METHOD OF DECOMPOSITION AND ITS APPLICATIONS TO UNCERTAIN DYNAMICAL SYSTEMS

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Abstract: A nonlinear dynamical system described by Lagrange's equations is considered. The system is subjected to uncertain external forces and controls bounded by geometric constraints. A feedback control that satisfies the imposed conditions and brings the system to a prescribed terminal state in a finite time is proposed. The control is based on the decomposition of the system and uses ideas of optimal control and differential games. Explicit formulas for the feedback control are presented. Applications to control of robots and underactuated systems are discussed. Computer simulation of motions of a double pendulum controlled by one torque is presented.

Keywords: Decomposition methods, Nonlinear control systems, Feedback control, Bang-bang control

1. INTRODUCTION

The problem of designing feedback controls for non-linear dynamical systems has been discussed in a number of papers. In the present paper, an approach to feedback control that brings Lagrangian systems to a prescribed terminal state in a finite time under uncertain disturbances is discussed. This approach is based on the decomposition of a non-linear dynamical system with n degrees of freedom and reduces this system to n independent linear systems with one degree of freedom each.

The statement of the problem is given in Section 2. A non-linear system described by Lagrange's equations and subjected to disturbances and control forces bounded by geometric constraints is considered. A feedback control law which guarantees that the system reaches the desired state in a finite time is sought for.

It is shown that under the assumptions discussed in Section 3 the original control problem can be replaced by control problems for subsystems with one degree of freedom each. Thus, decomposition of the system is feasible. For each of the subsystems, the approach of the theory of differential games is applied. To obtain a desired feedback control for the original non-linear system, an auxiliary optimal control problem is used. The obtained control law does not use exact expressions of non-linear terms and disturbances in the equations of motion. Only maximal possible values of these terms are essential. This control is robust and not sensitive to small variations in the parameters of the system, small additional forces, and other perturbations. In some cases it can be close to an optimal one. Note that determining a feedback solution for the optimal control problem under constraints is a hard task. Familiar techniques to achieve an optimal control in this case are mentioned in the introduction of the paper (Blanchini and Pellegrino, 2003).

A simplified version of the approach is described in Section 4.

The main assumption made in the present paper is valid for manipulation robots with electric drives with high gear ratios. In Section 5, possible applications of the obtained results to the control of robots are discussed.

In Section 6, the proposed method is used to swing up a double pendulum. This example demonstrates that this approach can be applied also to underactuated systems with uncertainties. The control law presented in this section differs from earlier versions developed by the authors.

2. CONTROL OF UNCERTAIN DYNAMICAL SYSTEM

Consider a nonlinear dynamical system described by Lagrange's equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i + P_i(\mathbf{q}, \dot{\mathbf{q}}, t), \quad i = 1, \ldots, n \quad (1)$$

and subject to the controls Q_i bounded by the constraints

$$|Q_i| \le Q_i^0, \quad i = 1, \ldots, n. \quad (2)$$

Here $\mathbf{q} = (q_1, \ldots, q_n)$ is the vector of generalized coordinates, P_i are external forces including uncertain disturbances, Q_i^0 are given constants, and T is the kinetic energy

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}(\mathbf{A}(\mathbf{q})\dot{\mathbf{q}}, \dot{\mathbf{q}}) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(\mathbf{q})\dot{q}_j\dot{q}_k. \quad (3)$$

Here a_{jk} are the elements of the symmetric positive definite $n \times n$ matrix $\mathbf{A}(\mathbf{q})$. System (1) can be rewritten as follows:

$$\begin{gathered} \mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{P}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathbf{S}(\mathbf{q}, \dot{\mathbf{q}}), \\ \mathbf{Q} = (Q_1, \ldots, Q_n), \quad \mathbf{P} = (P_1, \ldots, P_n), \\ \mathbf{S} = (S_1, \ldots, S_n), \quad S_i = \sum_{j,k=1}^{n} \Gamma_{ijk}(q)\dot{q}_j\dot{q}_k, \\ \Gamma_{ijk} = \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i} - \frac{\partial a_{ij}}{\partial q_k}. \end{gathered} \quad (4)$$

Problem 1. Find a feedback control $\mathbf{Q}(\mathbf{q}, \dot{\mathbf{q}})$ satisfying constraint (2) and bringing system (1) (or (4)) from any initial state

$$\mathbf{q}(0) = \mathbf{q}^0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}^0 \quad (5)$$

to the prescribed terminal state

$$\mathbf{q}(t_1) = \mathbf{q}^1, \quad \dot{\mathbf{q}}(t_1) = \mathbf{0} \quad (6)$$

in a finite time (the instant of time t_1 is not fixed).

