NONLINEAR CONTROL SYNTHESIS UNDER DOUBLE CONSTRAINTS [1]

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Abstract: Among the pending problems of control synthesis under set-membership uncertainty lies the case when the bounds on the disturbances and the controls are generated by different types of inequalities. This paper presents a scheme of designing strategies of nonlinear target control synthesis under joint quadratic and hard bounds on the controls and hard bounds on the disturbances. The techniques are based on a combination of dynamic programming methods with techniques introduced by N. N. Krasovski and L. S. Pontryagin.

Keywords: control synthesis, uncertainty, dynamic programming, double constraints, hard bound, soft bound

1. INTRODUCTION

It is usual that one considers problems of control under uncertainty assuming that the control and the disturbance are restricted by constraints of the similar type. For example, both are subject either to hard bounds, or soft bounds. However, in real applications sometimes arises the need to employ different constraint types for control and disturbance, or to impose several constraints simultaneously. This reflects applications with simultaneous restrictions on fuel resources and the ability of maneuvering.

In the present article a problem of control synthesis under uncertainty is considered for a linear system, where the control is subject to double constraints, namely hard bound and soft bound, and the disturbance is restricted only by hard bound. This allows to account both for the constructive properties of the control device which allow only certain range of control values, and for the limited amount of resources used by control.

A problem with double constraints on control in the absence of uncertainty was the subject of (Krasovski, 1965; Bondarenko *et al.*, 1965; Dar'in and Kurzhanskii, 2001). A particular case of the problem under uncertainty was investigated in (Ledyaev, 1985). Here the general case is considered.

The problem is treated by the approach (Kurzhanski, 1999; Kurzhanski and Melnikov, 2000) which, through the use of dynamic programming concepts (Bellman, 1957; Isaacs, 1965), combines the theory of the alternated integral (Pontryagin, 1967; Pontryagin, 1980) and the control theory of N. N. Krasovski (Krasovski, 1971; Krasovskii and Subbotin, 1988). This allows to develop constructive techniques aiming at solving the problem "to the end", that is, to devise effective numerical algorithms (for example, based on the ellipsoidal calculus (Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2002)). Other approaches to control

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under uncertainty belong to Başar and Bernhard (1991), Başar and Olsder (1982), Mitchell and Tomlin (2003). The considered problem may be interpreted as a special case of the viability problem (Aubin, 1991).

2. THE PROBLEM

Consider the controlled system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u + C(t)v, \\ \dot{k}(t) = -\|u\|^2_{R(t)} \end{cases} \tag{1}$$

on time interval $T = [t_0, t_1]$. Here $u \in \mathbb{R}^{n_p}$ is the control and $v \in \mathbb{R}^{n_q}$ is the unknown but bounded disturbance, both subject to hard bounds

$$u \in \mathcal{P}(t), \qquad v \in \mathcal{Q}(t), \qquad \forall t \in T. \tag{2}$$

There is also an additional state constraint

$$k(t) \geqslant 0, \qquad \forall t \in T \tag{3}$$

which is equivalent to a soft bound on control (this follows by integrating the second equation of (1)):

$$\int_{t_0}^{t_1} \|u(t)\|^2_{R(t)} \,\mathrm{d}t \leqslant k(t_0). \tag{4}$$

The set-valued mappings $\mathcal{P}(t)$, $\mathcal{Q}(t)$ are assumed continuous in the Hausdorff metric, and $R(t)$ is a positive definite continuous matrix function. The norm $\|u\|^2_R$ is equal to $\langle u, Ru\rangle$.

After a change of coordinates (see Kurzhanski and Valyi, 1997) the system (1) may be rewritten in a simplified form with $A(t) \equiv 0$, $C(t) \equiv I$, i.e.

$$\begin{cases} \dot{x}(t) = B(t)u + v, \\ \dot{k}(t) = -\|u\|^2_{R(t)}, \end{cases} \qquad t \in T = [t_0, t_1].$$

The control u may be chosen from one of the following classes.

(1) *Feedback strategies* $\mathcal{U}_{\mathrm{CL}}$ are set-valued functions $\mathcal{U}(t, x, k)$: $[t_0, t_1] \times \mathbb{R}^n \times \mathbb{R} \to \operatorname{conv} \mathbb{R}^n$, measurable in t and upper semicontinuous in (x, k), satisfying the conditions $\mathcal{U}(t, x, k) \subseteq \mathcal{P}(t)$, $\mathcal{U}(t, x, k)|_{k<0} = \{0\}$, which ensure the adherence to hard bound (2) and state constraint (3).
(2) *Open-loop controls* $\mathcal{U}_{\mathrm{OL}}$ are measurable functions satisfying a.e. (2) and (4). This class actually depends on initial reserve: $\mathcal{U}_{\mathrm{OL}} = \mathcal{U}_{\mathrm{OL}}(k(t_0))$.

