THE LADY, THE BANDITS AND THE BODY GUARDS – A TWO TEAM DYNAMIC GAME

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Abstract: A two team dynamic game, the Lady and the Body-Guards versus the Bandits is defined. The Bandits' team objective is to capture the Lady while the Lady and her Body-Guards objective is to prevent it. The Body-Guards are trying to intercept the Bandits prior to their arrival to the proximity of the Lady.
An approach to formulation and solution of the game is presented. The approach to the solution is based on the Multiple Objective Optimization and Differential Games theories. The approach to the solution is demonstrated for linear system and quadratic criterion. Closed loop Noninferior Nash equilibrium solution in linear strategies for specific game policy is derived for the players of the two teams.

Keywords: Game theory, Nash games, Differential games, Multiobjective optimization, Co-operative control, Non-cooperative control, Team games.

1. INTRODUCTION

The Lady and a single Bandit is the classical zero-sum two persons differential game (Ho, *et al*, 1965; Ben-Asher and Yaesh, 1998; Bryson and Ho, 1975). The Lady is the evader and the Bandit is the pursuer.
Here we consider a two team nonzero-sum game, the evaders'–pursuers' teams' game. The evaders' team objective is to avoid the capture of the Lady. This is accomplished by a cooperative action of the Body Guards that are intercepting the Bandits and by avoidance maneuvers of the Lady. The Bandits in their turn are trying to avoid the Body Guards or to neutralize them, while not jeopardizing their effort to intercept the Lady.
The differential game theory is an approach to solve the N-person noncooperative and cooperative (team) game problems. The subject of multi person games with conflicting objective is covered in (Basar and Olsder, 1982).
Here we deal with a noncooperative game of two teams. That is, the objective of the two teams is conflicting, therefore as teams they will act non-cooperatively. However, within each team the players act cooperatively to achieve the team's objective.
The subject of multi-team systems and their optimization within the cooperative and non-cooperative context for static games is dealt with in Liu and Simaan, 2004). This reference presents as well an up to date literature survey on the multi-team systems optimization.
Multi-team with discrete dynamics games are dealt with for military air operation assignment in (Liu *et al.*, 2003).
For non-cooperative games the Nash equilibrium is a common optimality paradigm. For cooperative (team) game it is the Multiple-Objective Optimization Theory (MOO) approach that gives the Pareto-optimal strategy.

A two team three person problems are considered in (Shinar and Silberman, 1995; Boyell, 1976). However, although (Shinar and Silberman, 1995)] considers three persons it solves a two person game as the defended ship is not cooperating with the defending missile. (Boyell, 1976) provides a solution under the assumption of collision course for the vehicles.

A two team three person game - the Lady, the Bandit and the Body-Guard (LBBG) game is presented and solved in (Rusnak, 2004).

The objectives of the individual players of the two team game are formulated at different times. The author did not find a formulation of N-person differential dynamic games where the objectives are formulated at different time instants and on different time intervals except of (Rusnak, 2004). Moreover, he did not find differential games problem where during the game the objective of the players change their character from maximizers (bandit is avoiding the body guard) to minimizers (bandit is intercepting the Lady).

In the paper the two team game, the Lady and the Body-Guards, versus the Bandits is defined. In the game there are many possibilities-contingencies for each team to achieve the Lady's survival or capture, respectively. Thus the problem presented is not well defined for the application of the existing approaches of solution and it is not clear how to derive the optimal strategies of for each player. In the paper the following solution procedure is devised: All relevant engagement possibilities-contingencies for the Bandits team and Body-Guards teams are enumerated. Each possibility-contingency is called a policy. For each policy of the Bandits team and each policy of the Body-Guards team (called the game policy), a well defined multi-person game is created. This enables the use of the differential games and Multi Objective Optimization theories for: (a) the derivation of the optimal non-inferior Nash Equilibrium strategies for each player; and (b) the derivation of the game cost. The set of costs over the set of the game policies defines a two players matrix game that enables the derivation of the non-inferior saddle point game policy (the teams are the players in this matrix game).