The approach to Problem 1 is based on the decomposition of system (4).

3. METHOD OF DECOMPOSITION

Denote $\mathbf{A}_1 = \mathbf{A}(\mathbf{q}^1)$ and multiply both sides of (4) by $\mathbf{A}_1\mathbf{A}^{-1}(\mathbf{q})$. The system takes the form

$$\begin{gathered} \mathbf{A}_1\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{R}(\mathbf{q}, \dot{\mathbf{q}}, t, \mathbf{Q}), \\ \mathbf{R} = \mathbf{A}_1\mathbf{A}^{-1}(\mathbf{q})(\mathbf{P} + \mathbf{S} + \mathbf{Q}) - \mathbf{Q}. \end{gathered} \quad (7)$$

Suppose that all motions under consideration lie in some domain W in the $2n$-dimensional $(\mathbf{q}, \dot{\mathbf{q}})$-space. Assume that

$$|R_i(\mathbf{q}, \dot{\mathbf{q}}, t, \mathbf{Q})| \le R_i^0 < Q_i^0, \quad i = 1, \ldots, n \quad (8)$$

for all $t \ge 0$, all $(\mathbf{q}, \dot{\mathbf{q}}) \in W$, and all $\mathbf{Q}$ satisfying (2). Conditions (8) can be verified by means of the following Lemma.

Lemma. Suppose that the inequalities

$$\begin{gathered} |\mathbf{A}_1\mathbf{z}| \ge \mu_1|\mathbf{z}|, \quad |[\mathbf{A}(\mathbf{q}) - \mathbf{A}_1]\mathbf{z}| \le \mu|\mathbf{z}|, \\ |P_i + S_i| \le \nu Q_i^0, \quad i = 1, \ldots, n, \\ 0 < \mu < \mu_1, \quad \nu > 0 \end{gathered} \quad (9)$$

hold for any $\mathbf{z} \in R^n$, all $t \ge 0$, and all $(\mathbf{q}, \dot{\mathbf{q}}) \in W$, where μ_1, μ, and ν are constants. Then, for all $t \ge 0$, all $(\mathbf{q}, \dot{\mathbf{q}}) \in W$, and all $\mathbf{Q}$ satisfying (2), the following estimates are fulfilled:

$$\begin{gathered} |R_i| \le \nu Q_i^0 + \mu(\mu_1 - \mu)^{-1}(1 + \nu)|\mathbf{Q^0}|, \\ \mathbf{Q}^0 = (Q_1^0, \ldots, Q_n^0). \end{gathered}$$

The proof of Lemma is given in (Chernousko, 1990).

Corollary. If $\nu < 1$ and μ is sufficiently small, then conditions (8) are satisfied. Consequently, to ensure (8) it is required to increase the control bounds Q_i^0 in (2) (and thus decrease ν) and diminish the domain W (and thus decrease μ).

After the change of variables $\mathbf{A}_1(\mathbf{q} - \mathbf{q}^1) = \mathbf{y}$, system (7) is reduced to

$$\ddot{y}_i = Q_i + R_i, \quad i = 1, \ldots, n. \quad (10)$$

The terms R_i can be regarded as independent uncertain disturbances subject to (8) and the approach of differential games can be applied to system (10). In this case, system (10) can be considered as a set of n subsystems with one degree of freedom each, whereas Q_i and R_i are the controls of the two players acting on the i-th subsystem.

