DYNAMIC OUTPUT FEEDBACK SYNTHESIS WITH GENERAL FREQUENCY DOMAIN SPECIFICATIONS [1]

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Abstract: This paper considers a control synthesis problem for linear systems to meet design specifications given in terms of multiple frequency domain inequalities in (semi)finite ranges. Dynamic output feedback controllers of order equal to the plant are considered. A new multiplier expansion is proposed to convert the synthesis condition to a linear matrix inequality (LMI) condition through the linearizing change of variables by Scherer, Masubuchi, de Oliveira *et al.* In the single objective setting, the LMI condition may or may not be conservative, depending upon the choice of the basis for the multiplier expansion. We provide a qualification for the basis matrix to yield nonconservative LMI conditions. It turns out to be difficult to determine the basis matrix meeting such qualification in general. However, it is shown that qualified bases can be found for some cases, and that the qualification can be used to find reasonable choices of the basis for other cases. Finally, the synthesis method is applied to a multiple objective control problem for an active magnetic bearing to demonstrate its utility.

Keywords: Multiobjective control, Frequency shaping, Linear matrix inequalities

1. INTRODUCTION

Frequency domain inequalities (FDIs) have played a crucial role in describing design specifications for feedback control designs. Due to the infinite dimensionality, however, FDIs are not directly useful for rigorous assessment/design of control systems. The Kalman-Yakubovich-Popov (KYP) lemma (Rantzer (1996)) has been proven to be a powerful tool to convert an FDI to a linear matrix inequality (LMI) which is numerically tractable. Many of the state space theories have been developed with the aid of the KYP lemma in one way or another. On the other hand, a drawback of the standard KYP lemma is that it does not exactly encompass the practical situation. Namely, it characterizes FDIs in the entire frequency range, while practical requirements are usually described by multiple FDIs in (semi)finite ranges; e.g., small sensitivity in a low frequency range and control roll-off in a high frequency range. The prevailing method for adjusting the discrepancy is the so-called weighting functions. However, the design iterations to search for

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good weighting functions can be time consuming, and the controller complexity tends to increase with the complexity of the weighting functions.

The objective of this paper is to develop a design method capable of directly treating multiple FDI specifications in various frequency ranges without introducing weighting functions. To our knowledge, this problem has not been addressed in the literature. Our approach is based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma (Iwasaki and Hara (2005)), recently developed by the authors, that provides an LMI characterization of FDIs in (semi)finite frequency ranges. We shall extend our previous result for the static gain feedback synthesis (Iwasaki and Hara (2004)) to the dynamic output feedback case. A multiplier method will be developed to render the synthesis conditions convex through a standard linearizing change of variables (de Oliveira et al. (2002); Masubuchi et al. (1998); Scherer et al. (1997)). In the single objective setting, a condition is provided for the multiplier basis to yield nonconservative design equations, and discuss how to choose the basis to satisfy the condition exactly for some cases and approximately for other cases. The synthesis method can be extended, with some conservatism, for the case of multi-objective specifications as has been done in Iwasaki and Hara (2004). Our method does not require weighting functions and the resulting controller order is equal to the original plant order, as illustrated by a design example.

Notation. For a matrix M, its transpose, and complex conjugate transpose are denoted by M^{T} and M^* respectively. The Hermitian part of a square matrix M is denoted by $\mathrm{He}(M) := M + M^*$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. For matrices Φ and P, $\Phi \otimes P$ means their Kronecker product. For matrices G and Π, the function $\sigma : \mathbb{C}^{n\times m} \times \mathbf{H}_{n+m} \to \mathbf{H}_m$ is defined by

$$\sigma(G,\Pi) := \begin{bmatrix} G \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

