DETERMINING THE DEGREE OF SYSTEM VARIABILITY FOR TIME-VARYING DISCRETE-TIME SYSTEMS

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Abstract: The paper develops the concept of a variability coefficient for linear time-varying (LTV), discrete-time (DT) systems. The main idea is to introduce frequency analysis tools for LTV systems which involve Singular Value Decomposition (SVD) and Discrete Fourier Transform (DFT) as well as Power Spectral Density (PSD). The general objective of this paper is to examine the first order system to show how the value of the proposed variability coefficient depends on the variability of particular system parameters and whether it is a good measure of the degree of the system variability. Especially we examine how the variability of two matrices of the state space model (scalars, in this case 1st order) influences the value of this coefficient. Three different cases of one-dimensional LTV DT system have been considered. The results of analysis for each case are shown in 2 diagrams: five proposed coefficients versus given parameter epsilon and step responses for given parameters. Examples are preceded by theoretical considerations and on the basis of these examples the most important conclusions are drawn and properties of the introduced concept summarized.

Keywords: discrete-time systems, time-varying systems, non-stationary systems.

1. INTRODUCTION

One of the first attempts to analyse LTV systems in the frequency domain was made by Zadeh (1950, 1961). The time-varying transfer function has been defined by extending the Laplace transform to the varying impulse response. Other works on the frequency aspects of LTV systems focus on the modal analysis. The ideas of varying eigenvalues or varying natural frequencies have been used without a rigorous definition by Bogoliubov (1961), Maia (1997). The concept of pseudo-modal parameters was introduced and described by Liu (1999), (Liu, Kujath 1999). The concept of frequency analysis of LTV systems using SVD-DFT approach (Singular Value Decomposition – Discrete Fourier Transform) with connection to Power Spectral Density (PSD) has been recently presented in (Orlowski 2003, 2004). The diagrams introduced in that work have similar properties to classical Bode diagrams for LTI systems. Approximated Bode diagrams are given by a finite set of frequencies (or singular vectors) and their corresponding gains.

An important extension of the SVD-DFT method can be obtained by defining two coefficients of variability of the LTV system and this is done in this paper. If the set of the system matrices have been obtained via the system identification (Liu 1999), the coefficients provide information on whether the system is LTI or LTV and if the variability can be omitted.

2. MODEL DESCRIPTION

Dynamic, discrete-time system can be given by set of difference equations, called the state space model

$$\mathbf{x}_p(k+1) = \mathbf{A}(k)\cdot\mathbf{x}_p(k) + \mathbf{B}(k)\cdot\mathbf{v}_p(k), \tag{1}$$

$$\mathbf{y}_p(k) = \mathbf{C}(k)\cdot\mathbf{x}_p(k), \quad k\in\mathbf{N}, \quad \mathbf{x}_p(0)=\mathbf{0}, \tag{2}$$

where $\mathbf{x}_p(\cdot)\in(\mathbf{R}^n)^N$ is nominal state, $\mathbf{v}_p(\cdot)\in(\mathbf{R}^m)^N$ is nominal control, $\mathbf{y}_p(\cdot)\in(\mathbf{R}^p)^N$ is nominal output, and $\mathbf{A}(k)\in\mathcal{L}(\mathbf{R}^n)$, $\mathbf{B}(k)\in\mathcal{L}(\mathbf{R}^m,\mathbf{R}^n)$, $\mathbf{C}(k)\in\mathcal{L}(\mathbf{R}^n,\mathbf{R}^p)$ are system's matrices.

Equivalently the system can be given using following operators

$$\hat{\mathbf{y}}_p = \hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}}\cdot\hat{\mathbf{v}}_p + \hat{\mathbf{C}}\hat{\mathbf{N}}\cdot\mathbf{x}_0 \tag{3}$$

The operator $\hat{\mathbf{C}}\hat{\mathbf{L}}\hat{\mathbf{B}}$ is a compact and Hilbert-Schmidt operator from l_2 into l_2 and actually maps boundedly signals $\mathbf{v}(k)\in\mathcal{V}=l_2[0,N]$ into signals $y\in\mathcal{Y}$.