By the change of variables

$$y_i = Q_i^0 x, \quad Q_i = Q_i^0 u, \quad R_i = Q_i^0 v,$$

$$\rho = R_i^0/Q_i^0 < 1, \quad \xi = [\mathbf{A}_1(\mathbf{q}^0 - \mathbf{q}^1)]_i/Q_i^0, \quad (11)$$

$$\eta = (\mathbf{A}_1 \dot{\mathbf{q}}^0)_i/Q_i^0, \quad i = 1, \ldots, n,$$

the i-th subsystem in (10), the constraints (2) and (8), and the boundary conditions (5) and (6) are reduced to the normalized form

$$\ddot{x} = u + v, \quad |u| \le 1, \quad |v| \le \rho < 1,$$
$$x(0) = \xi, \quad \dot{x}(0) = \eta, \quad x(t_1) = \dot{x}(t_1) = 0. \quad (12)$$

Using the approaches of differential games and optimal control (Krasovskii, 1970), one can readily obtain the feedback control $u(x, \dot{x})$ which brings system (12) to the prescribed terminal state under any admissible v in a minimal guaranteed time t_1. This control is given by

$$u(x, \dot{x}) = \operatorname{sign} \psi_\rho(x, \dot{x}), \quad \text{if} \quad \psi_\rho \ne 0,$$
$$u(x, \dot{x}) = \operatorname{sign} x = -\operatorname{sign} \dot{x}, \quad \text{if} \quad \psi_\rho = 0, \quad (13)$$

where ψ_ρ is the switching function defined by

$$\psi_\rho(x, \dot{x}) = -x - \dot{x}|\dot{x}|[2(1-\rho)]^{-1}. \quad (14)$$

By means of the change of variables (11), the feedback control solving Problem 1 can be represented as follows:

$$Q_i(\mathbf{q}, \dot{\mathbf{q}}) = Q_i^0 u(x, \dot{x}),$$
$$x = (Q_i^0)^{-1}[\mathbf{A}_1(\mathbf{q} - \mathbf{q}^1)]_i, \quad (15)$$
$$\dot{x} = (Q_i^0)^{-1}(\mathbf{A}_1 \dot{\mathbf{q}})_i, \quad i = 1, \ldots, n.$$

Here $u(x, \dot{x})$ is given by (13) and (14) with ρ defined by (11). The total time of motion is finite and does not exceed the maximum of all termination times corresponding to the degrees of freedom.

4. SIMPLIFIED APPROACH

The game approach can be replaced by other control methods. The simplest way is just to ignore the disturbances altogether, i.e., to put $v = 0$ in (12).

The dynamics of the system (12) under such a simplified control is described by

$$\ddot{x} = u(x, \dot{x}) + v, \quad |v| \le \rho < 1,$$
$$x(0) = \xi, \quad \dot{x}(0) = \eta. \quad (16)$$

Here the function $u(x, \dot{x})$ is given by (13) and (14) with $\rho = 0$.

