STABILITY AND STABILIZATION OF A CLASS OF NONLINEAR SYSTEMS WITH SATURATING ACTUATORS

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Abstract: This paper addresses the problem of controlling a certain class of continuous-time nonlinear systems subject to actuator saturations. The design of state feedback gains is done by considering a modelling of the nonlinear saturated system through deadzone nonlinearities satisfying a *modified sector condition*, which encompasses the classical sector-nonlinearity condition considered in previous works.

Keywords: Nonlinear systems, Saturations, Absolute stability, State feedback, Sector nonlinearities, LMIs.

1. INTRODUCTION

Absolute stability is a powerful tool for analysis and synthesis of nonlinear systems that can be represented as the feedback interconnection of a linear system with some sector bounded nonlinearities. The most widely known absolute stability results are the Popov Criterion, which is related to the use of a quadratic-plus-integral candidate Lyapunov function, and the Circle Criterion, which corresponds to using a quadratic candidate Lyapunov function.

Recently, the research on absolute stability methods has intensified due to the possibility of treating the related problems using the linear matrix inequality (LMI) framework, which involves mathematical tools (as Schur's Lemma and $\mathcal{S}$-procedure) and efficient computational tools (Boyd *et al.*, 1994). In this line, different analysis and synthesis results have been proposed in the control literature, allowing to efficiently compute solutions for different problems. In particular, absolute stability and the LMI framework have been used to treat the problems of stability and stabilization of linear systems subject to actuators saturations (Paim *et al.*, 2002; Hindi and Boyd, 1998; Pittet *et al.*, 1997). A computational difficulty that appeared in these works is that some bilinear matrix inequalities (BMIs) appear in the proposed stability/stabilization conditions. Thus some relaxation optimization methods could be used in order to search for solutions. However, by using a recently proposed *modified sector condition* associated to the saturation nonlinearity (Tarbouriech *et al.*, 2004; Gomes da Silva Jr. and Tarbouriech, 2005), LMI conditions can be obtained and the previous computational difficulty is overcome.

In the context briefly discussed above, the present work considers the problem of controlling a nonlinear system subject to actuator saturations. Thus, two sector bounded nonlinearities are con-

[1] Partially supported by CAPES/Brazil.

sidered, the first one related to the dynamics of the open-loop system, and the second one appearing from the modeling of the control saturations.

The considered saturating control law consists of the feedback of both the system states and the dynamic nonlinearity, through constant feedback gain matrices (Arcak and Kokotovic, 2001; Arcak *et al.*, 2003). Two complementary problems are investigated depending whether the feedback control gains are known (analysis problem), or have to be computed (synthesis problem). Thus, both local and global stability and stabilization results are proposed by considering a quadratic candidate Lyapunov function. Thanks to the use of the modified sector condition, the proposed conditions appear under LMI forms and can directly be cast into convex optimization problems and efficiently be solved.

The present work enlarges the applicability of absolute stability and LMI techniques for analyzing and synthesizing controllers for some cone bounded nonlinear systems subject to saturating actuators. The control of constrained linear systems appears as a special case of the considered problems. Also, relations between the proposed results and some existing results for unconstrained and constrained systems are discussed along the text.

Notations. Relative to a matrix $A \in \Re^{m\times n}$, A' denotes its transpose, and $A_{(i)}$, $i = 1,\dots m$, denotes its ith row. If $A = A' \in \Re^{n\times n}$, then $A < 0$ ($A \leq 0$) means that A is negative (semi-)definite. The components of any vector $x \in \Re^n$ are denoted $x_{(i)}$, $\forall i = 1,\dots n$. Inequalities between vectors are component-wise: $x \leq 0$ means that $x_{(i)} \leq 0$ and $x \leq y$ means that $x_{(i)} - y_{(i)} \leq 0$. I_n denotes the $n \times n$ identity matrix.