Matrices of operator are given as follow:

$$\hat{\mathbf{L}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \phi_1^1 & \mathbf{I} & \mathbf{0} & \vdots & \vdots \\ \vdots & \ddots & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \phi_1^{N-2} & \cdots & \phi_{N-2}^{N-2} & \mathbf{I} & \mathbf{0} \end{bmatrix} \quad \hat{\mathbf{N}} = \begin{bmatrix} \mathbf{I} \\ \phi_0^0 \\ \vdots \\ \phi_0^{N-2} \end{bmatrix} \tag{4}$$

where

$$\phi_i^k = \mathbf{A}(k)\cdot\mathbf{A}(k-1)\cdot\ldots\cdot\mathbf{A}(i) \tag{5}$$

and matrix operators $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ have diagonal form i.e.

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}(N-1) \end{bmatrix} \quad \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}(N-1) \end{bmatrix} \tag{6}$$

where vectors $\mathbf{x_p}(\cdot)$, $\mathbf{y_p}(\cdot)$ and $\mathbf{v_p}(\cdot)$ have following notation

$$\hat{\mathbf{x}}_\mathrm{p} = \begin{bmatrix} \mathbf{x}_\mathrm{p}(0) \\ \vdots \\ \mathbf{x}_\mathrm{p}(N-1) \end{bmatrix} \quad \hat{\mathbf{y}}_\mathrm{p} = \begin{bmatrix} \mathbf{y}_\mathrm{p}(0) \\ \vdots \\ \mathbf{y}_\mathrm{p}(N-1) \end{bmatrix} \quad \hat{\mathbf{v}}_\mathrm{p} = \begin{bmatrix} \mathbf{v}_\mathrm{p}(0) \\ \vdots \\ \mathbf{v}_\mathrm{p}(N-1) \end{bmatrix} \tag{7}$$

3. TRANSFORM THEOREMS

The method is based on Singular Value Decomposition of the system operator. This spectral decomposition is a generalisation for SVD of a matrix. For discrete-time systems and finite time horizon the operator is finite dimensional. In linear algebra, the SVD of a matrix describes it by a set of singular values σ_i and corresponding sets of singular input-vectors v_i and output vectors u_i. Any real or complex matrix X can be written as $X=U\cdot\Sigma\cdot V^*$, where $\Sigma=\mathrm{diag}\{\sigma_i\}$ and U and V are composed from u_i and v_i, respectively.

Theorem 1 Discrete power density spectrum of every orthogonal matrix computed as a sum of spectral density column vectors is constant and equal to **1**.

In particular, for matrix $\mathbf{V}=\{v_{ij}\}$, i,j=1…*N*,

$$\mathbf{S}_\mathrm{v}(\omega_k) = \sum_{j=1}^{N} S_j(\omega_k) = \frac{1}{N}\cdot\sum_{j=1}^{N}\left|\mathrm{DFT}_k[\mathbf{v}_j]\right|^2 = \mathbf{1} \tag{8}$$

where $\omega_k = \dfrac{k}{2\cdot T_p\cdot N}$, T_p – sampling period.

Theorem 2. Discrete input-output power density spectrum of system, can be computed as a sum of spectral density column vectors of product **U·S**.

The notation is following

$$\mathbf{S}_\mathrm{y}(\omega_k) = \frac{1}{N}\cdot\sum_{j=1}^{N}\left|\mathrm{DFT}_\mathrm{k}[\mathbf{u}_j\cdot s_{jj}]\right|^2 \tag{9}$$

where $\omega_k = \dfrac{k}{2\cdot T_p\cdot N}$, T_p – sampling period, $\sigma_i=s_{ii}$ – i – singular value of $\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{V}^\mathrm{T} = \hat{\mathbf{C}}\cdot\hat{\mathbf{L}}\cdot\hat{\mathbf{B}}$ decomposition.

