PID CONTROLLER APPROXIMATING GMVC WITH POLE-PLACEMENT USING STEADY-STATE PREDICTIVE OUTPUT

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Abstract: This paper proposes a new design method of a PID controller. We design the PID controller based on generalized minimum variance control (GMVC) with a steady state predictive output (GMVCS). The proposed method has better approximation than conventional design schemes since the orders of compensators of a GMVCS law do not increase even if a dead-time increases. Furthermore, in this paper it is shown that GMVCS has an advantage that it is possible to place poles of the closed-loop system at desired places. Hence, the PID controller is designed based on the pole-placement GMVCS. Finally, we illustrate numerical simulation results in order to show the effectiveness of the proposed control algorithm.

Keywords: PID control, generalized minimum variance control, steady state predictive output, pole-placement

1. INTRODUCTION

In this paper, a design method of a PID controller which has partial feedback compensators based on Generalized Minimum Variance Control (abbreviated as GMVC) (Clarke, 1984) with a steady state predictive output (GMVCS) (Sato *et al.*, 2001) is given. It is difficult that a controller of GMVC is approximated by a PID compensator because the order of the controller of GMVC depends on a dead-time which is higher than the order of the PID compensator, that is, 2. If a controlled plant is given by a second-order plus dead-time model, a numerator of a GMVC law is second-order and is approximated exactly by a numerator of the PID controller (Yamamoto *et al.*, 1998; Yamamoto *et al.*, 1999). Since most chemical processes are approximated and represented by second-order plus dead-time sufficiently, in this paper the controlled plant is expressed as the above. However increasing the dead-time of the controlled plant, the order of the controller of GMVC increases and the difficulty of approximation increases.

Since in designing GMVCS(Sato *et al.*, 2001) the steady state predictive output is used instead of the dead-time predictive output, the order of a GMVCS law is determined regardless of the dead-time. Hence, the increase of the order of GMVCS law is not caused by the increase of the dead-time and remains in relatively low-order. The method given in this paper designs the PID controller approximating GMVCS law. The PID controller used in this paper differs from an usual PID controller and has partial feedback compensators in order that the GMVCS law is approximated by the PID controller well. Thus an obtained PID controller will provide the same control performance as GMVCS law. The proposed method in this paper which is extended from the conventional method (Sato *et al.*, 2001) achieves placing poles of a closed-loop system at desired places.

Since conventional PID controllers (Yamamoto *et al.*, 1998; Yamamoto *et al.*, 1999) approximate the GMVC law by using a steady state gain of controller, closed-loop poles given by using the PID controller differ from those given by using the GMVC law when the PID controller is designed by the conventional methods. Hence the conventional methods do not achieve pole-placement accurately. On the other hand, the proposed PID controller can achieve pole-placement because GMVCS law can achieve pole-placement and the proposed PID controller approximates GMVCS law. The proposed PID controller is equal to the GMVCS law when the orders of coefficient polynomials of the GMVCS law except the dead-time are 2 or less.

This paper is organized as follows. In section 2, an ARMAX model to represent the controlled plant is given. In section 3, the PID controller to be tuned is obtained. GMVCS is introduced and extended to achieve pole-placement control in section 4. By comparing the PID controller with the GMVCS law, the PID controller is designed in section 5. Finally, numerical examples are given to compare the proposed method with the conventional PID control schemes and to evaluate the proposed method

2. PROBLEM STATEMENT

Let consider a plant described by the following discrete-time SISO model:

$$A[z^{-1}]y(t) = z^{-k_m}B[z^{-1}]u(t-1) + \xi(t) \quad (1)$$

$$A[z^{-1}] = 1 + a_1z^{-1} + a_2z^{-2} \quad (2)$$

$$B[z^{-1}] = b_0 + b_1z^{-1} + \cdots + b_mz^{-m} \quad (3)$$

where $u(t)$ is the input, $y(t)$ is the output, k_m is the dead-time and $\xi(t)$ is a noise. z^{-1} denotes the backward shift operator.

The following assumptions are required for the plant model described by (1):

[A.1] The dead time k_m, polynomials $A[z^{-1}]$ and $B[z^{-1}]$ are known.
[A.2] The polynomial $A[z^{-1}]$ is stable.
[A.3] The noise $\xi(t)$ is the white Gaussian noise with zero mean.

The control problem in this paper is to design PID parameters in the PID controller in order to make the output $y(t)$ follow a step-type reference input.