2. PROBLEM PRESENTATION

Consider a continuous-time nonlinear system represented by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + G\varphi(z(t)) + Bsat(u(t)) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $z(t) \in \Re^p$ and $\varphi(.) : \Re^p \to \Re^p$. A, B, G and L are real constant matrices of appropriate dimensions. The pairs (A, B) and (L, A) are controllable and observable, respectively.

The nonlinearity $\varphi(z(t))$ verifies a cone bounded sector condition (Johansson and Robertsson, 2002; Khalil, 2002), *i.e.:* there exists a symmetric positive definite matrix $\Omega = \Omega' \in \Re^{p\times p}$ such that

$$\varphi'(z(t))\Delta[\varphi(z(t)) - \Omega z(t)] \leq 0 \ , \quad \begin{matrix} \forall z \in \Re^p \\ \varphi(0) = 0 \end{matrix} \tag{2}$$

where $\Delta \in \Re^{p\times p}$ is any diagonal matrix defined by

$$\Delta \triangleq \begin{cases} diag\{\delta_l\} \ , \ \delta_l > 0 \ , & \begin{matrix}\forall l = 1,\dots,p, \text{ if } \varphi(.) \\ \text{is decentralized;}\end{matrix} \\ \delta I_p \quad\quad , \ \delta > 0 \ , & \text{otherwise.} \end{cases} \tag{3}$$

By definition, the nonlinearity globally satisfies the sector condition (2). Ω is given by the designer, and assumed to be known in the sequel. On the other hand, Δ represents a degree-of-freedom (then an optimization parameter).

The control inputs are bounded in amplitude, and the standard saturation function is considered:

$$sat(u_{(i)}(t)) = sign(u_{(i)}(t)) \ min(\rho_{(i)}, |u_{(i)}(t)|) \tag{4}$$

$\forall i = 1,\dots,m$, where $\rho_{(i)} > 0$ denotes the symmetric amplitude bound relative to the i-th control input.

Throughout this work, the following type of feedback control law is considered:

$$u(t) = Kx(t) + \Gamma\varphi(z(t)) \tag{5}$$

where $K \in \Re^{m\times n}$ and $\Gamma \in \Re^{m\times p}$. With $\Gamma \neq 0$, this feedback control law requires either the knowledge of the nonlinearity $\varphi(.)$ or its availability as a signal (Arcak *et al.*, 2003). The corresponding closed-loop system reads [2] :

$$\dot{x} = Ax + G\varphi(z) + Bsat\left(Kx + \Gamma\varphi(z)\right) \tag{6}$$

The following two problems are considered in the sequel:

Problem 1. (Absolute stability analysis)

Given feedback matrices K and Γ, determine a region $\mathcal{S}_0 \subseteq \Re^n$, as large as possible, such that for any initial condition $x_0 \in \mathcal{S}_0$ the origin of the closed-loop system (6) is asymptotically stable for any $\varphi(.)$ verifying the sector condition (2).

Problem 2. (Feedback synthesis)

Determine feedback matrices K and Γ and a region $\mathcal{S}_0 \subseteq \Re^n$, as large as possible, such that for any initial condition $x_0 \in \mathcal{S}_0$ the origin of the closed-loop system (6) is asymptotically stable for any $\varphi(.)$ verifying the sector condition (2).

The two problems above are complementary. The first one consists in a stability analysis problem. The idea is then to characterize an estimate, as large as possible, of the basin of attraction of the nonlinear system (6). The second one is a control design problem. In this case, the implicit objective

[2] For simplicity, the dependency of the variables on time t is omitted in the sequel.

is to design the gain matrices K and Γ in order to maximize the basin of attraction of the resulting closed-loop system (6).

Moreover, it is important to underline that although the cone bounded sector condition (2) is globally satisfied, the global stability or global stabilization of system (6) could be studied only if the open-loop matrix A satisfies some stability assumption (Sussmann *et al.*, 1994). Hence, in the sequel, even if Problems 2.1 and 2.2 will be mainly studied in a local context and some comments will be provided in a global context.