As an example a two teams five persons (Lady, two body guards, 2 bandits) for particular policy - is explicitly solved for linear systems and quadratic performance indices.

The approach of the solution of the game for specific policy makes a use of the impulse function in the objective. This leads to a solution in the form of Riccati equation that includes the impulse function in the indices. The solution of the corresponding Riccati equation is not continuous. Necessary and sufficient conditions for existence of the solution are presented.

2. NOTATION

The two-team game is a special case of multi-team game. In multi-team game we assume that there are (N) teams each team has n_i, i=1,2,…N players. So the multi-team game is a special case of $\sum_{i=1}^{N} n_i$ persons game. We use the following notation to specify this and call it the $(N,n_1,n_1,\ldots n_N)$ team game. Within this notation the two person game is a (2,1,1) game.

3. THE LADY, THE BODY GUARDS AND THE BANDITS GAME

In the section the two team game, the Lady and the Body-Guards, versus the Bandits is defined. As well we state the information pattern of the game we consider in this paper. The following $(2, n_e, n_p)$ game - a two team, game is considered:

The objective of the pursuer's team, the Bandits, is to capture the Lady. In this game we assume that capture means that at least one Bandit reached a close proximity of the Lady. Therefore, the objective of the Bandit's team is that at least one of Bandits will minimize his distance to the Lady, while the Lady is trying to maximize her distance from the members of the Bandits' team.

The Body-Guards' team objective is to prevent from the members of the Bandits' team the arrival into a close neighborhood of the Lady.

We assume that the information pattern (Basar and Olsder, 1982, pp.207) is the "closed loop perfect state" given by

$$\eta=\{x(s),\ 0\leq s\leq t\},$$

or the "memoryless perfect state" given by

$$\eta=\{x(s),\ x_o\}.$$

4. APPROACH TO THE SOLUTION

The most distinctive feature of the presented game is that the players have contradicting–different objectives during different stages of the game. The team objective is opposing to the objective of the individuals. The minimization or maximization of the respective miss distance is conflicting to the individual objective of energy preservation.

4.1 Engagement Policies

Within the definition of the game it is not clear how to formulate the different objectives of each player in the game. There are several possibilities-contingencies-engagement policies to achieve the goal above. For example:

Possible Bandits' team policies:

(a) All Bandits are trying to intercept the Lady. (Each Bandit is trying to achieve the objective individually.);

(b) Only a subset of the Bandits, the attacking Bandits, are trying to intercept the Lady while the others, the assistance Bandits, are trying to prevent from the Body-Guards to intercept the attacking Bandits; and more.

We denote the set of the Bandits' team policies $\Pi_p=\{\pi_{p1},\pi_{p2},\ldots.\}$.

Possible Body Guards' team policy:

(a) Each Body-Guard is trying to prevent from some Bandit the interception of the Lady;

(b) Groups of Body Guards are trying to prevent from some Bandits the interception of the Lady; and more

We denote the set of the Body Guards' team policies $\Pi_e=\{\pi_{e1},\pi_{e2},\ldots\}$.

The sequence of engagement-encounter is policy. The set of game policies is the set of all ordered pairs $\Pi_G=\{(\pi_{pi},\pi_{ej})|, \pi_{pi}\in\Pi_e, \pi_{pj}\in\Pi_p.\}$.We assume that the policy in a specific game of each team is known to both teams. There are N_π policies.

4.2 A two team balanced game policies

In this paper we assume a two team balanced game. Balanced game is defined as the case when the number of Body-Guards is equal to the number of Bandits. This means that we deal with a (2,n+1,n) game under a **Type I Policy. The Type I Policy** is defined as: All Bandits are trying to intercept the Lady and each Body-Guard is trying to prevent from different Bandit this interception.