Proofs of theorems 1 and 2 follows directly from the orthonormality of the SVD matrix and from unitary properties of the DFT transform. Detailed proof can be found in e.g. (Orlowski 2004).

4. AMPLITUDE AND PHASE DIAGRAMS APPROXIMATION

The relation between input and output power spectrum density and amplitude diagram is described following.

$$\mathbf{S}_\mathrm{y}(\omega_k) = \left|\mathbf{G}(\omega_k)\right|^2\cdot\mathbf{S}_\mathrm{x}(\omega_k) \tag{10}$$

taking into account theorem 1,

$$\left|\mathbf{G}(\omega_k)\right| = \sqrt{\mathbf{S}_\mathrm{y}(\omega_k)} \tag{11}$$

and finally

$$\left|\mathbf{G}(\omega_k)\right| = \sqrt{\frac{1}{N}\cdot\sum_{j=1}^{N}\sigma_j^2\cdot\left|\mathrm{DFT}_\mathrm{k}[\mathbf{u}_j]\right|^2} \tag{12}$$

Phase diagram, can be approximated by

$$\varphi(\omega_k) = \arg\left(\sum_{j=1}^{N} \sigma_j \cdot \frac{\mathrm{DFT_k}[\mathbf{u}_j]}{\mathrm{DFT_k}[\mathbf{v}_j]} \right) \qquad (13)$$

Operator's notation, for which has been defined fundamental frequency analysis tools, can be used for description and simulation both for time varying and time invariant systems. It assumed that, systems are defined on finite time horizon.

5. DEGREE OF SYSTEM TIME VARIABILITY

Dynamic linear systems can be variable either in the frequency domain (linear time invariant systems – LTI systems) or in the time domain (linear frequency invariant systems – LFI systems). The main difference between LTI and LFI systems can be recognized by comparison of the output function. The output function of an LTI system is a time domain convolution or frequency domain multiplication, whereas the output function of LFI system is a time domain multiplication or frequency domain convolution. Essentially, LFI is a static system with a time variant gain. In the most general case a dynamic linear system can be variable in both frequency and time domain. From the mathematical point of view a dichotomous classification into time or frequency variant systems is clearly defined. However, from the practical viewpoint determining the degree of time nonstationarity on a continuous scale is more significant than such a dichotomous classification (variant – invariant). This is due to the fact that scattering of system parameter values not necessarily must imply a change in system properties as a whole; if not, the changes can be insignificant.

To illustrate the problem of determining the degree of time variability let us consider a system with a periodically time variant gain. The system is given in the state space by eq. (1-2).

The system at hand is a single-input single-output, first order one. Thus system matrices are scalars. A simple analysis can be performed if $A(k)\equiv 0$. In such a case we have to do with a typical time-delay system. Depending on values the B and C matrices take the following four cases given in Table 1 may be distinguished.

In this example the degree of time nonstationarity is dependent on the ε parameter. The greater is ε, the greater is the degree of system time nonstationarity. As $\varepsilon \to 0$ the system response tends asymptotically to the response produced by a time invariant system. The output frequency spectrum varies with the degree of system time variability. In the output spectrum side bands with $\omega_0 - \omega_p, \omega_0 + \omega_p$ frequencies appear and, in addition, the amplitude of the main band ω_0 diminishes. These phenomena are caused by the system variability modulation.