2. PROBLEM STATEMENT/FORMULATION

2.1 Problem statement

Consider the plant $G(\lambda)$ described by

$$\begin{bmatrix} \lambda x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{1}$$

and a feedback controller $K(\lambda)$ given by

$$\begin{bmatrix} \lambda x_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} \tag{2}$$

where λ is the frequency variable (s for continuous-time and z for discrete-time cases), and $x(t) \in \mathbb{R}^{n_p}$, $x_c(t) \in \mathbb{R}^{n_c}$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, and $y(t) \in \mathbb{R}^{n_y}$. Denote by $H(\lambda)$ the closed-loop transfer function from w to z. A state space realization $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ of $H(\lambda)$ is given by

$$\left[\begin{array}{cc|c} A + B_2 D_c C_2 & B_2 C_c & B_1 + B_2 D_c D_{21} \\ B_c C_2 & A_c & B_c D_{21} \\ \hline C_1 + D_{12} D_c C_2 & D_{12} C_c & D_{11} + D_{12} D_c D_{21} \end{array}\right] \tag{3}$$

where the state dimension is $n := n_p + n_c$.

The control synthesis problem of our interest is, given $\Pi \in \mathbf{H}_{n_w+n_z}$ and $\Phi, \Psi \in \mathbf{H}_2$, find a full order $(n_c = n_p)$ controller $K(\lambda)$ such that

$$\det(\lambda I - \mathsf{A}) \neq 0, \quad \sigma(H(\lambda)^*, \Pi) < 0 \tag{4}$$

for all $\lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi)$ where

$$\mathbf{\Lambda}(\Phi,\Psi) := \{\, \lambda \in \mathbb{C} \mid \sigma(\lambda,\Phi) = 0,\ \sigma(\lambda,\Psi) \geq 0 \,\} \tag{5}$$

and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda}$ if $\mathbf{\Lambda}$ is bounded and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda} \cup \{\infty\}$ if unbounded. For clarity of exposition, we shall restrict our attention to this single-objective nominal control problem in the main body of our theoretical developments. Extensions can be made as in Iwasaki and Hara (2004) to a more general problem where there are pole constraints and multiple FDI constraints of the above form.

Throughout the paper, we shall impose the following for tractability and practicality:

Assumption 1.

(a) The upper left $n_w \times n_w$ block matrix of Π is positive semidefinite ($\Pi_{11} \geq 0$).
(b) The pair (Φ, Ψ) is chosen so that the set $\mathbf{\Lambda}(\Phi, \Psi)$ is not empty, nor a single point, nor the entire complex plane.

Item (a) ensures that the feasible set for the closed-loop transfer function $H(\lambda)$ is convex, and does not exclude such important specifications as the small gain and passivity. Item (b) excludes the trivial cases and ensures that $\mathbf{\Lambda}(\Phi, \Psi)$ is one of the following: (i) straight line or circle, (ii) half plane, or inside or outside of a circle, (iii) intersection of (i) and (ii). See Iwasaki and Hara (2005) for the details and a precise characterization of (Φ, Ψ) satisfying (b). Below, we will discuss some important cases.

For the continuous-time case, we have

$$\mathbf{\Lambda}(\Omega_c, \Psi_c) = \{\, j\omega :\ \tau(\omega - \varpi_1)(\omega - \varpi_2) \leq 0 \,\}$$

where

$$\Omega_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Psi_c = \tau \begin{bmatrix} -1 & j\varpi_c \\ -j\varpi_c & -\varpi_1\varpi_2 \end{bmatrix},$$

$$\tau = \pm 1, \quad \varpi_1, \varpi_2 \in \mathbb{R}, \quad \varpi_1 < \varpi_2,$$

$$\varpi_c := (\varpi_2 + \varpi_1)/2.$$

For the discrete-time case, we have

$$\mathbf{\Lambda}(\Omega_d, \Psi_d) = \{\, e^{j\theta} :\ (\theta - \vartheta_1)(\theta - \vartheta_2) \leq 0 \,\}$$

where

$$\Omega_d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Psi_d = \begin{bmatrix} 0 & e^{j\vartheta_c} \\ e^{-j\vartheta_c} & -2\cos\vartheta_o \end{bmatrix}$$

$$\vartheta_1, \vartheta_2 \in \mathbb{R}, \quad 0 < \vartheta_2 - \vartheta_1 \leq 2\pi,$$

$$\vartheta_c := (\vartheta_2 + \vartheta_1)/2, \quad \vartheta_o := (\vartheta_2 - \vartheta_1)/2.$$

2.2 *Problem formulation via a dual GKYP lemma*

Consider the transfer function $H(\lambda)$ specified by (3). The GKYP lemma in Iwasaki and Hara (2005) provides a characterization of the FDI: $\sigma(H(\lambda), \Pi) < 0$ for all $\lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi)$. The following result provides a dual version of the GKYP lemma.