3. PID CONTROLLER

The proposed discrete-time PID controller is given by:

$$\Delta u(t) = C_1[z^{-1}]e(t) - \left(C_2[z^{-1}] - z^{-k_m}C_3[z^{-1}]\right)u(t-1) \quad (4)$$

$$e(t) = w(t) - y(t) \quad (5)$$

$$\Delta = 1 - z^{-1} \quad (6)$$

where $w(t)$ is the reference input to be followed by the output $y(t)$. Although an usual PID controller uses the only error between the reference input and the output, this PID controller (4) contains two partial feedback compensators $C_2[z^{-1}]$ and $C_3[z^{-1}]$ to approximate a controller of GMVCS (Sato *et al.*, 2001).

The compensators $C_1[z^{-1}]$, $C_2[z^{-1}]$ and $C_3[z^{-1}]$ are the following

$$C_i[z^{-1}] = k_{c_i}\left(\Delta + \frac{T_s}{T_{I_i}} + \frac{T_{D_i}}{T_s}\Delta^2\right) \quad (7)$$

where, $i = 1, 2, 3$. Parameters k_{c_i}, T_{I_i} and T_{D_i} are the gain, the integral time and the derivative time, respectively. The sampling interval is denoted by T_s. The tuning problem is to design suitable values of parameters k_{c_i}, T_{I_i} and T_{D_i} for the plant model (1). This paper solves the design problem by designing the PID controller (4) based on the GMVCS law.

The proposed PID controller based on the GMVCS law has three PID compensators while a conventional method based on the GMVC law has one PID compensator. Since the conventional method uses the steady state gain of a controller of GMVC law instead of the controller of the GMVC law, control performance of the conventional method differs from one of the original GMVC and control performance may deteriorate. In the conventional method, in order to obtain approximation exactly, if more PID compensators are used instead of the steady state gain, the more a dead-time increases, the more the number of the PID compensator increases. Then, the controller becomes complex. On the other hand in this paper using only three compensators, the PID controller can approximate the GMVCS law.

4. GENERALIZED MINIMUM VARIANCE CONTROL WITH POLE-PLACEMENT USING STEADY STATE PREDICTIVE OUTPUT

In this section, to be compared with the PID controller, the GMVCS law (Sato *et al.*, 2001) is derived first. And then, GMVCS is extended to a pole-placement controller.

The performance index of GMVCS is the following variance of a generalized output

$$J = E[\Phi_s(t)^2]. \tag{8}$$

GMVCS derives a control law by minimizing the generalized output, the dead-time predictive output $y(t+k_m+1)$ of which is replaced by the steady state predictive output $y(s|t)$.

$$\begin{aligned}\Phi_s(t) &= P[z^{-1}]y(s|t) + Q[z^{-1}]\Delta u(t)\\ &\quad - R[z^{-1}]w(t) \qquad (9)\\ P[z^{-1}] &= p_0 + p_1 z^{-1} + \cdots + p_{n_p} z^{-n_p} \qquad (10)\\ Q[z^{-1}] &= q_0 + q_1 z^{-1} + \cdots + q_{n_q} z^{-n_q} \qquad (11)\\ R[z^{-1}] &= r_0 + r_1 z^{-1} + \cdots + r_{n_r} z^{-n_r} \qquad (12)\end{aligned}$$

where, polynomials $P[z^{-1}]$, $Q[z^{-1}]$ and $R[z^{-1}]$ are the design parameters. $y(s|t)$ is defined as

$$y(s|t) = \lim_{j\to\infty} \hat{y}(t+j|t), \tag{13}$$

and is calculated as follows (Kwok and Shah, 1994)

$$\begin{aligned}y(s|t) &= g_s \Delta u(t) + F_s[z^{-1}]\frac{y(t)}{P[z^{-1}]}\\ &\quad + G_s[z^{-1}]\frac{u(t-1)}{P[z^{-1}]} \qquad (14)\\ g_s &= \frac{B[1]}{A[1]} \qquad (15)\\ e_s &= \frac{P[1]}{A[1]} \qquad (16)\\ G_s[z^{-1}] &= g_s P[z^{-1}] - e_s z^{-k_m} B[z^{-1}] \qquad (17)\\ F_s[z^{-1}] &= e_s A[z^{-1}]. \qquad (18)\end{aligned}$$