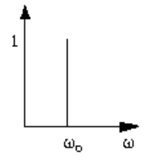
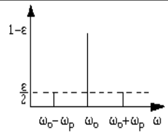
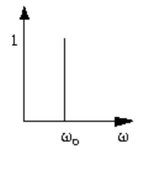
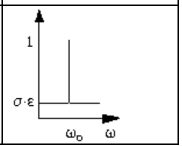
	B(*k*), C(*k*)	Output spectra
Stationary	B(*k*)=1 C(*k*)=1	
Non-stationary	B(*k*)= $1-\varepsilon+\varepsilon\cdot\cos\left(\omega_p\cdot k\cdot T_s\right)$ C(*k*)=1	
Quasi-stationary	B(*k*)= $1-\varepsilon+\varepsilon\cdot\cos\left(\omega_p\cdot k\cdot T_s\right)$ C(*k*)= $\dfrac{1}{1-\varepsilon+\varepsilon\cdot\cos\left(\omega_p(k-1)T_s\right)}$	
Quasi-non-stationary	B(*k*)= $1+\varepsilon\cdot\mathrm{randn}$ C(*k*)=1	

Table 1. Scattering of the signal spectrum for various systems.

A little different effect is produced by white noise modulation. The noise modulated by any other signal still remains the noise, only the parameters are changed. In the quasi-nonstationary system the noise has been modulated by a sinusoidal input. As a result, a white noise with another variance is produced. A change in variance entails a change in standard deviation, in proportion to which is the value of the zero-frequency component in the system output magnitude-frequency response.

Time and frequency variability for continuous systems was dealt with in (Bello 1963, Coates 1998 Debnath 2001, Kozek 1997). An approach has been proposed there that is based on the impulse response of a time variant system, which presents an outgrowth of the approach originated by Zadeh.

This approach has been used successfully for analysis of time variant communication channels. The measure of potential time-frequency shifting, which system can impart is spreading function. For example, asymetrical spreading function or delay Doppler spread function introduced by Bello (1963) is obtained:

$$S_H^{(1/2)}\left(\tau,\nu\right) = \int_t h(t,t-\tau)\cdot e^{-i\cdot 2\cdot\pi\cdot\nu\cdot t}\mathrm{dt} \qquad (14)$$

For the spreading function to be evaluated, the knowledge of the set of system impulse responses $h(t,t\text{-}\tau)$ is needed, where t is the determined time instant, and t-τ is the point in time at which the impulse has been generated.

The generalised spreading function of an LTI system with kernel $h(t,t-\tau) = g\left(t-\tau\right) = g\left(\tau\right)$

$$S_H(\tau,\nu) = g(\tau)\cdot\delta(\nu) \tag{15}$$

and is concentrated along τ axis reflecting the system can only cause time shifts. Dirac function is denoted by $\delta(\nu)$.

Employing the spreading function for system analysis makes it possible to determine the system variability in both time and frequency domains. To do this, however, the knowledge of system responses obtained with appropriate resolution over a wide time horizon is needed. An attempt made by the author to determine the spreading function for a discrete system defined over a finite time horizon failed. The numerical algorithm turned out to be unstable, and the results obtained were almost independent of actual changes in system parameters.

Another approach that may be useful to determine the system time variability is the modal analysis (Maia 1997, Liu 1999). It yields a relatively great body of data the interpretation of which requires some knowledge and experience. The cardinal virtues of PMP are its numerical stability independent of the time horizon and a very wide field of application.

Analysis of output amplitude spectra carried out by the author for different systems allows one to define certain functions that enable the system time variability to be measured. The test input should be chosen first. In the light of SVD properties, especially those revealed by Theorem 1, the optimal test signal is represented by the system singular vectors v_i with their corresponding weights σ_i. If so, the measure of the system nonstationarity may be defined as:

(I) Weighted main band attenuation.

The less the output is affected by the parameter nonstationarity, the smaller is the coefficient (I). Rate of parameter changes is here of secondary importance.

Numerically, the coefficient (I) can be evaluated as a sum of squared differences between consecutive discrete values for power spectral density of normed characteristic vectors for input and output spectra:

$$S_{\mathrm{var}1} = \sum_{i=1}^{N} \frac{\sigma_i}{\sigma_1}\cdot \left\| \left|\mathrm{DFT}[\mathbf{v}_i]\right| - \left|\mathrm{DFT}[\mathbf{u}_i]\right| \right\|_2 \tag{16}$$

(II) Weighted relative distance between the side bands and the main band.