3. PRELIMINARIES

Let us consider the generic nonlinearity $\psi(v) = v - sat_{v_0}(v)$, $\psi(v) \in \Re^m$ and define the following associated set:

$$S(v_0) = \{v \in \Re^m; w \in \Re^m; -v_0 \preceq v - w \preceq v_0\} \quad (7)$$

Lemma 3. (Tarbouriech *et al.*, 2004) If v and w are elements of $S(v_0)$ then the nonlinearity $\psi(v)$ satisfies the following inequality:

$$\psi(v)'T(\psi(v) - w) \le 0 \quad (8)$$

for any diagonal positive definite matrix $T \in \Re^{m\times m}$.

Throughout the paper we define a decentralized deadzone nonlinearity:

$$\psi(u) = u - sat(u)$$

where from (4) one gets: $u = Kx + \Gamma\varphi(z)$. Thus, the closed-loop system reads:

$$\begin{array}{l} \dot{x} = (A + BK)x + (G + B\Gamma)\varphi(z) - B\psi(u) \\ z = Lx \end{array} \quad (9)$$

System (9) is then subject to two nonlinearities $\varphi(z)$ and $\psi(u)$, the last one depending on the first one by definition.

In the sequel, we will see how Lemma 3 may be applied to the nonlinearity $\psi(u)$ by considering adequate vectors v, w and v_0. Furthermore, the links between $\varphi(z)$ and $\psi(u)$, and their implications in terms of stability region will be addressed with more details in the next section.

4. MAIN RESULTS

In this section, some stability and stabilization conditions are proposed by considering quadratic candidate Lyapunov functions.

4.1 Absolute stability analysis

The first result concerns the closed-loop local asymptotic stability, for matrices K and Γ a priori given.

Proposition 4. If there exist a symmetric positive definite matrix $W \in \Re^{n\times n}$, two positive diagonal matrices $S_\Delta \in \Re^{p\times p}$, $S \in \Re^{m\times m}$, two matrices $Z_1 \in \Re^{m\times n}$ and $Z_2 \in \Re^{m\times p}$ satisfying [3]:

$$\begin{bmatrix} \begin{array}{c} W(A+BK)' \\ +(A+BK)W \end{array} & \begin{array}{c} (G+B\Gamma)S_\Delta \\ +WL'\Omega \end{array} & Z_1' - BS \\ \star & -2S_\Delta & Z_2' \\ \star & \star & -2S \end{bmatrix} < 0 \quad (10)$$

$$\begin{bmatrix} W & -WL'\Omega & WK'_{(i)} - Z'_{1(i)} \\ \star & 2S_\Delta & S_\Delta\Gamma'_{(i)} - Z'_{2(i)} \\ \star & \star & \rho^2_{(i)} \end{bmatrix} \ge 0, i = 1, ..., m \quad (11)$$

then the ellipsoid $\mathcal{E}(P) = \{x \in \Re^n; x'Px \le 1\}$, with $P = W^{-1}$, is a region of asymptotic stability for the closed-loop system (9), for any nonlinearity $\varphi(z)$ satisfying relation (2) with $\Delta = S_\Delta^{-1}$.

Proof. Lemma 3 may be applied to the nonlinearity $\psi(u)$ by considering adequate vectors v, w and v_0 as follows: $v = Kx + \Gamma\varphi(z)$; $w = E_1x + E_2\varphi(z)$; $v_0 = \rho$; $S(v_0) = S(\rho)$, where $E_1 \in \Re^{m\times n}$ and $E_2 \in \Re^{m\times p}$.