The control law which minimizes the generalized output (9) is given by the following without solving Diophantine equation

$$\begin{aligned}G_1[z^{-1}]\Delta u(t) &= R[z^{-1}]w(t) - F_s[z^{-1}]y(t)\\ &\quad - G_s[z^{-1}]u(t-1) \qquad (19)\\ G_1[z^{-1}] &= g_s P[z^{-1}] + Q[z^{-1}] \qquad (20)\\ &= g_{1,0} + g_{1,1}z^{-1} + \cdots + g_{1,n_{g1}} z^{-n_{g1}}\\ n_{g1} &= \max\{n_p, n_q\} \qquad (21)\end{aligned}$$

To calculate this controller, the leading term and the remaining terms in polynomial $G_1[z^{-1}]$ multiplied by $\Delta u(t)$ of (20) are separated as

$$G_1[z^{-1}] = g_{1,0} + z^{-1}G_1'[z^{-1}] \tag{22}$$

Then the controller (19) is rewritten by

$$\begin{aligned}\Delta u(t) &= \frac{1}{g_{1,0}}\big[R[z^{-1}]w(t) - F_s[z^{-1}]y(t) \qquad (23)\\ &\quad -(G_2[z^{-1}] - e_s z^{-k_m} B[z^{-1}])u(t-1)\big]\\ G_2[z^{-1}] &= \Delta G_1'[z^{-1}] + g_s P[z^{-1}] \qquad (24)\\ &= g_{2,0} + g_{2,1}z^{-1} + \cdots + g_{2,n_{g2}} z^{-n_{g2}}\end{aligned}$$

where the order of $G_2[z^{-1}]$ is same as the one of $G_1[z^{-1}]$, that is, $n_{g1} = n_{g2}$ and it is determined regardless of the dead-time k_m.

Substituting the control law (23) into the plant (1), a closed-loop system is given by

$$\begin{aligned}y(t) &= \frac{z^{-(k_m+1)}B[z^{-1}]R[z^{-1}]}{T[z^{-1}]A[z^{-1}]}w(t) \qquad (25)\\ &\quad + \frac{g_s P[z^{-1}] + \Delta Q - e_s z^{-(k_m+1)}B[z^{-1}]}{T[z^{-1}]A[z^{-1}]}\xi(t)\\ T[z^{-1}] &= g_s P[z^{-1}] + \Delta Q[z^{-1}]. \qquad (26)\end{aligned}$$

In this paper, $R[z^{-1}]$ is designed as

$$R[z^{-1}] = F_s[z^{-1}], \tag{27}$$

then the closed-loop system becomes

$$\begin{aligned}y(t) &= \frac{z^{-(k_m+1)}e_s B[z^{-1}]}{T[z^{-1}]}w(t) \qquad (28)\\ &\quad + \frac{g_s P[z^{-1}] + \Delta Q - e_s z^{-(k_m+1)}B[z^{-1}]}{T[z^{-1}]A[z^{-1}]}\xi(t)\end{aligned}$$

and then it follows from (16),(26) that a steady state error is eliminated.

Since g_s is a scalar and $\Delta = 1 - z^{-1}$, it follows from (26) that a pole-placement controller can be achieved easily (Åström and Wittenmark, 1997). To design $P[z^{-1}]$ and $Q[z^{-1}]$ uniquely, the first term p_0 in $P[z^{-1}]$ is fixed on 1, that is,

$$P[z^{-1}] = 1 + p_1 z^{-1} + \cdots + p_{n_p} z^{-n_p}. \tag{29}$$

And using the desired closed-loop characteristic polynomial

$$T_d[z^{-1}] = 1 + t_1 z^{-1} + \cdots + t_{n_t} z^{-n_t}, \tag{30}$$

$P[z^{-1}]$ and $Q[z^{-1}]$ are designed by the following

$$\begin{aligned}Q[z^{-1}] &= 1 - g_s \qquad (31)\\ P[z^{-1}] &= \frac{1}{g_s}(T_d[z^{-1}] - \Delta Q[z^{-1}]). \qquad (32)\end{aligned}$$

Then, $Q[z^{-1}]$ is a scalar and the order of $P[z^{-1}]$ is $\max\{n_t, 1\}$. To obtain stability of the closed-loop system, unstable zeros of $B[z^{-1}] = 0$ must not be included in $T_d[z^{-1}]$.