Theorem 1. *Let* $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, *and* $H(\lambda)$ *in (3) be given and consider* $\mathbf{\Lambda}(\Phi, \Psi)$ *defined by (5). Suppose Assumption 1 holds. The following statements are equivalent.*

(i) $\det(\lambda I - \mathsf{A}) \neq 0$ *and* $\sigma(H(\lambda)^*, \Pi) < 0$ *hold for all* $\lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi)$.

(ii) There exist $P = P^*$ *and* $Q = Q^* > 0$ *such that*

$$\begin{bmatrix} \mathsf{A} & I \\ \mathsf{C} & 0 \end{bmatrix} (\Phi^{\mathsf{T}} \otimes P + \Psi^{\mathsf{T}} \otimes Q) \begin{bmatrix} \mathsf{A} & I \\ \mathsf{C} & 0 \end{bmatrix}^* + \begin{bmatrix} \mathsf{B} & 0 \\ \mathsf{D} & I \end{bmatrix} \Pi \begin{bmatrix} \mathsf{B} & 0 \\ \mathsf{D} & I \end{bmatrix}^* < 0. \quad (6)$$

Proof. The result basically follows by dualizing Theorem 3 of Iwasaki and Hara (2005). ■

With the result of Theorem 1, the synthesis problem can be formulated as the search for the parameters $Q > 0$, P, and $K(s)$ satisfying (6) where the state space matrices are defined by (3). The resulting condition is not convex due to the product terms between P, Q, and the controller parameters. We shall develop a multiplier method to re-parametrize the condition so that the problem becomes convex.

3. SYNTHESIS

3.1 *Multiplier expansion*

The following result provides an alternative condition to (6) by introducing a multiplier through the projection lemma. To state the result, let us define $J \in \mathbb{R}^{(2n+n_z)\times 2n}$, $\mathsf{H} \in \mathbb{C}^{(2n+n_z)\times(n_w+n_z)}$, and $\mathsf{L} \in \mathbb{C}^{(2n+n_z)\times n}$ by

$$J := \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \mathsf{H} := \begin{bmatrix} 0 & 0 \\ \mathsf{B} & 0 \\ \mathsf{D} & I \end{bmatrix}, \quad \mathsf{L} := \begin{bmatrix} -I \\ \mathsf{A} \\ \mathsf{C} \end{bmatrix}.$$

Lemma 1. Let $P, Q \in \mathbf{H}_n$, $R \in \mathbb{C}^{n\times(2n+n_z)}$, $\Phi, \Psi \in \mathbf{H}_2$, $\Pi \in \mathbf{H}_{n_w+n_z}$, and $H(\lambda)$ in (3) be given. Let N be the null space of R. The following statements are equivalent.

(i) The condition in (6) holds and

$$N^* \left(J(\Phi^{\mathsf{T}} \otimes P + \Psi^{\mathsf{T}} \otimes Q)J^* + \mathsf{H}\Pi\mathsf{H}^* \right) N < 0. \quad (7)$$

(ii) There exists $W \in \mathbb{C}^{n\times n}$ such that

$$J(\Phi^{\mathsf{T}} \otimes P + \Psi^{\mathsf{T}} \otimes Q)J^* + \mathsf{H}\Pi\mathsf{H}^* < \mathrm{He}(\mathsf{L}WR). \quad (8)$$

Proof. The result follows from the projection lemma (Iwasaki and Skelton (1994)) once we notice that L is the null space of $\begin{bmatrix} \mathsf{A} & I_n & 0 \\ \mathsf{C} & 0 & I_{n_z} \end{bmatrix}$. ■