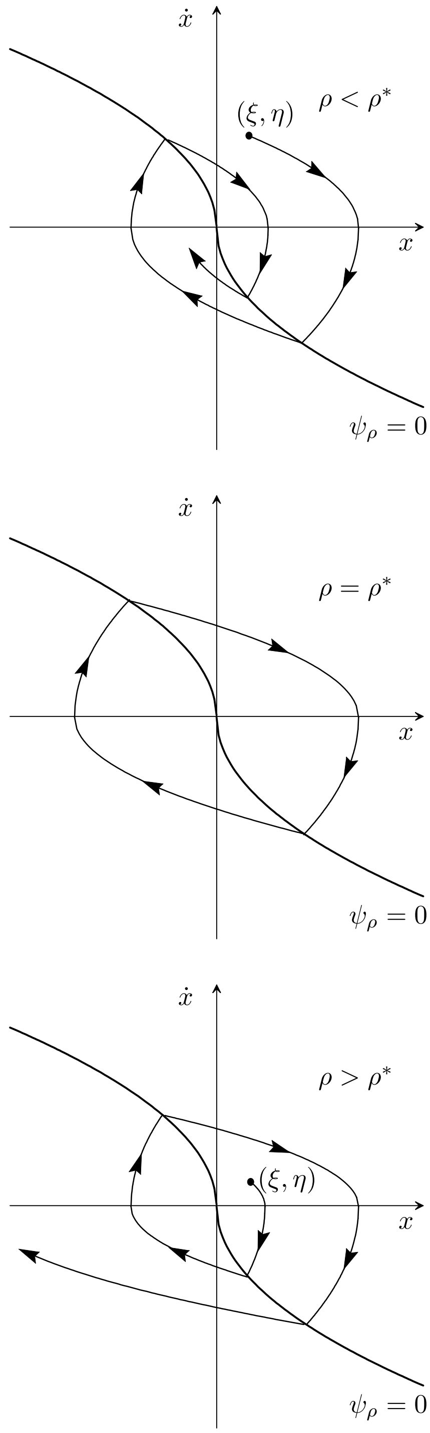

Fig. 1. Trajectories for different values of ρ.

It has been shown in (Chernousko, 1990) that the behavior of the nonlinear system (16) depends on the parameter ρ in the following way (see figure 1). The critical value of ρ is equal to the golden section ratio

$$\rho^* = 5^{1/2} - 1 = 0.618\ldots. \quad (17)$$

If $\rho < \rho^*$, then all trajectories of the system (16) approach the point $x = \dot{x} = 0$ and reach it for any admissible disturbance $v(t)$. The number of switches of the control $u(x, \dot{x})$ may be infinite, but the time of the motion is finite.

If $\rho = \rho^*$, there exist special admissible disturbances such that the system stays in a bounded domain but never reaches the point $x = \dot{x} = 0$. Here periodical motions are possible.

If $\rho > \rho^*$, then there exist admissible disturbances $v(t)$ such that the corresponding trajectories are unbounded and tend to infinity.

Therefore, the simplified control ignoring the disturbances solves Problem 1 only if $\rho < \rho^*$, where ρ is given by (17). The game control described in this paper gives better results: it solves Problem 1 for $\rho < 1$.

Other versions of the decomposition approach and their applications were suggested and discussed in (Chernousko, 1992; Chernousko and Reshmin, 1998).

5. APPLICATION TO ROBOTS

Consider a manipulation robot which consists of n rigid links connected consecutively by revolute or prismatic joints. The angles of relative rotation of the links and their relative linear displacements for revolute and prismatic joints, respectively, are denoted by $\mathbf{q} = (q_1, \ldots, q_n)$. The equations of motion for the manipulator can be presented in the form (1) with the kinetic energy T given by (3). The generalized forces here are torques for revolute joints and forces for prismatic joints. The terms Q_i in (1) are the control torques and forces created by the actuators in revolute and prismatic joints, respectively. The terms P_i include all external and internal forces except the controls, namely, the weight, resistance, friction, etc.

Assume that the robot is driven by independent electric DC (direct current) actuators placed at the joints. Note that state constraints $q_i^- \leq q_i \leq q_i^+$, $i = 1, \ldots, n$, which are often imposed in practical problems can also be taken into account in the decomposition approach. The equation of balance of voltages in the circuit of the ith actuator has the form

$$L_i \frac{dj_i}{dt} + R_i j_i + N_i k_i^E \dot{q}_i = u_i. \qquad (18)$$

Here, L_i is the inductance, R_i is the electrical resistance, k_i^E is a constant coefficient, N_i is the gear ratio, and u_i is the electric voltage in the circuit of the ith actuator. The electromagnetic torque $M_i = N_i^{-1} Q_i$ is proportional to the electric current j_i in the ith actuator

$$M_i = k_i^M j_i, \qquad k_i^M = \text{const} > 0. \qquad (19)$$

The first term in (18) is usually small compared with other terms and, hence, can be omitted. Then (18) and (19) yield