5. PID CONTROLLER BASED ON GMVCS

In this section the PID parameters of the PID controller (4) are designed based on the GMVCS law (19). Comparing the PID controller (4) with the GMVCS law (23), the following relations are obtained.

$$C_1[z^{-1}] = \frac{1}{g_{1,0}} F_s[z^{-1}] \tag{33}$$

$$C_2[z^{-1}] = \frac{1}{g_{1,0}} G_2[z^{-1}] \tag{34}$$

$$C_3[z^{-1}] = \frac{1}{g_{1,0}} e_s B[z^{-1}] \tag{35}$$

Solving (33), the PID parameters of $C_1[z^{-1}]$ are designed by the following equations

$$k_{c_1} = -\frac{1}{g_{1,0}}(f_1 + 2f_2) \tag{36}$$

$$T_{I_1} = -\frac{f_1 + 2f_2}{f_0 + f_1 + f_2} T_s \tag{37}$$

$$T_{D_1} = -\frac{f_2}{f_1 + 2f_2} T_s \tag{38}$$

where,

$$F_s[z^{-1}] = f_0 + f_1 z^{-1} + f_2 z^{-2}. \tag{39}$$

If the order of the desired characteristic polynomial $T_d[z^{-1}]$ is 2 or less, (34) is solvable. The assumption is reasonable in most case and the PID parameters of $C_2[z^{-1}]$ are given as follows

$$k_{c_2} = -\frac{1}{g_{1,0}}(g_{2,1} + 2g_{2,2}) \tag{40}$$

$$T_{I_2} = -\frac{g_{2,1} + 2g_{2,2}}{g_{2,0} + g_{2,1} + g_{2,2}} T_s \tag{41}$$

$$T_{D_2} = -\frac{g_{2,2}}{g_{2,1} + 2g_{2,2}} T_s. \tag{42}$$

When $m \le 2$, it follows from (35) that the PID parameters of $C_3[z^{-1}]$ are designed by the following equations

$$k_{c_3} = -\frac{e_s}{g_{1,0}}(b_1 + 2b_2) \tag{43}$$

$$T_{I_3} = -\frac{b_1 + 2b_2}{b_0 + b_1 + b_2} T_s \tag{44}$$

$$T_{D_3} = -\frac{b_2}{b_1 + 2b_2} T_s. \tag{45}$$

The proposed PID controller is equal to the GMVCS law under the condition; $\max\{m, n_t\} \le 2$. If m or n_t are higher than 2, $B[z^{-1}]$, $F_s[z^{-1}]$ and $G_2[z^{-1}]$ are approximated by polynomials having suitable order using reference (Sato *et al.*, 2000) and the PID parameters are designed using the approximated polynomials.

6. NUMERICAL EXAMPLE

The proposed PID controller is designed for the following continuous system having a long dead-time.

$$G(s) = \frac{3s + 1}{(s+1)(s+2)} e^{-20s} \tag{46}$$

Using $T_s = 1$[s], (46) is transformed into a discrete-time system. Then, the controlled plant in discrete-time domain is represented by

$$\begin{aligned}(1 - 0.50z^{-1} + 0.050z^{-2})y(t) \\ = z^{-20}(0.90 - 0.62z^{-1})u(t-1) + \xi(t).\end{aligned} \tag{47}$$

In this example, the closed-loop poles are assigned to 0.88 and -0.1.

To design the PID controller, the GMVCS law is derived first. The generalized output is designed by:

$$\begin{aligned}\Phi_s(t) = (1 - 0.56z^{-1} - 0.18z^{-2})y(s|t) \\ + 0.5\Delta u(t) - R[z^{-1}]w(t).\end{aligned} \tag{48}$$

Then, the GMVCS law is given as

$$\begin{aligned}&(1 - 0.28z^{-1} - 0.088z^{-2})\Delta u(t) \\ &= (0.48 - 0.24z^{-1} + 0.024z^{-2})e(t) \\ &-\{0.5 - 0.28z^{-1} - 0.088z^{-2} \\ &-z^{-20}(0.43 - 0.30z^{-1})\}u(t-1).\end{aligned} \tag{49}$$

The proposed PID controller based on this control law is given by following

$$\begin{aligned}\Delta u(t) = (0.48 - 0.24z^{-1} + 0.024z^{-2})e(t) \\ - \{(0.22 - 0.088z^{-1}) \\ -z^{-20}(0.43 - 0.30z^{-1})\}u(t-1).\end{aligned} \tag{50}$$

To be compared with the proposed method, a PID controller designed by the conventional method (Yamamoto *et al.*, 1999) is obtained. To design the conventional PID controller, the GMVC law (Clarke, 1984) is also designed first. Then, the generalized output is designed by

$$\begin{aligned}\Phi(t+21) = (-0.92 + 1.6z^{-1} - 0.15z^{-2})y(t+21) \\ + 1.8\Delta u(t) - R[z^{-1}]w(t).\end{aligned} \tag{51}$$