Similar multiplier expansion techniques have been used in the literature; de Oliveira et al. (1999); Henrion et al. (2003); Peaucelle and Arzelier (2001); Peaucelle et al. (2000). The expanded equation (8) will be used as a basis for our synthesis. In particular, the equation will be equivalently converted to an LMI synthesis condition in the next section. Hence, conservatism associated with (8) needs to be carefully analyzed. For an arbitrary R, (8) gives a sufficient condition for (6). On the other hand, (6) and (8) become equivalent if R is chosen to satisfy (7). Thus, condition (7) precisely captures the gap (conservatism) between the synthesis condition (8) and the original design objective.

For synthesis, it is desired that matrix R be chosen to satisfy (7). Note that condition (7) involves the yet unknown controller parameters and hence has to be properly interpreted to give a condition useful for synthesis:

Condition 1. Condition (7) holds for some matrices $(P, Q, A_c, B_c, C_c, D_c)$ satisfying $P, Q \in \mathbf{H}_n$, $Q > 0$, and (6), where $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ are defined by (3).

This condition is independent of the unknown parameters P, Q and (A_c, B_c, C_c, D_c), and thus can be used to fix R *before* the control design. With R satisfying Condition 1, there exists a controller that meets the specification (4) if and only if there exist matrices $P, Q \in \mathbf{H}_n$, $W \in \mathbb{C}^{n\times n}$, and (A_c, B_c, C_c, D_c) such that $Q > 0$ and (8) hold. We will show how to solve the synthesis problem (8) in the next section. How to choose an appropriate R will be addressed in the section that follows.

Finally, we give a remark on the relation between the multiplier expansion described in this section and the one used in our prior work on the static gain synthesis (Iwasaki and Hara (2004)). In particular, the former can be considered as a partial expansion, and a further expansion of the quadratic term in H in (8) would yield the full multiplier expansion in Iwasaki and Hara (2004) that avoids direct product of H and Π. The advantage here is that the additional information $\Pi_{11} \geq 0$ can be exploited to convexify the problem and that the number of multiplier parameters is smaller for better computational efficiency.

3.2 *Reduction to LMIs*

The synthesis problem described by (8) is nonconvex due to the product term between the multiplier W and the controller parameters. Below, we show that the change of variable introduced by de Oliveira *et al.* (de Oliveira et al. (2002)) works perfectly to convert the problem to an LMI problem, provided R satisfies an additional structural constraint.

Let X, Y, U and V be defined by

$$W = \begin{bmatrix} X & * \\ U & * \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} Y & V \\ * & * \end{bmatrix}^*.$$

Note that, given any $X, Y, U, V \in \mathbb{C}^{n_p\times n_p}$ with U and V invertible, the blanks "$*$" can be filled to satisfy the above two equalities for some W. In particular, we have

$$\begin{bmatrix} X & (I-XY^*)V^{-*} \\ U & -UY^*V^{-*} \end{bmatrix} \begin{bmatrix} Y^* & (I-Y^*X)U^{-1} \\ V^* & -V^*XU^{-1} \end{bmatrix} = I.$$

Defining the new variables

$$\begin{bmatrix} M & G \\ H & L \end{bmatrix} := \begin{bmatrix} YAX & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V & YB_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} U & 0 \\ C_2X & I \end{bmatrix}, \quad (9)$$

we have

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} := \begin{bmatrix} F\mathsf{A}WF^* & F\mathsf{B} \\ \mathsf{C}WF^* & \mathsf{D} \end{bmatrix} \quad (10)$$

$$= \left[\begin{array}{cc|c} AX+B_2H & A+B_2LC_2 & B_1+B_2LD_{21} \\ M & YA+GC_2 & YB_1+GD_{21} \\ \hline C_1X+D_{12}H & C_1+D_{12}LC_2 & D_{11}+D_{12}LD_{21} \end{array}\right]$$

$$\mathcal{W} := FWF^* = \begin{bmatrix} X & I \\ Z & Y \end{bmatrix} \quad (11)$$

$$F := \begin{bmatrix} I & 0 \\ Y & V \end{bmatrix}, \quad Z := YX + VU. \quad (12)$$