The value of the coefficient (II) depends on the rate of parameter changes. The slower the changes, the smaller the coefficient (II).

The coefficient (II) is evaluated from

$$S_{\mathrm{var}2} = \Delta f\cdot\sum_{i=1}^{N}\left|k_{mv}(i) - k_{mu}(i)\right|\cdot\frac{\sigma_i}{\sigma_1} \tag{17}$$

where

$k_{mv}(i)$ is index of maximal value in vector $\mathbf{VV}_i$=|DFT[$\mathbf{v}_i$]|, $k_{mu}(i)$ is index of maximal value in vector $\mathbf{UU}_i$=|DFT[$\mathbf{u}_i$]|, resolution in frequency domain $\Delta f = \frac{1}{T\cdot N}$ is normalisation factor for eq. (17).

(III) Main band attenuation.

Similar to coeff. (I), but simplified to the most significant vector only.

$$S_{\mathrm{var}3} = \left\| \left|\mathrm{DFT}[\mathbf{v}_1]\right| - \left|\mathrm{DFT}[\mathbf{u}_1]\right| \right\|_2 \cdot 1000 \tag{18}$$

(IV) Quadratic weighted main band attenuation.

Similar to coeff. (I), with only the difference, that linear terms are replaced by quadratic terms.

$$S_{\mathrm{var}4} = \sqrt{\sum_{i=1}^{N}\frac{\sigma_i^2}{\sigma_1^2}\cdot\left\| \left|\mathrm{DFT}[\mathbf{v}_i]\right|^2 - \left|\mathrm{DFT}[\mathbf{u}_i]\right|^2 \right\|_2} \tag{19}$$

(V) Weighted relative distance between corresponding bands.

Coefficient (V) is similar to coeff. (II). The improvement is that the algorithm for computation take into account not only first maximal value in vectors **U**, **V** but also consecutive values sorted in decreasing order with corresponding weights

$$S_{\mathrm{var}5} = \Delta f\cdot\sum_{i=1}^{N}\left[\frac{\sigma_i}{\sigma_1}\cdot\sum_{j=1}^{N/2}\left(\left|k_v(j,i) - k_u(j,i)\right|\cdot\frac{\mathbf{VV}_{j,i}+\mathbf{UU}_{j,i}}{\mathbf{VV}_{j,1}+\mathbf{UU}_{j,1}}\right)\right]_{\mathrm{sorted}} \tag{20}$$

where $\mathbf{VV}_{i,j}$, $\mathbf{UU}_{i,j}$ are column sorted matrices of DFT **v** and **u** respectively. $k_v(j,i)$ and $k_u(j,i)$ indexes of sorted matrices.

For time invariant systems all coefficients should be equal to zero. In the next section an attempt will be made to check whether the proposed relationships present a good measure of the system nonstationarity.

6. NUMERICAL EXAMPLES

In the light of how the variability coefficients (16,20) are defined, an analysis of the influence the ε parameter changes of a non-stationary system exert on the whole system properties may be of interest. To carry out a numerical analysis four models of systems with a specified magnitude of parameter variations that has been presented and discussed in Section 5 are used. As a measure of nonstationarity the definitions (16-20), and for the purpose of comparison the graphical shape of step responses have been taken. In this section coefficients of eq. (16-20) vs. ε

parameter are evaluated for similar systems by way of simulation. For purposes of simulation it is assumed that the system is also nonstationary in the frequency domain and represents a time lag $A(k)$ = 0.1. Sampling period is scaled to improve resolution in time domain and is equal to T_s=3.2/N=0.0128s, k=1,2,…,N, $\omega_p = 20$ where N=250 is time horizon length. The remaining system parameters are the same as given in Table 1.