Consider now $E_1 = Z_1W^{-1}$ and $E_2 = Z_2S_\Delta^{-1}$. The ellipsoid $\mathcal{E}(P) = \{x \in \Re^n; x'Px \le 1\}$, with $P = W^{-1}$, is included in $S(\rho)$ if

$$[x' \ \varphi'] \left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathbb{K}'_{(i)} \\ \mathbb{G}'_{(i)} \end{bmatrix} \frac{1}{\rho^2_{(i)}} \begin{bmatrix} \mathbb{K}_{(i)} & \mathbb{G}_{(i)} \end{bmatrix} \right) \begin{bmatrix} x \\ \varphi \end{bmatrix} \ge 0 \quad (12)$$
$$\forall i = 1, ..., m$$
$$\forall x, \varphi \text{ such that } 2\varphi'\Delta(\varphi - \Omega Lx) \le 0$$

where $\begin{bmatrix} \mathbb{K} & \mathbb{G} \end{bmatrix} = \begin{bmatrix} K - E_1 & \Gamma - E_2 \end{bmatrix}$). Using the S-procedure and the Schur's complement it follows that (12) is satisfied if

$$\begin{bmatrix} P & -L'\Omega\Delta & K'_{(i)} - E'_{1(i)} \\ \star & 2\Delta & \Gamma'_{(i)} - E'_{2(i)} \\ \star & \star & \rho^2_{(i)} \end{bmatrix} \ge 0, \ i = 1, ..., m \quad (13)$$

Pre- and post-multiplying (13) by the matrix $\begin{bmatrix} W & 0 & 0 \\ \star & S_\Delta & 0 \\ \star & \star & 1 \end{bmatrix}$, with $S_\Delta = \Delta^{-1}$, it follows that the resulting inequality is equivalent to relation (11). Hence, the satisfaction of relation (11) means that the ellipsoid $\mathcal{E}(P)$ is included in $S(\rho)$ and therefore the nonlinearity $\psi(u)$ satisfies the sector condition (8) for any $x \in \mathcal{E}(P)$.

[3] The symbol $\star$ stands for symmetric blocks.

Consider now the quadratic function $V(x) = x'Px$. Its time-derivative along the trajectories of the closed-loop system (9) reads:

$$\begin{aligned}\dot{V}(x) &= x'((A+BK)'P + P(A+BK))x \\ &\quad +2x'P(G+B\Gamma)\varphi - 2x'PB\psi\end{aligned}$$

From the satisfaction of sector conditions (8) and (2), for any $x \in \mathcal{E}(P)$ it follows:

$$\begin{aligned}\dot{V}(x) &\leq x'((A+BK)'P + P(A+BK))x \\ &\quad +2x'P(G+B\Gamma)\varphi - 2x'PB\psi \\ &\quad -2\psi'T(\psi - E_1x - E_2\varphi) - 2\varphi'\Delta(\varphi - \Omega Lx)\end{aligned}$$

The inequality above can be written as

$$\dot{V}(x) \leq \begin{bmatrix} x' & \varphi' & \psi' \end{bmatrix} \mathcal{M} \begin{bmatrix} x \\ \varphi \\ \psi \end{bmatrix}$$

with

$$\mathcal{M} = \begin{bmatrix} \begin{matrix}(A+BK)'P \\ +P(A+BK)\end{matrix} & \begin{matrix}P(G+B\Gamma) \\ +L'\Omega\Delta\end{matrix} & E_1'T - PB \\ \star & -2\Delta & E_2'T \\ \star & \star & -2T \end{bmatrix}$$

By pre- and post-multiplying the matrix $\mathcal{M}$ above defined by the matrix $\begin{bmatrix} W & 0 & 0 \\ \star & S_\Delta & 0 \\ \star & \star & S \end{bmatrix}$, with $W = P^{-1}$, $S_\Delta = \Delta^{-1}$, $T = S^{-1}$, it follows that if relation (10) is satisfied, one obtain $\mathcal{M} < 0$ and therefore $\dot{V}(x) < 0$, $\forall x \in \mathcal{E}(P)$, $x \neq 0$. Hence, one can conclude that $\mathcal{E}(P)$ is a contractive set along the trajectories of system (9) and then is a region of asymptotic stability, which finishes the proof. □

From Proposition 4, the following basic convex programming problem can be formulated to find solutions for Problem 1:

$$\begin{array}{c} \min\limits_{W,S_\Delta,S,Z_1,Z_2} J(.) \\ \text{subject to} \\ \text{LMIs (10) and (11)} \end{array} \tag{14}$$

where $J(.)$ is some linear optimization criterion associated to the size of $\mathcal{E}(P)$, with $P = W^{-1}$;*e.g.:*

- the volume: $J = -log(det(W))$, or
- the size of the minor axis: $J = -\lambda$, with $W \geq \lambda I_n$.