Now, suppose R in (8) has been chosen to satisfy the following:

Condition 2. There exists a fixed matrix $\mathcal{R} \in \mathbb{C}^{n\times(2n+n_z)}$ satisfying

$$\begin{array}{l} R\mathcal{F}^* = F^*\mathcal{R}, \\ \mathcal{F} := \mathrm{diag}(F, F, I_{n_z}), \end{array} \quad F := \begin{bmatrix} I & 0 \\ Y & V \end{bmatrix} \quad (13)$$

for all matrices $Y, V \in \mathbb{C}^{n_p\times n_p}$.

We will discuss how to choose such R later. Then, through the congruence transformation of (8) by $\mathcal{F}$, we obtain

$$J(\Phi^{\mathsf{T}}\otimes\mathcal{P}+\Psi^{\mathsf{T}}\otimes\mathcal{Q})J^*+\mathcal{H}\Pi\mathcal{H}^* < \mathrm{He}(\mathcal{L}\mathcal{R}) \quad (14)$$

where

$$\mathcal{P} := FPF^*, \quad \mathcal{Q} := FQF^*, \quad (15)$$

$$\mathcal{H} := \begin{bmatrix} 0 & 0 \\ \mathcal{B} & 0 \\ \mathcal{D} & I \end{bmatrix}, \quad \mathcal{L} := \begin{bmatrix} -\mathcal{W} \\ \mathcal{A} \\ \mathcal{C} \end{bmatrix}. \quad (16)$$

Summarizing the above, we have the following.

Lemma 2. Consider the plant $G(\lambda)$ in (1) and the controller $K(\lambda)$ in (2) with $n_c = n_p$, and let $P, Q \in \mathbf{H}_n$, $R \in \mathbb{C}^{n\times(2n+n_z)}$, $\Phi, \Psi \in \mathbf{H}_2$, and $\Pi \in \mathbf{H}_{n_w+n_z}$ be given where $n := 2n_p$. Suppose R satisfies Condition 2. Then the following statements are equivalent.

- (i) There exist matrices $P, Q \in \mathbf{H}_n$, a multiplier $W \in \mathbb{C}^{n\times n}$ and a controller (A_c, B_c, C_c, D_c) such that (8) is satisfied, where $(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ are defined in (3).
- (ii) There exist matrices X, Y, Z, M, G, H, L, and $\mathcal{P}, \mathcal{Q} \in \mathbf{H}_n$ satisfying (14) where $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ and $\mathcal{W}$ are defined by (10) and (11), respectively.

Moreover, the parameters $(P, Q, A_c, B_c, C_c, D_c, W)$ and $(\mathcal{P}, \mathcal{Q}, M, G, H, L, X, Y, Z)$ are related through the bijective mapping defined by (9), (11), (12), and (15).

As a direct consequence of Theorem 1 and Lemmas 1 and 2, we have the following result.

Theorem 2. Consider the plant $G(\lambda)$ of order n_p in (1). Let $R \in \mathbb{C}^{n\times(2n+n_z)}$, $\Phi, \Psi \in \mathbf{H}_2$, and $\Pi \in \mathbf{H}_{n_w+n_z}$ be given where $n := 2n_p$. Suppose Assumption 1 holds and R satisfies Conditions 1 and 2. Then the following statements are equivalent.

- (i) There exists a dynamic output feedback controller $K(\lambda)$ in (2) with $n_c = n_p$ satisfying the specification in (4).
- (ii) There exist matrices X, Y, Z, M, G, H, L, and $\mathcal{P}, \mathcal{Q} \in \mathbf{H}_n$ satisfying $\mathcal{Q} > 0$ and (14) where $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ and $\mathcal{W}$ are defined by (10) and (11), respectively.