6.1. Nonstationary system (cosinusoidal)

System coefficients are as follows:

$$A(k) = 0.1, C(k) = 1, D(k) = 0,$$
$$B(k) = 1 - \varepsilon + \varepsilon \cdot \cos\left(\omega_p \cdot k \cdot T_s\right)$$

It is assumed here that ε varies within the range $\varepsilon \in \left\langle 10^{-8}, 1 \right\rangle$, from where 10 values of ε have been chosen to evaluate the system variability coefficient. The results obtained are summarized in Fig. 1. Results yielded by eq. (16) are marked with "x" (thin line). Results yielded by eq. (17) are marked with "+"(thin line). For $\varepsilon < 10^{-1}$ S_{var2} takes the value 0. Results yielded by eq. (18) are marked with "o" (thick line). Results yielded by eq. (19) are marked with "□"(thick line). Results yielded by eq. (20) are marked with "◊"(thick line).

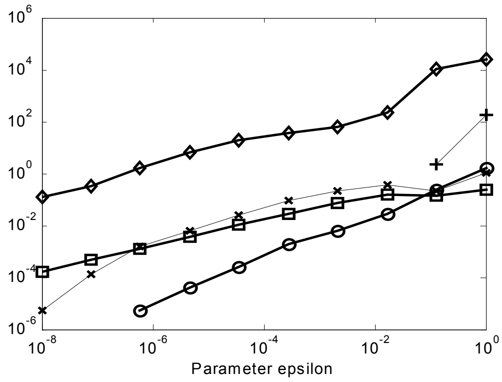


Figure 1. Variability coefficient vs. ε for a nonstationary system (cosinusoidal).

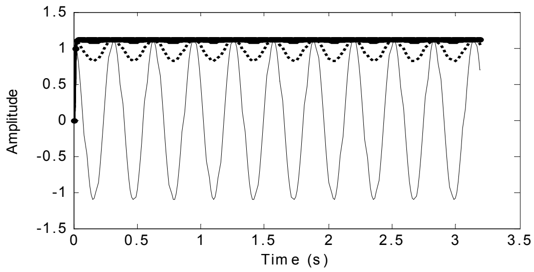


Figure 2. Step responses obtained for ten biggest values of ε. (nonstationary system).

Step responses corresponding to ten highest values of ε are depicted in Fig. 2.

6.2. Quasi-stationary system

System coefficients are as follows:

$$A(k) = 0.1, \quad B(k) = 1 - \varepsilon + \varepsilon \cdot \cos\left(\omega_p \cdot k \cdot T_s\right),$$

$$C(k) = \frac{1}{1 - \varepsilon + \varepsilon \cdot \cos\left(\omega_p \cdot (k-1) \cdot T_s\right)}, D(k) = 0$$

It is assumed here that ε varies within the range $\varepsilon \in \left\langle 10^{-6}, 1 \right\rangle$, from where 10 values of ε have been chosen to evaluate the system variability coefficient. The results obtained are summarized in Fig. 3. S_{var2} takes the nonzero value only for ε=1.

Step responses corresponding to ten highest values of ε are depicted in Fig. 4.

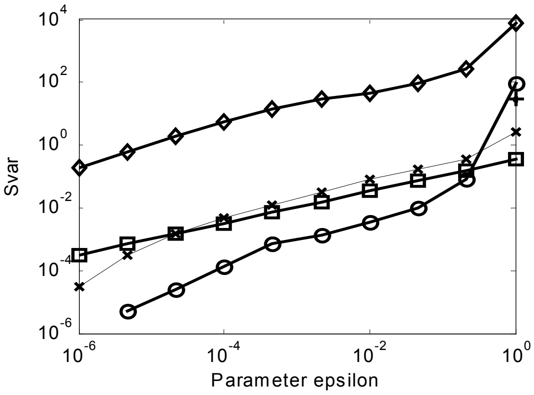


Figure 3. Variability coefficient vs. ε for a quasi-stationary system.