Proposition 4 ensures the absolute stability in a local context. In particular, the proposed LMI conditions are well-adapted for stability analysis of systems for which matrix A is unstable since, in this case, only local stability can be achieved in the presence of control bounds. However, if matrix A is stable, global stability analysis may be of interest. Thus, the following result can be stated by letting $E_1 = K$ and $E_2 = \Gamma$ in Proposition 4.

Proposition 5. If there exist a symmetric positive definite matrix $W \in \Re^{n\times n}$, two positive diagonal matrices $S_\Delta \in \Re^{p\times p}$ and $S \in \Re^{m\times m}$ satisfying:

$$\begin{bmatrix} \begin{matrix}W(A+BK)' \\ +(A+BK)W\end{matrix} & \begin{matrix}(G+B\Gamma)S_\Delta \\ +WL'\Omega\end{matrix} & WK' - BS \\ \star & -2S_\Delta & S_\Delta\Gamma' \\ \star & \star & -2S \end{bmatrix} < 0 \tag{15}$$

then the closed-loop system (9) is globally asymptotically stable for any nonlinearity $\varphi(z)$ satisfying relation (2) with $\Delta = S_\Delta^{-1}$. □

Notice that if LMI (15) is feasible, then the radially unbounded function $V(x) = x'W^{-1}x$ verifies $\dot{V}(x) < 0$, $\forall x \in \Re^n$, $x \neq 0$.

The following remarks consider two particular instances of system (1) under the control law (5). Thus, they show how the proposed results can be related to some existing results considered in the literature.

Remark 6. Consider the case where the nonlinearity $\varphi(.)$ does not apply, *i.e.:*

$$\dot{x} = Ax + Bsat(Kx) \tag{16}$$

where K is a given state feedback matrix. Using a similar approach as the one considered in the proof of Proposition 4, it is not difficult to show that the existence of $W = W' > 0$, Z_1 and a diagonal matrix $S > 0$ verifying the two following LMIs,

$$\begin{bmatrix} W(A+BK)' + (A+BK)W & Z_1' - BS \\ \star & -2S \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} W & WK_{(i)}' - Z_{1(i)}' \\ \star & \rho_{(i)}^2 \end{bmatrix} \geq 0, i = 1, ..., m \tag{18}$$

implies that the ellipsoid $\mathcal{E}(P) = \{x \in \Re^n; x'Px \leq 1\}$, with $P = W^{-1}$, is a region of asymptotic stability for the system (16).

The above LMIs can be cast into convex programming optimization problems to determine regions of stability for the saturating system (16). In this way, this approach can be viewed as an alternative to the LMI approach proposed in (Hu *et al.*, 2002) for analysis of linear systems under saturating actuators, which requires the evaluation of 2^m LMIs associated to the vertices of the polytope of matrices modeling the saturations. Notice that our approach reduces this computational burden since a single LMI (17) is now associated to the saturations through the used sector nonlinearity representation and Lemma 3. Relative to the results proposed in (Pittet *et al.*, 1997) and (Hindi and Boyd, 1998), in which the saturations are also considered as sector nonlinearities, the present approach overcomes the necessity of using BMI techniques for treating the nonlinearities between some optimization variables.

At last, as for Proposition 5, global stability of system (16) can be checked by the existence of $W = W' > 0$ and S satisfying:

$$\begin{bmatrix} W(A+BK)' \\ +(A+BK)W & WK' - BS \\ \star & -2S \end{bmatrix} < 0 \qquad (19)$$

It is important to remark that global asymptotic stability of the closed-loop system can then be addressed through an LMI test. The use of classical sector condition as in (Paim *et al.*, 2002) leads to BMI conditions, whereas it is not possible to test the global asymptotic stability when considering a polytopic model as in (Hu *et al.*, 2002).