4.2.1 A (2,3,2) game with Type I Policy

The Body-Guards team objective is to intercept all the members of the Bandit's team prior to their arrival into a close neighborhood of the Lady. That is, the objective of each Body-Guard (BG) is minimizing his distance to some Bandit (B) while the objective of this Bandit is maximizing his distance from that Body-Guard and at the same time not to jeopardize his effort to intercept the Lady. Each Body-Guard has only one chance to intercept a Bandit. During this phase of the game the Bandit is the evader and the Body Guard is the pursuer. A Body-Guard ceases to exist after a Bandit interception. However, it is assumed that the success of the Body-Guard, i.e. Bandit's annihilation, is not guaranteed. Figure 1 presents the (2,3,2) game with Type I Policy implementation. Therefore, the game does not terminate neither at the expected interception time, t_{f1}, of Bandit #1 by Body Guard #1, nor at the expected interception time, t_{f3}, of Bandit #2 by Body Guard #2, but continues until Bandit #2 is expected to intercept the Lady at t_{f4}. The time instant t_{f2} is the expected interception time of the Lady by Bandit #1. We assume that the time intervals $[t_o,t_{f1}]$,$[t_o,t_{f2}]$, $[t_o,t_{f3}]$ and $[t_o,t_{f4}]$ are defined a priory and they define the different stages of the game.

Notice: (1) Even in the relatively simple **Type I Policy** of the balanced (232) game there are four contingencies. That is, we assumed that, in figure 1, first BG#1 engages B#1 and BG#2 engages B#2, denoted as [(11),(22)]. However, additional three combinations are possible, these are the [(12),(21)], [(22),(11)], [(21),(12)] engagements. The difference between them is the level of commitment and skill of each player. These contingencies degenerate if the Body-Guards and the Bandits, respectively, are of equal commitment and skill.

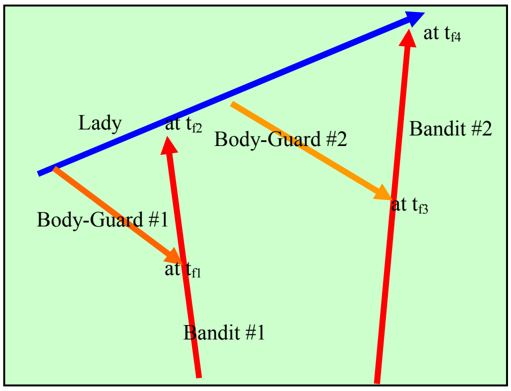

Figure 1: Illustration of a (2,3,2) game under execution of **Type I Policy**.

(2) For $t>t_{f2}$ the (232) game reduces to the (221) game, which is the Lady, the Bandit and the Body-Guard game presented and solved in (Rusnak, 2004).

4.2.2 A (2,3,2) game with non Type I Policy

As an example Figure 2 presents an example of a (2,3,2) game under a **non Type I policy** realization. Here B#1 has been selected to pursue the Lady. BG#1 is heading toward interception of B#1. B#2 has been selected as the assistant. Therefore, B#2 is heading toward BG#1 in order to prevent him to intercept B#1. BG#2 is the defender of BG#1.

4.3 Structure of the solution

We assume that the value of the (2, n_e, n_p) game for each game policy from Π_G has been computed. This defines a matrix game for two players on the set of the game policies. The policy of each team is the player in this two person matrix game.

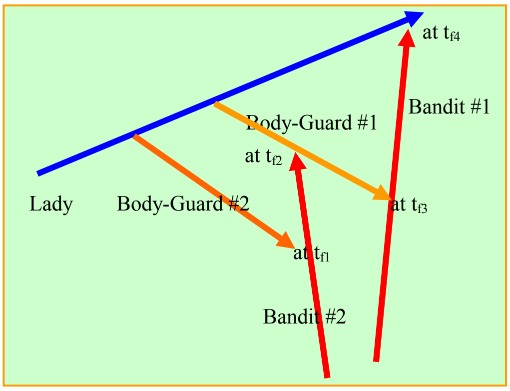

Figure 2: Illustration of a (2,3,2) game under a **non Type I policy** execution.