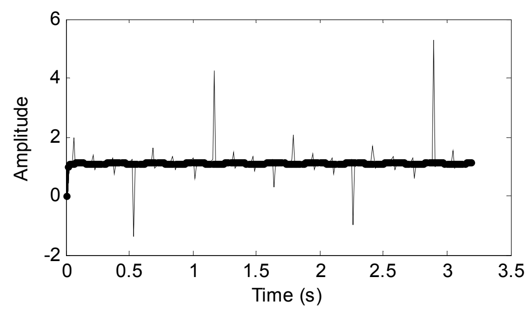


Figure 4. Step responses obtained for ten biggest values of ε (quasi-stationary system).

6.3. Quasi-nonstationary

System coefficients are as follows:

$$A(k) = 0.1, B(k) = 1 + \varepsilon \cdot \text{randn}, C(k) = 1, D(k) = 0,$$

where randn is random noise described by normal distribution with parameters $N(0,1)$.

It is assumed here that ε varies within the range $\varepsilon \in \left\langle 10^{-8}, 1 \right\rangle$, from where 10 values of ε have been chosen to evaluate the system variability coefficient.

The results obtained are summarized in Fig. 5. For $\varepsilon < 0.1$ S_{var2} takes the value 0.

Step responses corresponding to ten highest values of ε are depicted in Fig. 6.

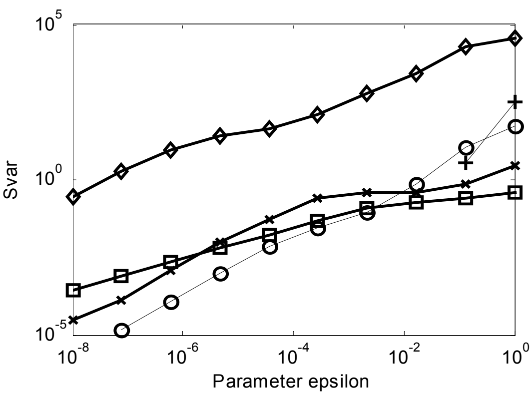


Figure 5. Variability coefficient vs. ε for a quasi-nonstationary system.

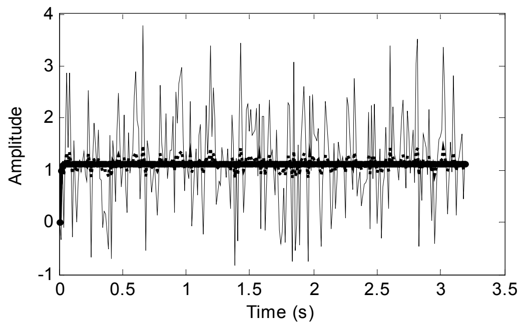


Figure 6. Step responses obtained for ten biggest values of ε (quasi-nonstationary system).

7. CONCLUSION

The paper has introduced a new method for estimating the degree of variability of the system.

It is shown that the time-variability coefficient of the system can be defined using data from the SVD-DFT analysis of the system. The key idea of the proposed method is to modify the input-output spectra of LTV systems. The quantity of the modification can be a measure of time-variability of the system.

Three examples with numerical data provide illustrations of how the variability coefficient can help to understand the degree of time variations due to different parameter values in the system model.

In the course of the study some drawbacks of the proposed methods have been revealed. This fact raises some open questions that will direct further studies.

- The coefficient of the system variability S_{var1} changes its properties whenever any of the poles goes beyond the unit circle. Would it be possible to generalize these observations to arbitrary systems and to define, on this basis, a stability concept for discrete-time systems determined over a finite time horizon ? (Notice that the response of such systems is always bounded)
- The coefficient of the system variability S_{var2} gives results only for large ε and exhibits a certain numerical instability (lower for S_{var3}).
- The feasibility of S_{var2} evaluation is conditioned largely by the resolution obtained in the frequency domain. Requirements to be met here are much higher than those for e.g. S_{var1}.
- It may be concluded from figures 1, 3, 5 that coefficients S_{var5}, S_{var4}, and also S_{var1} are more stable numerically than S_{var2}, S_{var3}, even on short time horizons.

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