Moreover, (ii) implies (i) for any choice of R. If statement (ii) is true, the parameters (A_c, B_c, C_c, D_c) of controller $K(\lambda)$ in statement (i) can be calculated by solving (9) and (12).

Since $\Pi_{11} \geq 0$, the condition in (14) can be made linear in $\mathcal{B}$ and $\mathcal{D}$ via the Schur complement. The resulting equation is an LMI in terms of variables $X, Y, Z, M, G, H, L, \mathcal{P}$, and $\mathcal{Q}$. Once we solve the LMI with the additional condition $\mathcal{Q} > 0$, the controller parameters can be recovered as follows. First let U and V be any factor such that $VU = Z - YX$ where nonsingularity of $Z - YX$ can be assumed without loss of generality due to the strictness of the LMIs. The controller parameters can then be obtained by solving (9) for (A_c, B_c, C_c, D_c).

4. NONCONSERVATIVE/REASONABLE CHOICES OF R

In this section, we would like to choose R such that feasibility of (14) and $\mathcal{Q} > 0$ is necessary and sufficient for the existence of a controller (2) that meets the specification (4). In view of Theorem 2, such R can be characterized by Conditions 1 and 2. It can readily be verified that R satisfies Condition 2 if and only if it has the following structure:

$$R = \begin{bmatrix} aI & 0 & bI & 0 & \Gamma \\ 0 & aI & 0 & bI & 0 \end{bmatrix} \in \mathbb{C}^{2n_p\times(4n_p+n_z)} \quad (17)$$

where $a, b \in \mathbb{C}$ and $\Gamma \in \mathbb{C}^{n_p\times n_z}$. In this case, we have $\mathcal{R} = R$. On the other hand, the set of R satisfying Condition 1 does not seem to have a simple parametrization, and it turns out to be difficult in general to find R satisfying both conditions exactly.

However, one can find such R for some special cases, and the conditions can be used to find reasonable (but potentially conservative) choices of R for other cases. We will show in the next subsection how to choose R satisfying Conditions 1 and 2 for the case where the frequency range is not restricted and closed-loop stability is required, i.e., $\mathbf{\Lambda}(\Phi, \Psi)$ is either the closed right half plane or outside of the unit circle. The subsection that follows will suggest reasonable choices of R for the general restricted frequency case.

4.1 Case 1: The entire frequency range

We consider the case $\Phi = 0$ and $\Psi = \Omega_c$ or Ω_d so that $\mathbf{\Lambda}(\Phi, \Psi)$ is the instability region on the complex plane for the continuous-time or discrete-time setting. In this case, the specification in (4) requires the closed-loop stability in addition to the frequency domain inequality on the entire frequency range. The variable P then disappears from equations (6) and (7), and the former becomes a standard LMI that arises in the classical KYP lemma.

It can be verified that each choice of R in Table 1 satisfies Condition 1, where $\epsilon > 0$ is a sufficiently small number. The proof is omitted due to the space limitation.

Table 1. Nonconservative choices of R
(entire frequency range)

$\mathbf{\Lambda}(\Phi, \Psi)$	R
$\{ s \in \mathbb{C} \mid s + s^* \geq 0 \}$	$[\ \epsilon I \quad I \quad 0\ ]$
$\{ z \in \mathbb{C} \mid \lvert z \rvert \geq 1 \}$	$[\ I \quad 0 \quad 0\ ]$

4.2 Case 2: The restricted frequency range

For this case, it seems difficult to choose R so that Conditions 1 and 2 are satisfied, and hence we shall look for "reasonable" choices of R.

We propose a heuristic choice of R given by (17) with $a = c\eta_2$, $b = -c\eta_1$, and $\Gamma = 0$ where $c \in \mathbb{C}$ is an arbitrary scalar and $\eta := \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ is an arbitrary vector satisfying $\eta^* \Phi^{\mathsf{T}} \eta = 0$ and $\eta^* \Psi^{\mathsf{T}} \eta < 0$. The existence of such vector η is guaranteed by the assumption that $\mathbf{\Lambda}(\Phi, \Psi)$ represents curve(s). This choice of R is reasonable in the sense that the terms in (7) associated with P and Q are negative semidefinite for any $P, Q \in \mathbf{H}_n$ such that $Q > 0$.