The preceding observations and approach partitions the problem of finding the optimal strategies for all players (engagement sequence and actions) of a $(N,n_1,n_1,\ldots n_N)$ game into the following stages:

1) Specify all relevant policies (possible, acceptable policies, all policies);
2) Find the non-inferior Nash equilibrium for each

policy and the respective cost;

3) Find the saddle point (if exists) of the policies of the matrix game.

The preceding approach transforms the original problem to a problem of solving N_π (number of policies) dynamic multi-person games each with $N_{BG}+N_B+1$ players and one static matrix two players game.

5. THE (232) GAME UNDER TYPE I POLICY

This section presents a formal definition of a (232) dynamic game under Type I Policy within the domain of the linear systems and quadratic performance indices. Let us introduce the following notations

y_{u1} – the position of the Bandit #1
y_{u2} – the position of the Bandit #2
y_v – the position of the Lady
y_{w1} – the position of the Body-Guard#1
y_{w2} – the position of the Body-Guard#2

We assume $t_{f1}<t_{f2}<t_{f3}<t_{f4}$. There are four terminal objectives at different time instants as follows from the section 3 and figure 1:

I) at the expected interception moment (EIM) of the Lady by the Bandit #2, t_{f4},

$$\min_{u2}\max_{v}\left[y_{u2}(t_{f4})-y_v(t_{f4})\right]^2; \tag{5.1}$$

II) at the EIM of Bandit #2 by the Body-Guard #2, t_{f3},

$$\min_{w2}\max_{u2}\left[y_{u2}(t_{f3})-y_{w2}(t_{f3})\right]^2=\min_{u2}\max_{w2}-\left[y_{u2}(t_{f3})-y_{w2}(t_{f3})\right]^2 \tag{5.2}$$

III) at the EIM of the Lady by the Bandit #1, t_{f2},

$$\min_{u1}\max_{v}\left[y_{u1}(t_{f2})-y_v(t_{f2})\right]^2 \tag{5.3}$$

IV) at the EIM of Bandit #1 by Body-Guard #1, t_{f1},

$$\min_{w1}\max_{u1}\left[y_{u1}(t_{f1})-y_{w1}(t_{f1})\right]^2=\min_{u1}\max_{w1}-\left[y_{u1}(t_{f1})-y_{w1}(t_{f1})\right]^2 \tag{5.4}$$

Further, each participant wishes, simultaneously, to minimize its energy expenditure formulated as

V)

$$\min_{u_1}\int_{t_o}^{t_{f2}} u_1^T R_{u1} u_1 dt \tag{5.5}$$

VI)

$$\min_{u_2}\int_{t_o}^{t_{f4}} u_2^T R_{u2} u_2 dt \tag{5.6}$$

VII)

$$\min_{v}\int_{t_o}^{t_{f4}} v^T R_v v dt \tag{5.7}$$

VIII)

$$\min_{w_1}\int_{t_o}^{t_{f1}} w_1^T R_{w1} w_1 dt \tag{5.8}$$

IX)

$$\min_{w_2}\int_{t_o}^{t_{f3}} w_2^T R_{w2} w_2 dt \tag{5.9}$$

where

v – the Lady's control (evader)
w_1 – the Body-Guard #1 control (pursuer)
w_2 – the Body-Guard #2 control (pursuer)
u_1 – the Bandit #1 control
u_2 – the Bandit #2 control
t_o – the moment the game starts
t_{f1} – the EIM of B#1 by BG #1
t_{f2} – the EIM of Lady by B #1
t_{f3} – the EIM of B#2 by BG #2
t_{f4} – the EIM of the Lady by B#2

Notice that the energy expenditure of the game participants is defined on different time intervals.