Remark 7. Consider the case where the control saturations are not taken into account, *i.e.*:

$$\dot{x} = (A+BK)x + (G+B\Gamma)\varphi \qquad (20)$$

where K and Γ are given matrices. In this case, it can be shown that the closed-loop system (20) is globally asymptotically stable for any nonlinearity $\varphi(z)$ satisfying relation (2), if there exist $W = W'$ and a diagonal matrix $S_\Delta = \Delta^{-1}$ satisfying:

$$\begin{bmatrix} W(A+BK)' & (G+B\Gamma)S_\Delta \\ +(A+BK)W & +WL'\Omega \\ \star & -2S_\Delta \end{bmatrix} < 0 \qquad (21)$$

Now, the LMI (21) can be viewed as an alternative test for absolute stability for the nonlinear closed-loop system (20). Notice that a main difference between this approach and the Circle criterion approach investigated in (Arcak and Kokotovic, 2001) and (Arcak *et al.*, 2003) concerns the consideration of the cone bounded sector property (2), *i.e.:* $\varphi(.) \in [\,0\,,\,\Omega\,]$, instead of the classic sector property $\varphi' z \geq 0$, *i.e.:* $\varphi(.) \in [\,0\,,\,\infty\,]$. It is not difficult to show that the verification of (21) is equivalent to have the following, for $P = W^{-1}$:

$$\begin{cases} (A+BK)'P + P(A+BK) + \mathcal{L}'\mathcal{L} < 0 \\ P(G+B\Gamma) = L'\Omega\Delta - \mathcal{L}'\mathcal{W} \\ \mathcal{W}'\mathcal{W} = \Delta + \Delta' \end{cases}$$

Thus, from Kalman-Yakubovich-Popov Lemma, condition (21) is verified if and only if the following transfer function matrix is strictly real positive: $G(s) = \Delta\Omega L\,(sI - A - BK)^{-1}\,(G+B\Gamma) + \Delta$.

We finally remark that, in general, we may have $\varphi(.) \in [\Omega_1\,,\,\Omega_2]$, with $\Omega_2 - \Omega_1 > 0$. Thus whenever $\Omega_1 \geq \epsilon I$, with $\epsilon > 0$, or $\|\Omega_2\|$ is bounded, the cone bounded sector property (2) can be easily recovered by a loop transformation (Khalil, 2002), and the proposed LMI approach can be used for testing the absolute stability of the closed-loop system (20).

4.2 Stabilization results

LMI-type stabilization conditions are now proposed. They can be obtained from the previous stability results, using classical changes of variables.

Proposition 8. If there exist a symmetric positive definite matrix $W \in \Re^{n\times n}$, two positive diagonal matrices $S_\Delta \in \Re^{p\times p}$, $S \in \Re^{m\times m}$, four matrices $Z_1 \in \Re^{m\times n}$, $Z_2 \in \Re^{m\times p}$, $Y_1 \in \Re^{m\times n}$ and $Y_2 \in \Re^{m\times p}$ satisfying:

$$\begin{bmatrix} WA + Y_1'B' & GS_\Delta + BY_2 & \\ +AW + BY_1 & +WL'\Omega & Z_1' - BS \\ \star & -2S_\Delta & Z_2' \\ \star & \star & -2S \end{bmatrix} < 0 \qquad (22)$$

$$\begin{bmatrix} W & -WL'\Omega & Y_{1(i)}' - Z_{1(i)}' \\ \star & 2S_\Delta & Y_{2(i)}' - Z_{2(i)}' \\ \star & \star & \rho_{(i)}^2 \end{bmatrix} \geq 0,\, i = 1, ..., m \qquad (23)$$

then the gains $K = Y_1 W^{-1}$, $\Gamma = Y_2 S_\Delta^{-1}$ and the ellipsoid $\mathcal{E}(P) = \{x \in \Re^n; x'Px \leq 1\}$, with $P = W^{-1}$, are solutions to Problem 2.2 for any nonlinearity $\varphi(z)$ satisfying relation (2) with $\Delta = S_\Delta^{-1}$.