In the conventional method, the design polynomial $R[z^{-1}]$ is designed as $F[z^{-1}]$ and in this case

$$\begin{aligned}R[z^{-1}] &= F[z^{-1}] \\ &= 0.88 - 0.44z^{-1} + 0.044z^{-2}.\end{aligned} \tag{52}$$

Then, the GMVC law is given by

$$(1+0.72z^{-1}+0.45z^{-2}+0.32z^{-3}+0.27z^{-4}$$
$$+0.25z^{-5}+0.25z^{-6}+0.24z^{-7}+0.24z^{-8}$$
$$+0.24z^{-9}+0.24z^{-10}+0.24z^{-11}+0.24z^{-12}$$
$$+0.24z^{-13}+0.24z^{-14}+0.24z^{-15}+0.24z^{-16}$$
$$+0.24z^{-17}+0.24z^{-18}+0.24z^{-19}+0.24z^{-20}$$
$$-0.55z^{-21})\Delta u(t)$$
$$=F[z^{-1}]e(t). \tag{53}$$

This GMVC law differs from the GMVCS law proposed in this paper, and the order is too large to be approximated by the PID controller. The conventional PID controller based on this control law is given by following

$$\Delta u(t)=\frac{1}{6.1}F[z^{-1}]e(t). \tag{54}$$

By using the conventional method, approximation error is too large since the higher order polynomial is approximated by a static gain although the order of the polynomial of the GMVC law is 21.

Simulation is conducted under the conditions that the reference input $w(t)$ is a rectangular wave with amplitude 1.0 over a period of 100 steps and the variance of random disturbance $\xi(t)$ is 0.003. The obtained PID parameters are shown in Table 1.

The closed-loop poles by each method are shown in Table 2. By using GMVC, the closed-loop poles are desired and stable. However, using the conventional PID controller (Yamamoto *et al.*, 1999) based on the GMVC law, the number of poles increases and some poles are near by unit circle badly. Furthermore, the poles by the conventional PID controller are not equal to those by the GMVC. On the other hand, the poles of GMVCS are desired and stable and the poles are the same as those by the proposed PID controller based on the GMVCS law.

Output results are shown in Fig. 1. The output result by the conventional PID controller is shown by the dashed line and does not diverge. However, over-shoot emerges since the closed-loop poles differ from those of the original GMVC. While, the output result by the proposed PID controller based on the GMVCS law is shown by the solid line. As compared with the result by the conventional method, the output converges to the reference input faster than that by the conventional method and the over-shoot is reduced. Furthermore, the output result by the proposed PID is same as an output result that by the GMVCS law although omitted on account of space.

7. CONCLUSIONS

In this paper, a new design method of a PID controller is proposed. PID parameters of the PID controller are derived by designing the PID controller based on a GMVCS law. The GMVCS law is approximated by the PID controller using three PID compensators exactly since the order of the GMVCS law is independent of a dead-time. Because the GMVCS is extended to a pole-placement control system, pole-placement control can be achieved by using the proposed PID controller. Numerical examples are given to show the effectiveness of the proposed method and to compare with a conventional PID controller based on a GMVC law with a static gain.

In this study it is assumed that plant parameters are known. Therefore a self-tuning controller will be needed to deal with the case in which the plant parameters are unknown.

REFERENCES

Clarke, D.W. (1984). Self-tuning control of nonminimum-phase systems. *Automatica* **20**(5), 501–517.

Åström, K.J. and B. Wittenmark (1997). *Computer-Controlled Systems: Theory & Design.* third ed.. Prentice Hall.

Kwok, K.Y. and S.L. Shah (1994). Long-range predictive control with a terminal matching condition. *Chemical Engineering Science.*

Sato, T., A. Inoue and Y. Hirashima (2001). Self-tuning two-degree-of-freedom PID controller approximating to generalized minimum variance control with steady state predictive output. In: *2001 International Symposium on Adaptive and Intelligent Systems and Control.* pp. 56–57.

Sato, T., A. Inoue, T. Yamamoto and S.L. Shah (2000). Self-tuning PID controllers based on the strongly stable generalized minimum variance control law. In: *IFAC Workshop on Digital Control: Past, Present and Future of PID Control.* Terrassa. pp. 511–516.

Yamamoto, T., A. Inoue and S.L. Shah (1999). Generalized minimum variance self-tuning pole-assignment controller with a PID structure. In: *Proc. of 1999 IEEE Int Conf. on Control Applications.* Hawaii. pp. 125–130.