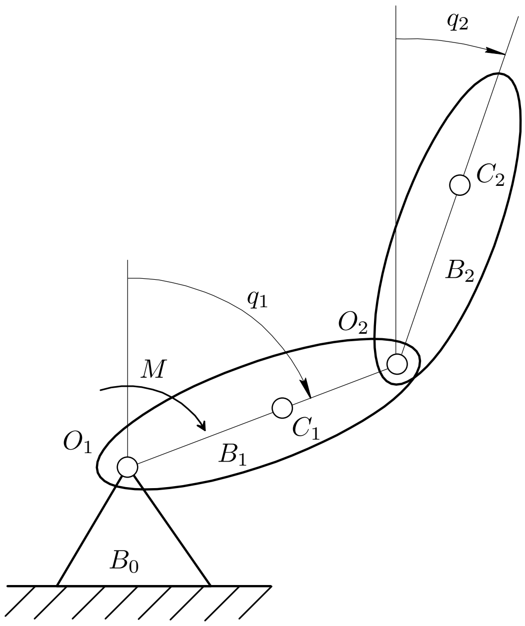

Fig. 2. Double pendulum.

$$M_i = k_i^M R_i^{-1}(u_i - N_i k_i^E \dot{q}_i), \quad Q_i = N_i M_i. \quad (20)$$

Substituting Q_i from (20) into equation (1), one obtains

$$\mathbf{A}\ddot{\mathbf{q}} = \mathbf{Q}^* + \mathbf{S}^* + \mathbf{P}, \qquad \mathbf{S}^* = \mathbf{S} - \mathbf{\Lambda}\dot{\mathbf{q}},$$

$$\mathbf{\Lambda} = \text{diag}(N_1^2 k_1^M k_1^E R_1^{-1}, \ldots, N_n^2 k_n^M k_n^E R_n^{-1}), \quad (21)$$

$$\mathbf{Q}^* = (N_1 k_1^M R_1^{-1} u_1, \ldots, N_n k_n^M R_n^{-1} u_n).$$

The voltages u_i of the actuators are usually restricted by the constraints $|u_i| \leq u_i^0$ (u_i^0 are constants) which are transformed into constraints imposed on the components Q_i^* of the vector $\mathbf{Q}^*$ from (21)

$$|Q_i^*| \leq Q_i^{*0} = N_i k_i^M R_i^{-1} u_i^0, \quad i = 1, \ldots, n. \quad (22)$$

The results of Section 3 can be applied to system (21) with constraints (22), and the feedback control voltages can be obtained in an explicit form. More detailed analysis and numerous examples for the control of manipulation robots are presented in (Chernousko and Reshmin, 1998).

6. CONTROL OF A DOUBLE PENDULUM

The decomposition approach described above is applicable to systems with n degrees of freedom equipped with n actuators. Much more complicated control problems arise if the system is underactuated, i.e., if the number of actuators is less than the number of degrees of freedom. As an example, a double pendulum controlled by a torque applied to the suspension axis is considered, see figure 2. The pendulum consists of two rigid links B_1 and B_2. The revolute joint O_1 with a horizontal axis attaches the link B_1 to a fixed base B_0. The links B_1 and B_2 are connected by the revolute joint O_2 the axis of which is parallel to that of O_1. The motion of such a system occurs in a vertical plane. The center of mass C_1 of the link B_1 lies on the ray O_1O_2. The center of mass C_2 of the link B_2 does not lie on the axis of the joint O_2. The system is controlled by the torque M applied to the joint O_1.

The motion of this system is governed by Lagrange's equations

$$
\begin{gathered}
(m_2 l_1^2 + I_1)\, \ddot{q}_1 + m_2 l_1 l_{g2} \cos(q_2 - q_1)\, \ddot{q}_2 \\
-m_2 l_1 l_{g2} \sin(q_2 - q_1)\, \dot{q}_2^2 = M + G_1 + v_1, \\
\\
m_2 l_1 l_{g2} \cos(q_2 - q_1)\, \ddot{q}_1 + I_2\, \ddot{q}_2 \\
+m_2 l_1 l_{g2} \sin(q_2 - q_1)\, \dot{q}_1^2 = G_2 + v_2, \\
\\
G_1 = g(m_1 l_{g1} + m_2 l_1) \sin q_1, \\
G_2 = g m_2 l_{g2} \sin q_2,
\end{gathered}
\tag{23}
$$

where q_i is the angle of the deflection of the link B_i from the vertical; l_{gi} is the length of the segment O_iC_i; l_1 is the length of the segment O_1O_2; m_i is the mass of the link B_i; I_i is the moment of inertia of the link B_i about the axis of the joint O_i; G_i is the torque created by the gravity force at the joint O_i; v_i is the torque created by the disturbances at the joint O_i; and g is the acceleration due to gravity.