Proof. It follows the same lines than that one of Proposition 4 by using the supplementary change of variables $KW = Y_1$ and $\Gamma S_\Delta = Y_2$. □

Using Proposition 8, the basic convex programming problem (14) can directly be adapted to find solutions for Problem 2, as follows:

$$\begin{array}{c} \min\limits_{W,S_\Delta,S,Y_1,Y_2,Z_1,Z_2} J(.) \\ \text{subject to} \\ \text{LMIs (22) and (23)} \end{array} \qquad (24)$$

Problem (24) can be modified to consider additional convex constraints (for instance, associated to some constraints on norm gains or structural ones) or a different optimization criterion. Let us consider, for optimization of the size of set $\mathcal{S}_0 = \mathcal{E}(P)$, a given *shape set* $\Xi_0 \in \Re^n$ and a scaling factor β, where: $\Xi_0 = Co\{v_r \in \Re^n\ ;\ r = 1, \ldots, n_r\}$. We want to verify $\beta\Xi_0 \subset \mathcal{E}(P)$, the goal consisting of maximizing the scaling factor β (Gomes da Silva Jr. and Tarbouriech, 2001). This can be accomplished by solving the following convex programming problem:

$$\begin{array}{c} \min_{W,S_\Delta,S,Y_1,Y_2,Z_1,Z_2} \mu \\ \text{subject to} \\ \text{LMIs (22) and (23)} \\ \begin{bmatrix} \mu & v_r' \\ v_r & W \end{bmatrix} \geq 0 \ , \ r = 1, \ldots, n_r \end{array} \quad (25)$$

Thus, by considering $\beta = 1/\sqrt{\mu}$, the minimization of μ implies the maximization of β.

As in the previous section, by considering $Y_1 = Z_1$ and $Z_2 = Y_2$, the following global stabilization result can be stated from Proposition 8.

Proposition 9. If there exist a symmetric positive definite matrix $W \in \Re^{n\times n}$, two positive diagonal matrices $S_\Delta \in \Re^{p\times p}$, $S \in \Re^{m\times m}$, two matrices $Y_1 \in \Re^{m\times n}$ and $Y_2 \in \Re^{m\times p}$ satisfying:

$$\begin{bmatrix} WA + Y_1'B' & GS_\Delta + BY_2 & \\ +AW + BY_1 & +WL'\Omega & Y_1' - BS \\ \star & -2S_\Delta & Y_2' \\ \star & \star & -2S \end{bmatrix} < 0 \quad (26)$$

then the gains $K = Y_1 W^{-1}$, $\Gamma = Y_2 S_\Delta^{-1}$ globally asymptotically stabilize the closed-loop system (9) for any nonlinearity $\varphi(z)$ satisfying relation (2) with $\Delta = S_\Delta^{-1}$. □

Remark 10. The stability conditions commented in Remarks 6 and 7 can easily be extended to obtain stabilization conditions for the two considered instances of system (1) under control law (5). Thus, these stabilization conditions can also be related to existing results for stabilization of linear systems with saturating actuators and for stabilization of sector bounded nonlinear systems via Circle criterion.

5. CONCLUSION

The problem of controlling a nonlinear system under saturating actuators has been investigated. A quadratic candidate Lyapunov function and a modified sector condition allowed to derive LMI stability and stabilization conditions for the nonlinear system represented by the interconnection of a linear system and a cone bounded sector nonlinearity, and also subject to saturating actuators. The presented results can be thought as absolute stability/stabilization results for systems subject to saturating actuators. Some relations with existing works have been commented and some numerical issues have been pointed out. Possible extension to using quadratic plus integral (Popov) candidate Lyapunov functions will be presented in a future extend version of this paper. Further work will also be devoted to numerical implementation to evaluate the potential conservatism of the approach and to compare with other similar results (Hu *et al.*, 2002), (Pittet *et al.*, 1997), (Hindi and Boyd, 1998).

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