The general choice suggested above can be specialized to find appropriate R for restricted frequency range cases in the continuous- and discrete-time settings. Table 2 summarizes these cases where (ϖ_1, ϖ_2) and $(\vartheta_1, \vartheta_2)$ are real scalars specifying the frequency ranges and satisfy $\varpi_1 < \varpi_2$ and $0 < \vartheta_2 - \vartheta_1 < 2\pi$, respectively, and $\varpi_c := (\varpi_1 + \varpi_2)/2$ and and $\vartheta_c := (\vartheta_1 + \vartheta_2)/2$.

Table 2. Reasonable choices of R
(restricted frequency range)

$\mathbf{\Lambda}(\Phi, \Psi)$	R
$\{ j\omega \mid \varpi_1 \leq \omega \leq \varpi_2 \}$	$[\ 0 \quad I \quad 0\ ]$
$\{ j\omega \mid \omega \leq \varpi_1 \text{ or } \varpi_2 \leq \omega \}$	$[\ I \quad j\varpi_c I \quad 0\ ]$
$\{ e^{j\theta} \mid \vartheta_1 \leq \theta \leq \vartheta_2 \}$	$[\ I \quad e^{-j\vartheta_c} I \quad 0\ ]$

As a byproduct of the above development, we have the following (the proof is omitted again due to the space limitation).

Proposition 1. Suppose $\Phi_{11} = 0$ and $\Psi_{11} < 0$ hold and let $\mathcal{R} := [\ 0 \quad I \quad 0\ ]$. If the condition in (14) is infeasible, then there is no controller (2) that meets the specification (4) with D_c satisfying $\sigma(\mathcal{D}^*, \Pi) < 0$.

The proposition captures the continuous-time, convex bounded frequency interval case. Suppose D_c has been fixed so that $\sigma(\mathcal{D}^*, \Pi) < 0$ holds. A typical situation would be the case where a strictly proper controller is to be designed to meet a small gain requirement for a system with $D_{11} = 0$. Then, (14) with $\mathcal{R} := [\ 0 \ I \ 0\ ]$ provides a *necessary and sufficient condition* for the existence of such controller.

5. EXAMPLE

The main objective of this section is to illustrate the proposed design procedure. In particular, we will design a controller using statement (ii) of Theorem 2 with the heuristic choices of R described in Table 2. The design method is potentially conservative and another aim of this example is to show that the degree of conservatism can be small enough for some applications to allow for direct design of controllers to meet multiple specifications in different frequency ranges.

We consider the control of an active magnetic bearing (AMB). With a constant biasing, the normalized dynamics of an AMB, from the voltage input to the displacement output, can be described by

$$P(s) = \frac{1}{s^3 + \sigma s^2 - \sigma}$$

where σ ranges between about 0.3 and 3 for physically reasonable AMB designs (Maslen et al. (2005)). Below, we take $\sigma = 0.5$.

The problem is to design a stabilizing controller $K(s)$ to meet the following specifications:

$$\begin{aligned} |P(j\omega)S(j\omega)| &< \gamma_o \quad \forall\, |\omega| \leq \varpi_o \\ |K(j\omega)S(j\omega)| &< \gamma_1 \quad \forall\, |\omega| \geq \varpi_1 \\ |K(j\omega)P(j\omega)S(j\omega)| &< \gamma_2 \quad \forall\, |\omega| \geq \varpi_2 \end{aligned} \tag{18}$$

where $S := 1/(1 - PK)$ is the sensitivity function. These three specifications address position regulation against input-port disturbance, sensitivity of the control input to the sensor noise, and robustness against the multiplicative plant uncertainty.

