KUČERA PARAMETRIZATION

We have the following corollary of Proposition 1.

Corollary 2. (1) If $P \in K^{q\times r}$ admits a left-coprime factorization

$$P = D^{-1}\,N, \quad D\,X - N\,Y = I_q,$$

with $(X^T \;\; Y^T)^T \in A^{(q+r)\times q}$ and $\det X \neq 0$, then $L = ((X\,D)^T \;\; (Y\,D)^T)^T \in A^{(q+r)\times q}$ satisfies 1.a and 1.b of Proposition 1 and $C = Y\,X^{-1}$ is a stabilizing controller of P.

(2) If $P \in K^{q\times r}$ admits a right-coprime factorization

$$P = \tilde{N}\,\tilde{D}^{-1}, \quad -\tilde{Y}\,\tilde{N} + \tilde{X}\,\tilde{D} = I_r,$$

with $(-\tilde{Y} \;\; \tilde{X}) \in A^{r\times(q+r)}$ and $\det \tilde{X} \neq 0$, then $\tilde{L} = (-\tilde{D}\,\tilde{Y} \;\; \tilde{D}\,\tilde{X}) \in A^{r\times(q+r)}$ satisfies 2.a and 2.b of Proposition 1 and $C = \tilde{X}^{-1}\,\tilde{Y}$ is a stabilizing controller of P.

PROOF. If $P = D^{-1}\,N$, $D\,X - N\,Y = I_q$, is a left-coprime factorization of P, then we have

$$\begin{cases} D\,X - N\,Y = I_q \Rightarrow X - D^{-1}\,N\,Y = D^{-1} \\ \Rightarrow X - P\,Y = D^{-1} \Rightarrow (X\,D) - P\,(Y\,D) = I_q, \\ (X\,D)\,P = X\,N \in A^{q\times r}, \\ (Y\,D)\,P = Y\,N \in A^{r\times r}, \end{cases}$$

i.e., $L = ((X\,D)^T \;\; (Y\,D)^T)^T \in A^{(q+r)\times q}$ satisfies 1.a and 1.b of Proposition 1, and thus, the controller $C = (Y\,D)\,(X\,D)^{-1} = Y\,X^{-1}$ internally stabilizes P. 2 can be proved similarly.

From Corollary 2, the existence of a doubly coprime factorization of P is a sufficient but not a necessary condition for internal stabilizability.

Proposition 4. If $P \in K^{q\times r}$ admits the doubly coprime factorization

$$\begin{cases} P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}, \\ \begin{pmatrix} D & -N \\ -\tilde{Y} & \tilde{X} \end{pmatrix} \begin{pmatrix} X & \tilde{N} \\ Y & \tilde{D} \end{pmatrix} = I_{q+r}, \end{cases} \tag{6}$$

then Ω defined by (2) satisfies $\Omega = \tilde{D}\,A^{r\times q}\,D$.

PROOF. Let $Q \in \tilde{D}\,A^{r\times q}\,D$, i.e., Q has the form $Q = \tilde{D}\,\Lambda\,D$ for a certain $\Lambda \in A^{r\times q}$, then we have:

$$\begin{cases} Q = \tilde{D}\,\Lambda\,D \in A^{r\times q}, \\ Q\,P = \tilde{D}\,\Lambda\,N \in A^{r\times r}, \\ P\,Q = \tilde{N}\,\Lambda\,D \in A^{q\times q}, \\ P\,Q\,P = \tilde{N}\,\Lambda\,N \in A^{q\times r}, \end{cases} \Rightarrow Q \in \Omega.$$

Conversely, let $Q \in \Omega$ and let us define the matrix $\Lambda = \tilde{D}^{-1}\,Q\,D^{-1} \in K^{r\times q}$. From (6), we obtain

$$\begin{cases} D^{-1} = X - P\,Y, \\ \tilde{D}^{-1} = -\tilde{Y}\,P + \tilde{X}, \end{cases}$$

$$\begin{aligned} \Rightarrow \Lambda = \tilde{D}^{-1}\,Q\,D^{-1} &= (-\tilde{Y}\,P + \tilde{X})\,Q\,(X - P\,Y) \\ &= -\tilde{Y}\,(P\,Q)\,X + \tilde{Y}\,(P\,Q\,P)\,Y + \tilde{X}\,Q\,X \\ &\quad -\tilde{X}\,(Q\,P)\,Y \in A^{r\times q}, \end{aligned}$$

because X, Y, $\tilde{X}$ and $\tilde{Y}$ are matrices with entries in A and $Q \in \Omega$. Therefore, we have $Q = \tilde{D}\,\Lambda\,D$ for a certain $\Lambda \in A^{r \times q}$, and thus, $Q \in \tilde{D}\,A^{r\times q}\,D$.