6. REFORMULATION OF THE (232) GAME UNDER TYPE I POLICY

In order to solve the (232) team Game problem we scalarize the objectives, defined in section 5, as suggested by the MOO theory. As the criteria are not convex, once a solution is derived the optimality of the solution must be verified. This leads to conditions required for existence and optimality of the solution. Thus, the objective of the game is

$$J=\frac{1}{2}\left\{\begin{array}{l} x^T(t_{f4})G_4x(t_{f4})-x^T(t_{f3})G_3x(t_{f3}) \\ +x^T(t_{f2})G_2x(t_{f2})-x^T(t_{f1})G_1x(t_{f1}) \\ +\int_{t_o}^{t_{f2}} u_1^T R_{u1}u_1dt+\int_{t_o}^{t_{f4}} u_2^T R_{u2}u_2dt \\ -\int_{t_o}^{t_{f4}} v^T R_v v dt-\int_{t_o}^{t_{f1}} w_1^T R_{w1}w_2dt-\int_{t_o}^{t_{f3}} w_2^T R_{w2}w_2dt \end{array}\right\} \tag{6.1}$$

The problem being considered here is the optimization of J, that is

$$\max_{w_1}\max_{w_2}\ \min_{u_1}\min_{u_2}\max_{v} J \tag{6.2}$$

subject to the differential equation

$$\begin{aligned} &\dot{x}(t)=Ax(t)+B_1u_1(t)+B_2u_2(t)+Cv(t) \\ &+D_1(t)w_1(t)+D_2(t)w_2(t), x_o(t_o)=x_o, \\ &z_i(t)=N_ix(t), i=1,2,3,4. \end{aligned} \tag{6.3}$$

where

$$D_1(t)=\begin{cases}D_1, & t\le t_{f1} \\ 0, & t_{f1}<t\end{cases},\quad D_2(t)=\begin{cases}D_2, & t\le t_{f3} \\ 0, & t_{f3}<t\end{cases}, \tag{6.4}$$

$$B_1(t)=\begin{cases}B_1, & t\le t_{f2} \\ 0, & t_{f2}<t\end{cases},$$

and $x(t)=[y_{u1}\ y_{u2}\ y_v\ y_{w1}\ y_{w2}\ \ldots\ldots]^T$,

$$\begin{aligned} z_4&=y_v-y_{u2}; \\ z_3&=y_{u2}-y_{w2};. \\ z_2&=y_v-y_{u1}; \\ z_1&=y_{u1}-y_{w1}; \end{aligned} \tag{6.5}$$

Equations (6.4) reflect the fact that Body-Guard #1 ceases to exist for $t>t_{f1}$, Body-Guards #2 ceases to exist for $t>t_{f3}$ and Bandit #1 ceases to exist for $t>t_{f2}$.

The objective (6.1) is rewritten by the use of the impulse function $\delta(t)$, (a generalized function) defined by

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)\cdot \qquad (6.5)$$

Then (6.1) is

$$(6.6)$$

$$J = \frac{1}{2}\int_{t_o}^{\infty}\left\{\begin{array}{l} x^T(t)\begin{bmatrix} G_4\delta(t-t_{f4}) - G_3\delta(t-t_{f3}) \\ + G_2\delta(t-t_{f2}) - G_1\delta(t-t_{f1}) \end{bmatrix} x(t) \\ + u_1^T R_{u1} u_1 + u_2^T R_{u2} u_2 - v^T R_v v \\ - w_1^T R_{w1} w_2 - w_2^T R_{w2} w_2 \end{array}\right\} dt$$

where

$$G_i = N_i^T g_i N_i, i = 1,2,3,4. \qquad (6.7)$$

The problem can be formulated in the discrete domain. Then all indices are finite and no generalized functions are needed. The continuous solution presented in the following is then derived by limiting the time interval by procedures presented for example in (Gelb, Ed. 1974), thus justifying the use of a generalized functions.