The control torque M is subjected to the constraint

$$
|M| \le M^0, \tag{24}
$$

where M^0 is a positive constant. Constraints are also imposed on the disturbances

$$
|v_1| \le v_1^0, \qquad |v_2| \le v_2^0, \tag{25}
$$

where $v_1^0 \ge 0$ and $v_2^0 \ge 0$ are given constants.

The following control problem can be formulated.

Problem 2. Find a feedback control $M(q_1, \dot{q}_1, q_2, \dot{q}_2)$ that satisfies (24) and steers the system (23) from the given initial state

$$
\begin{gathered}
q_1(0) = \pi, \qquad q_2(0) = \pi, \\
\dot{q}_1(0) = 0, \qquad \dot{q}_2(0) = 0
\end{gathered}
\tag{26}
$$

to the prescribed neighborhood of the upper equilibrium position

$$
\begin{gathered}
|q_1 - 2\pi k| < \varepsilon, \quad |q_2 - 2\pi m| < \varepsilon, \\
k, m = 0, \pm 1, \pm 2, \ldots, \\
|\dot{q}_1| < \varepsilon', \quad |\dot{q}_2| < \varepsilon',
\end{gathered}
\tag{27}
$$

where ε and ε' are given constants which can be arbitrarily small.

Problem 2 is solved under certain simplifying assumptions concerning the control M and the disturbances v_1 and v_2. It is assumed that on the one hand, the constant M^0 in (24) is not too small and, on the other hand, the constants v_1^0 and v_2^0 in (25) are not very large. In this case, a bounded feedback control $M(q_1, \dot{q}_1,\ q_2, \dot{q}_2)$ which satisfies (24) and brings the system (23) from the initial state (26) to the terminal state (27) in a finite time for any admissible disturbances v_1 and v_2 satisfying (25) can be taken in the form

$$
\begin{gathered}
M = \operatorname{sign}(\dot{x} - \tilde{\psi}), \qquad \dot{x} \ne \tilde{\psi}; \\
M = \operatorname{sign}(\dot{x}), \qquad \dot{x} = \tilde{\psi},
\end{gathered}
\tag{28}
$$

where x is the angle between the links

$$
x = q_2 - q_1, \qquad \dot{x} = \dot{q}_2 - \dot{q}_1 \tag{29}
$$

and $\tilde{\psi}(x)$ is a switching function defined by the following relations:

$$
\begin{gathered}
\tilde{\psi}(x) = \psi(x - \tilde{x}), \\
\psi(\cdot) = -(2X|\cdot|)^{1/2} \operatorname{sign}(\cdot), \\
\tilde{x} = -f \operatorname{sign} \dot{q}_1, \qquad (\operatorname{sign} 0 = -1).
\end{gathered}
\tag{30}
$$

Here, X and f are positive control parameters which are to be found. This control is of the bang-bang type and switches between the two limiting vales, i.e., $M = \pm M^0$. The switching curve $\dot{x} = \tilde{\psi}(x)$ consists of two parabolic arcs symmetric with respect to the point $(\tilde{x}, 0)$. Note that the variable $\tilde{x}$ and the velocity $\dot{q}_1$ change in sign simultaneously. It has been proved that the control law (28)–(30) can be used for the solution of Problem 2 and a system of inequalities (bounds) has been obtained for the admissible values of the control parameters X and f. A specific procedure for choosing or calculating these parameters has been also proposed.

The control of (28)–(30) can be modified for solving a more complicated problem.

Problem 3. Find a bounded control $M(q_1, \dot{q}_1, q_2, \dot{q}_2)$ which brings the system from any initial position to the unstable equilibrium point $q_1 = 2\pi k$, $q_2 = 2\pi m$, $k, m = 0, \pm 1, \pm 2, \ldots$, with zero velocities $\dot{q}_1 = \dot{q}_2 = 0$ (in the absence of the disturbance v_2).