Corollary 3. If $P \in K^{q \times r}$ admits the doubly coprime factorization (6), then all stabilizing controllers of P are of the form

$$\begin{aligned} C(\Lambda) &= (Y + \tilde{D}\,\Lambda)\,(X + \tilde{N}\,\Lambda)^{-1}, \\ &= (\tilde{X} + \Lambda\,N)^{-1}\,(\tilde{Y} + \Lambda\,D), \end{aligned}$$

where Λ is any matrix of $A^{r\times q}$ which satisfies:

$$\begin{cases} \det(X + \tilde{N}\,\Lambda) \neq 0, \\ \det(\tilde{X} + \Lambda\,N) \neq 0. \end{cases} \tag{7}$$

PROOF. If P admits a doubly coprime factorization $P = D^{-1}\,N = \tilde{N}\,\tilde{D}^{-1}$, then, by Proposition 4, we have $\Omega = \tilde{D}\,A^{r\times q}\,D$. Moreover, by 1 of Corollary 2, we know that $C = (Y\,D)\,(X\,D)^{-1} = Y\,X^{-1}$ is a stabilizing controller of P. Moreover, by 2 of Corollary 2, $C' = (\tilde{D}\,\tilde{X})^{-1}\,(\tilde{D}\,\tilde{Y}) = \tilde{X}^{-1}\,\tilde{Y}$ is also a stabilizing controller of P. Then, using (6), we obtain that $-\tilde{Y}\,X + \tilde{X}\,Y = 0$, and thus, $C' = C$. Therefore, by Proposition 2 or Theorem 1, we obtain that all stabilizing controllers of P are of the form

$$\begin{aligned} C(\Lambda) &= (Y\,D + \tilde{D}\,\Lambda\,D)\,(X\,D + P\,\tilde{D}\,\Lambda\,D)^{-1} \\ &= (Y\,D + \tilde{D}\,\Lambda\,D)\,(X\,D + \tilde{N}\,\Lambda\,D)^{-1} \\ &= (Y + \tilde{D}\,\Lambda)\,D\,D^{-1}\,(X + \tilde{N}\,\Lambda)^{-1} \\ &= (Y + \tilde{D}\,\Lambda)\,(X + \tilde{N}\,\Lambda)^{-1} \end{aligned}$$

$$\begin{aligned} C(\Lambda) &= (\tilde{D}\,\tilde{X} + \tilde{D}\,\Lambda\,D\,P)^{-1}\,(\tilde{D}\,\tilde{Y} + \tilde{D}\,\Lambda\,D) \\ &= (\tilde{D}\,\tilde{X} + \tilde{D}\,\Lambda\,N)^{-1}\,(\tilde{D}\,\tilde{Y} + \tilde{D}\,\Lambda\,D) \\ &= (\tilde{X} + \Lambda\,N)^{-1}\,\tilde{D}^{-1}\,\tilde{D}\,(\tilde{Y} + \Lambda\,D) \\ &= (\tilde{X} + \Lambda\,N)^{-1}\,(\tilde{Y} + \Lambda\,D) \end{aligned}$$

where $\Lambda \in A^{r\times q}$ is any matrix which satisfies (7).

Example 1. In the literature of differential time-delay systems, it is well-known that the unstable plant $p = e^{-s}/(s-1)$ is internally stabilized by the controller $c = -2\,e\,(s-1)/(s+1-2\,e^{-(s-1)})$ involving a *distributed delay.* This result can be directly checked by computing:

$$\begin{cases} u = \dfrac{1}{(1-p\,c)} = \dfrac{(s+1-2\,e^{-(s-1)})}{(s+1)} \in H_\infty(\mathbb{C}_+), \\ v = \dfrac{c}{1-p\,c} = -\dfrac{2\,e\,(s-1)}{(s+1)} \in H_\infty(\mathbb{C}_+), \\ u\,p = \dfrac{p}{(1-p\,c)} = \dfrac{e^{-s}}{(s+1)} \\ \qquad\qquad \dfrac{(s+1-2\,e^{-(s-1)})}{(s-1)} \in H_\infty(\mathbb{C}_+). \end{cases}$$

By Theorem 1, all stabilizing controllers of p are parametrized by (1), where a free parameter $q \in \Omega$ has the form $q = q_1\,u^2 + q_2\,v^2 + q_3\,u\,v$ where q_1, q_2 and $q_3 \in H_\infty(\mathbb{C}_+)$. After a few computations, we obtain that all stabilizing controllers of p have the form:

$$\begin{aligned} c(l) &= \frac{-2\,e + l\,\frac{(s-1)}{(s+1)}}{1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) + l\,\frac{e^{-s}}{(s+1)}}, \\ l &= \left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right)^2 q_1 + 4\,e^2\,q_2 \\ &\quad + q_3\,2\,e\left(1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right). \end{aligned}$$

The previous parametrization is in fact the Youla-Kučera parametrization obtained from the following coprime factorization $p = n/d$:

$$\begin{cases} n = \dfrac{e^{-s}}{(s+1)}, \quad d = \dfrac{(s-1)}{(s+1)}, \\ (-2\,e)\,n - \left(1 + 2\left(\dfrac{1-e^{-(s-1)}}{s-1}\right)\right)\,d = 1. \end{cases}$$

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