7. CANDIDATE SOLUTION OF THE (232) GAME UNDER TYPE I POLICY

To arrive at a candidate solution we proceed by constructing the Hamiltonian (Lewin, 1994; Ben-Asher and Yaesh, 1998):

$$(7.1)$$

$$H = \frac{1}{2}\left\{\begin{array}{l} x^T(t)\begin{bmatrix} G_4\delta(t-t_{f4}) - G_3\delta(t-t_{f3}) \\ + G_2\delta(t-t_{f2}) - G_1\delta(t-t_{f1}) \end{bmatrix} x(t) \\ + u_1^T R_{u1} u_1 + u_2^T R_{u2} u_2 - v^T R_v v - w_1^T R_{w1} w_2 - w_2^T R_{w2} w_2 \end{array}\right\}$$
$$+ \lambda^T \begin{bmatrix} Ax(t) + B_1 u_1(t) + B_2 u_2(t) \\ + Cv(t) + D_1(t) w_1(t) + D_2(t) w_2(t) \end{bmatrix}$$

where $\lambda(t)$ is the costate, and the terminal condition is $\lambda^T(\infty) = 0$ .Then, requiring the necessary conditions, we arrive at

$$\begin{array}{ll} H_{u1} = u_1^T R_{u1} + \lambda^T B_1 = 0 & u_1 = -R_{u1}^{-1} B_1^T \lambda \\ H_{u2} = u_2^T R_{u2} + \lambda^T B_2 = 0 \quad \text{and} & u_2 = -R_{u2}^{-1} B_2^T \lambda \\ H_v = -v^T R_v + \lambda^T C = 0 & v = R_v^{-1} C^T \lambda \\ H_{w1} = -w_1^T R_{w1} + \lambda^T D_1 = 0 & w_1 = R_{w1}^{-1} D_1^T \lambda \\ H_{w2} = -w_2^T R_{w2} + \lambda^T D_2 = 0 & w_2 = R_{w2}^{-1} D_2^T \lambda \end{array} \qquad (7.3)$$

Notice that the inclusion of the w_1,w_2 and u_1 terms under the integral in (6.7) is justified by the last result (7.3) that shows that $w_1(t)=0$ for $t_{f1}<t$, $w_2(t)=0$ for $t_{f3}<t$ and $u_1(t)=0$ for $t_{f2}<t$.

The Two Point Boundary Value Problem is

$$(7.5)$$

$$\dot{x}(t) = Ax(t) - \begin{bmatrix} B_1 R_{u1}^{-1} B_1^T + B_2 R_{u2}^{-1} B_2^T - C R_v^{-1} C^T \\ - D_1 R_{w1}^{-1} D_1^T - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} \lambda(t),$$

$$\dot{\lambda}(t) = \begin{bmatrix} G_4\delta(t-t_{f4}) - G_3\delta(t-t_{f3}) \\ + G_2\delta(t-t_{f2}) - G_1\delta(t-t_{f1}) \end{bmatrix} x(t) - A^T\lambda(t),$$

$$x(t_o) = x_o,\ \lambda(\infty) = 0,$$

As the uniqueness of feedback Nash equilibrium for linear systems and quadratic indices has been verified only when the solution belongs to the set of linear strategies (Basar and Olsder, 1982), we seek solutions within this set. That is, we assume the existence of a matrix, P, such that

$$\lambda(t) = P(t)x(t), \qquad (7.6)$$

The formal solution is given by the time varying Riccati equation with indices that include the impulse function.

$$(7.7)$$

$$-\dot{P} = PA + A^T P + G_4\delta(t-t_{f4}) - G_3\delta(t-t_{f3})$$
$$+ G_2\delta(t-t_{f2}) - G_1\delta(t-t_{f1})$$
$$- P\begin{bmatrix} B_1 R_{u1}^{-1} B_1^T + B_2 R_{u2}^{-1} B_2^T - C R_v^{-1} C^T \\ - D_1 R_{w1}^{-1} D_1^T - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} P,\ P(\infty) = 0.$$