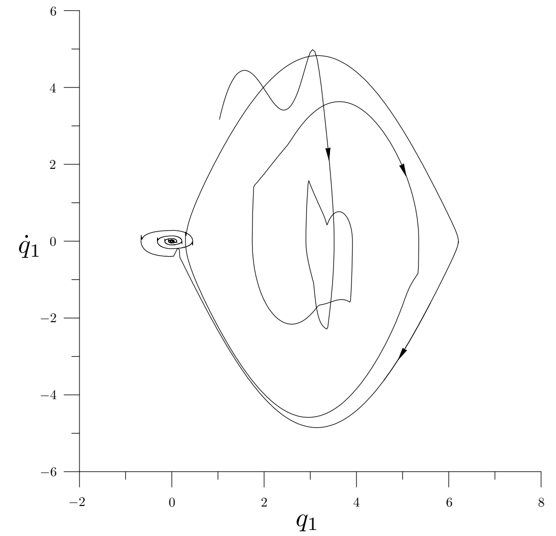

Fig. 3. Projection of the trajectory onto the plane $(q_1, \dot{q}_1)$.

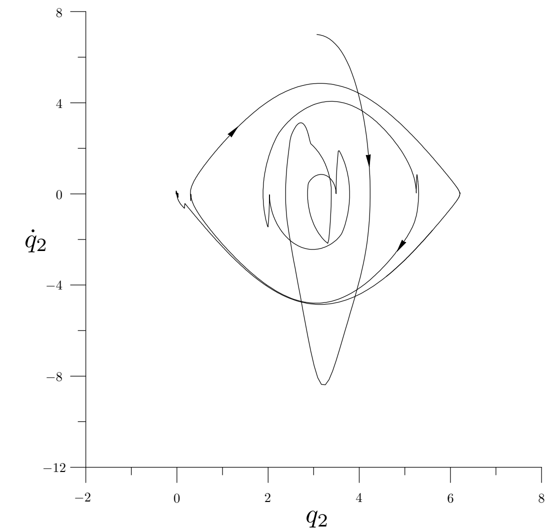

Fig. 4. Projection of the trajectory onto the plane $(q_2, \dot{q}_2)$.

A typical behavior of the projections of the phase trajectory onto the planes $(q_1, \dot{q}_1)$, $(q_2, \dot{q}_2)$, and $(q_2 - q_1, \dot{q}_2 - \dot{q}_1)$ for the control law that solves Problem 3 is presented in figures 3, 4, and 5.

7. CONCLUSION

The approach described above makes it possible to design feedback controls that can bring nonlinear dynamical systems to a prescribed terminal state in a finite time.

The approach is based on the decomposition of the system into simple subsystems. Methods of optimal control and differential games are used to obtain explicit formulas for feedback controls.

This approach does not assume the external forces to be known; they can be uncertain, and only bounds on them are essential. The obtained feedback controls are robust, i.e., they can cope with additional small disturbances and parameter variations. To ensure the robustness, we should only increase the parameter $\rho < 1$ in (12)–(14), thus creating a sufficient margin in control possibilities.

The approach presented can be applied also to the underactuated systems with uncertainties.

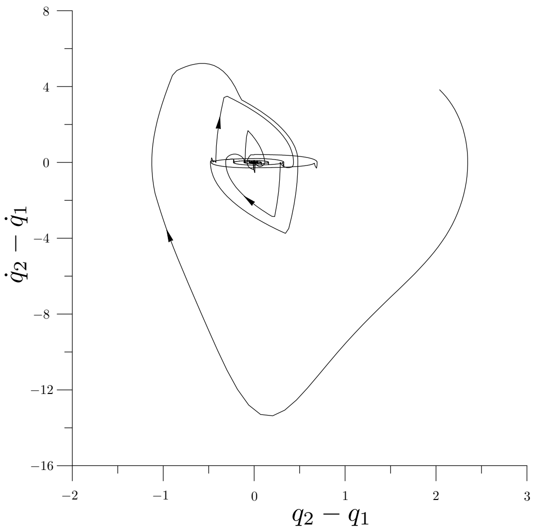

Fig. 5. Projection of the trajectory onto the plane $(q_2 - q_1, \dot{q}_2 - \dot{q}_1)$.

8. ACKNOWLEDGEMENTS

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