We set the parameter values

$$\varpi_o = 0.5, \ \ \varpi_1 = 3, \ \ \varpi_2 = 0.8, \ \ \gamma_1 = 4, \ \ \gamma_2 = 10$$

and minimize γ_o subject to the above constraints over a set of stabilizing full order controllers using Theorem 2 and Corollary 4 of Iwasaki and Hara (2004). The optimal value is found to be $\gamma_o = 4.45$ and the controller is

$$K(s) = -\frac{2.2207(s + 5.975)(s^2 + 1.234s + 0.9334)}{(s + 3.369)(s^2 + 2.235s + 4.492)}.$$

The resulting close-loop frequency responses are plotted in Fig. 1, where the specification bound for the solid curve is indicated by the shaded region with a solid boundary, and similarly for the dashed and dash-dotted curves. We see that the bounds on $|PS|$ and $|KS|$ are fairly tight, suggesting effectiveness of the design method.

On the other hand, the bound on $|KPS|$ is not very tight, indicating a possible drawback (conservatism). The design would be difficult if the resulting frequency response is insensitive to the change in the bound specification when the bound is active. If the result is sensitive, however, frequency shaping can still be done even when the bound is conservative, by iteratively revising the design specification. To illustrate this point, let us consider the case where the specification on $|KPS|$ is relaxed to $\varpi_2 = 1$. In this case, the optimal value of $\gamma_o = 2.97$ is achieved by

$$K(s) = -\frac{2.6793(s + 2.911)(s^2 + 1.122s + 0.9738)}{(s + 1.705)(s^2 + 2.104s + 4.242)}.$$

The resulting frequency responses are plotted in Fig. 2. We see that the slight change introduced to the specification yielded a significant change in the closed-loop responses. Due to this high sensitivity, we can tune the specification to meet the original goal. For example, if we want to minimize γ_o subject to (18), we may adjust ϖ_2 between 0.8 and 1 so that the constraint on $|KPS|$ in (18) becomes tight. In fact, choosing $\varpi_2 = 0.913$ gives the peak value $\|KPS\|_\infty = 9.97$ at $\omega = 0.825$, while achieving $\gamma_o = 3.56$.

6. CONCLUSION

We have developed a method for synthesizing dynamic output feedback controllers to achieve multiple FDI specifications in (semi)finite frequency ranges. A sufficient condition for existence of feasible controllers are given in terms of LMIs, and some special cases, where the condition becomes also necessary, are discussed. An example of the active magnetic bearing illustrated the proposed design method and demonstrated its effectiveness.

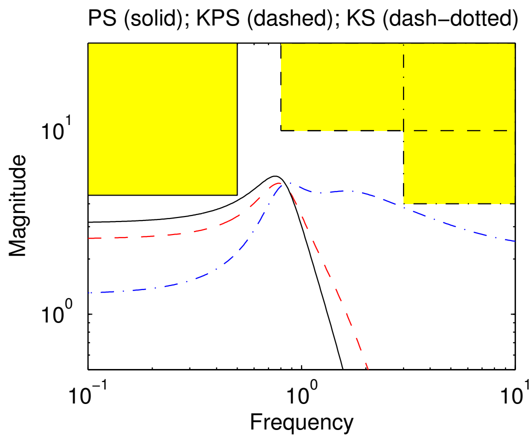

Fig. 1. Design result (Case 1: $\varpi_2 = 0.8$)

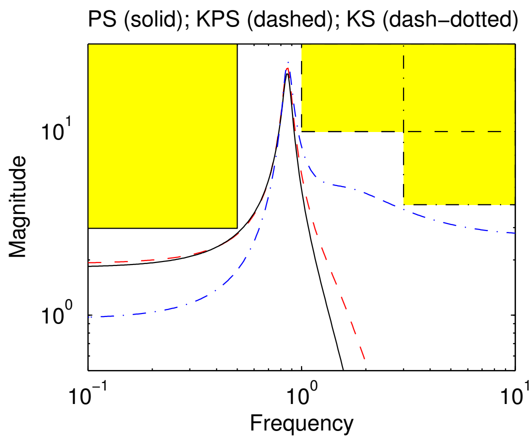

Fig. 2. Design result (Case 2: $\varpi_2 = 1$)

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