The optimal linear strategies in closed loop are

$$\begin{array}{l} u_1^* = -R_{u1}^{-1} B_1^T P(t)x(t) \\ u_2^* = -R_{u2}^{-1} B_2^T P(t)x(t)\ . \\ v^* = R_v^{-1} C^T P(t)x(t) \\ w_1^* = R_{w1}^{-1} D_1^T P(t)x(t) \\ w_2^* = R_{w2}^{-1} D_2^T P(t)x(t) \end{array} \qquad (7.8)$$

8. SUFFICIENT CONDITIONS FOR THE EXISTANCE OF THE (232) GAME UNDER TYPE I POLICY

By following closely (Ben-Asher and Yaesh, 1998) it is possible to show that sufficient conditions for existence of solution are

$$\begin{array}{l} R_{u1} > 0, R_{u2} > 0, R_v > 0, R_{w1} > 0, \\ R_{w2} > 0, G_1 \geq 0, G_2 \geq 0, G_3 \geq 0, G_4 \geq 0. \end{array} \qquad (8.1)$$

and the existence of a solution of the Riccati equation (7.7). This condition is that no conjugate points exist in the time interval $[0,t_{f4}]$.

9. SPECIFIC SOLUTION OF THE (232) GAME UNDER TYPE I POLICY

A specific solution of the (232) game under Type I Policy is achieved by integrating backward from the terminal condition the Riccati equation (7.7). When integrating backwards eq. (7.7) from infinity toward t_{f4}, the solution remains zero

We assume $t_{f1}<t_{f2}<t_{f3}<t_{f4}$.

$$P_{4\infty} = P(t) = 0, t_{f4} < t\,. \qquad (9.1)$$

Then, due to the impulse function

$$P(t_{f4}^+) = 0, \quad P(t_{f4}^-) = G_4\,. \qquad (9.2)$$

The solution for $t_{f3}^+ \leq t \leq t_{f4}^-$ is the solution of the Riccati equation

$$(9.3)$$

$$-\dot{P} = PA + A^T P - P\left[B_2 R_{u2}^{-1} B_2^T - C R_v^{-1} C^T\right] P, P(t_{f4}) = G_4$$

This is a two person zero-sum differential game solved in (Ho, *et al*, 1965; Ben-Asher and Yaesh, 1998) [3, 4]. The representation of the solution of (9.3) used here is (Rusnak, 1998)

$$(9.4)$$

$$P_{34} = P(t) = \Phi^T(t_{f4}, t)$$
$$\left\{ I + \int_t^{t_{f2}} G_4 \Phi(t_{f4}, \tau)\left[B_2 R_{u2}^{-1} B_2^T - C R_v^{-1} C^T\right]\Phi^T(t_{f4}, \tau) d\tau \right\}^{-1}$$
$$G_4 \Phi(t_{f4}, t)$$

where

$$\dot{\Phi}(t, t_o) = A\Phi(t, t_o), \dot{\Phi}(t_o, t_o) = I. \qquad (9.5)$$

Further, at t_{f3} additional impulse function induces jump in the solution, that is

$$P(t_{f3}^-) = P(t_{f3}^+) - G_3, \tag{9.6}$$

and the solution for $t_{f2}^+ \le t \le t_{f3}^-$ is the solution of the Riccati equation

$$\begin{aligned} -\dot{P} &= PA + A^T P \\ &- P\left[B_2 R_{u2}^{-1} B_2^T - C R_v^{-1} C^T - D_2 R_{w2}^{-1} D_2^T\right] P, \\ P(t_{f3}) &= P(t_{f3}^+) - G_3 \end{aligned} \tag{9.7}$$

The representation of the solution of (9.7) is

$$\begin{aligned} P_{23} &= P(t) = \Phi^T(t_{f3}, t) \\ &\left\{ I + \int_t^{t_{f3}} P(t_{f3}^-)\Phi(t_{f3}, \tau) \begin{bmatrix} B_2 R_{u2}^{-1} B_2^T - C R_v^{-1} C^T \\ - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} \Phi^T(t_{f3}, \tau) d\tau \right\}^{-1} \\ &P(t_{f3})\Phi(t_{f3}, t) \end{aligned} \tag{9.8}$$