Yamamoto, T., K. Fujii and M. Kaneda (1998). Design and implementation of a self-tuning PID controller. In: *Preprints of IFAC Workshop on Adaptive Control and Signal Processing.* Glasgow. pp. 58–63.

Table 1. Obtained PID parameters

Design method	Proportional gain	Integral time	Derivative time
Conventional method (Yamamoto *et al.*, 1999)	0.059	0.74	0.12
$C_1[z^{-1}]$ of the proposed method	0.20	0.74	0.12
$C_2[z^{-1}]$ of the proposed method	0.088	0.67	0
$C_3[z^{-1}]$ of the proposed method	0.30	2.3	0

Table 2. Closed-loop poles

Design method	Closed-loop Poles
GMVC (Clarke, 1984)	$0.88,\ -0.1$
PID based on GMVC (Yamamoto *et al.*, 1999)	$0.97 \pm 0.061i,\ 0.83 \pm 0.36i,\ 0.67 \pm 0.60i,\ 0.46 \pm 0.77i,$ $0.21 \pm 0.88i,\ -0.060 \pm 0.90i,\ -0.32 \pm 0.84i,\ -0.56 \pm 0.71i,$ $-0.74 \pm 0.51i,\ -0.86 \pm 0.27i,\ -0.90,\ 0.70,\ 0.37,\ 0.14$
GMVCS	$0.88,\ -0.1$
PID based on GMVCS	$0.88,\ -0.1$

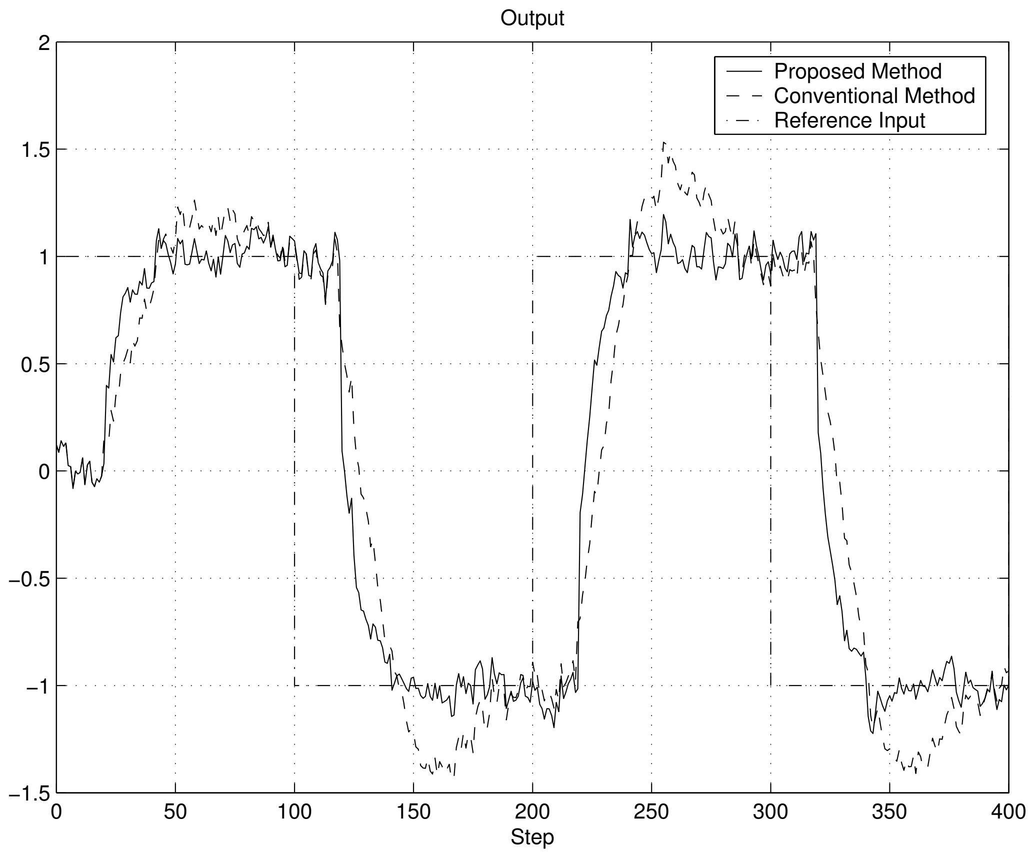

Fig. 1. Output results by the conventional and the proposed methods