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her, at t_{f2} additional impulse function induces jump in the solution, that is

$$P(t_{f2}^-) = P(t_{f2}^+) + G_2, \tag{9.9}$$

and the solution for $t_{f1}^+ \le t \le t_{f2}^-$ is the solution

$$\begin{aligned} -\dot{P} &= PA + A^T P - P \begin{bmatrix} B_1 R_{u1}^{-1} B_1^T + B_2 R_{u2}^{-1} B_2^T \\ - C R_v^{-1} C^T - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} P, \\ P(t_{f2}) &= P(t_{f2}^+) + G_2 \end{aligned} \tag{9.10}$$

The representation of the solution of (9.10) is

$$\begin{aligned} P_{12} &= P(t) = \Phi^T(t_{f2}, t) \\ &\left\{ I + \int_t^{t_{f2}} P(t_{f2})\Phi(t_{f3}, \tau) \begin{bmatrix} B_1 R_{u1}^{-1} B_1^T + B_2 R_{u2}^{-1} B_2^T \\ - C R_v^{-1} C^T - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} \Phi^T(t_{f2}, \tau) d\tau \right\}^{-1} \\ &P(t_{f2})\Phi(t_{f2}, t) \end{aligned} \tag{9.11}$$

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her, at t_{f1} additional impulse function induces jump in the solution, that is

$$P(t_{f1}^-) = P(t_{f1}^+) - G_1, \tag{9.12}$$

and the solution for $t \le t_{f1}^-$ is the solution of

$$\begin{aligned} -\dot{P} &= PA + A^T P \\ &- P \begin{bmatrix} B_1 R_{u1}^{-1} B_1^T + B_2 R_{u2}^{-1} B_2^T \\ - C R_v^{-1} C^T - D_1 R_{w1}^{-1} D_1^T - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} P, \\ P(t_{f1}) &= P(t_{f1}^+) - G_1 \end{aligned} \tag{9.13}$$

The representation of the solution of (9.13) is

$$\begin{aligned} P_{01} &= P(t) = \Phi^T(t_{f1}, t) \\ &\left\{ I + \int_t^{t_{f1}} P(t_{f1})\Phi(t_{f1}, \tau) \begin{bmatrix} B_1 R_{u1}^{-1} B_1^T + B_2 R_{u2}^{-1} B_2^T \\ - C R_v^{-1} C^T - D_1 R_{w1}^{-1} D_1^T \\ - D_2 R_{w2}^{-1} D_2^T \end{bmatrix} \Phi^T(t_{f1}, \tau) d\tau \right\}^{-1} \\ &P(t_{f1})\Phi(t_{f1}, t) \end{aligned} \tag{9.14}$$

Notice that the solution $P(t)$, $t_o \le t \le t_{f4}$ is discontinuous at the instants t_{f1}, t_{f1}, t_{f3}, t_{f4}.

10. SPECIFIC SUFFICIENT CONDITIONS FOR THE EXISTANCE OF THE SOLUTION OF THE (232) GAME

From (9.4, 9.8, 9.11 and 9.14) we can get the following sufficient conditions for solution of the LBBG game:

(i) $R_{w1}, R_{w2}, R_v, R_{u1}, R_{u2} > 0, \quad G_1, G_2, G_3, G_4 \ge 0,$

(ii) the non-singularity of the matrices in (9.4, 9.8, 9.11, 9.14) that are inverted.

11. CONCLUSIONS

A two team dynamic game, the Lady, the Bandits and the Body-Guards has been defined, an approach to the solution has been presented and a solution has been derived for linear system and quadratic criterion for a single policy in a balanced game. This is a closed loop non-inferior Nash equilibrium solution in linear strategies for the members of the team players. The solution is not continuous. Necessary and sufficient conditions for existence of the solution are presented. It follows that the lady and her Body-Guards must be coordinated in order to achieve optimal cost. If there is a lack of coordination the Bandits can use it for their own benefit.

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