Controls from $\mathcal{U}_{CL}$ ensure the existence of solutions to the following differential inclusion (Filippov, 1988):

$$\begin{bmatrix} \dot{x}(t) \\ \dot{k}(t) \end{bmatrix} \in \left[\operatorname{conv} \bigcup_{u \in \mathcal{U}} \left\{ \begin{bmatrix} B(t)u \\ -\|u\|^2_{R(t)} \end{bmatrix} \right\} \right] + \mathcal{Q}(t). \tag{5}$$

Here the closed convex hull in the right-hand side does not enhance the capabilities of control since it only adds "non-effective" points.

Definition 1. Let $\mathcal{N}$ be a set in space $\mathbb{R}^{n+1}$ of variables (x, k). The values of the set-valued mapping $\mathcal{N}(k) = \{ x \in \mathbb{R}^n \mid (x, k) \in \mathcal{N} \}$ will be called *cross-sections of the* $\mathcal{N}$. (Since the set $\mathcal{N}$ can be reconstructed from the set-valued mapping $\mathcal{N}(\cdot)$ and vice versa, we shall sometimes treat $\mathcal{N}$ and $\mathcal{N}(\cdot)$ as the same item.)

Let $\mathcal{M}$ be a non-empty closed *target set* satisfying the following assumptions:

- A_1 $\mathcal{M}(k_1) \subseteq \mathcal{M}(k_2)$, if $k_1 \leqslant k_2$;
- A_2 $\mathcal{M}(k) = \varnothing$ if $k < 0$;
- A_3 $\mathcal{M}(k)$ are compact.

The class of such mappings $\mathbb{R} \to \operatorname{conv} \mathbb{R}^n$ will be denoted as $\mathfrak{M}$. In some cases assumption A_3 will be replaced by a stronger one

- A_3' $\mathcal{M}$ is convex.

The corresponding class of mappings is $\mathfrak{M}'$.

Problem 2. For a given target set $\mathcal{M} \in \mathfrak{M}$ find the *solvability domain* $\mathcal{W}[t_0] \subseteq \mathbb{R}^{n+1}$ and a feedback strategy $\mathcal{U}(t, x, k) \in \mathcal{U}_{\mathrm{CL}}$, such that all the solutions to differential inclusion (5) which start at any position $(t, x(t), k(t))$, $t_0 \leqslant t \leqslant t_1$, $(x(t), k(t)) \in \mathcal{W}[t]$ end up in the target set: $x(t_1) \in \mathcal{M}(k(t_1))$. (Notation $\mathcal{W}[t] = \mathcal{W}(t; t_1, \mathcal{M})$ sometimes will be used to underline the dependence of the solvability domain on the target set and the terminal time. The cross-sections of $\mathcal{W}[t]$ will be denoted by $\mathcal{W}[k, t] = \mathcal{W}(k, t; t_1, \mathcal{M}(\cdot))$).

3. THE ALTERNATED INTEGRAL

Definition 3. The *max-min type solvability domain* $W^+(t; t_1, \mathcal{M})$ is the set of positions $(x, k) \in \mathbb{R}^{n+1}$, such that for every disturbance $v(\cdot)$ there exists an admissible open-loop control $u(\cdot) \in \mathcal{U}_{\mathrm{OL}}(k)$ ensuring $x(t_1) \in \mathcal{M}(k(t_1))$ whenever $x(t) = x$, $k(t) = k$.

Similarly, the *min-max type solvability domain* $W^-(t; t_1, \mathcal{M})$ is the set of positions for which there exists an admissible open-loop control which ensures the inclusion $x(t_1) \in \mathcal{M}(k(t_1))$ for all the disturbances $v(\cdot)$.

Lemma 4. The following inclusion holds:

$$W^-(t; t_1, \mathcal{M}) \subseteq \mathcal{W}(t; t_1, \mathcal{M}) \subseteq W^+(t; t_1, \mathcal{M}),$$

and the cross-sections of min-max and max-min solvability domains may be expressed as

$$W^+(k, t; t_1, \mathcal{M}(\cdot)) = \Bigg[\bigcup_{0 \leqslant \gamma \leqslant k} (\mathcal{M}(\gamma) - \\ - \mathcal{X}_{\mathrm{GI}}(t, t_1, k - \gamma)) \Bigg] \dot{-} \int_t^{t_1} \mathcal{Q}(\tau) \,\mathrm{d}\tau, \tag{6}$$

$$W^-(k,t;t_1,\mathcal{M}(\cdot)) = \bigcup_{0\leqslant\gamma\leqslant k}\Bigg[\Bigg(\mathcal{M}(\gamma)\,\dot{-}\,\dot{-}\int_t^{t_1}\mathcal{Q}(\tau)\,\mathrm{d}\tau\Bigg) - \mathcal{X}_{\mathrm{GI}}(t,t_1,k-\gamma)\Bigg], \quad (7)$$

where $\mathcal{X}_{\mathrm{GI}}$ denotes the reachability set under double constraint from the origin (Dar'in and Kurzhanskii, 2001):

$$\mathcal{X}_{\mathrm{GI}}(t,t_1,\Delta k) = \Bigg\{\int_t^{t_1} B(\tau)u(\tau)\,\mathrm{d}\tau \;\Bigg|\; \int_t^{t_1}\|u(\tau)\|^2_{R(\tau)}\,\mathrm{d}\tau \leqslant \Delta k,\, u(\tau)\in\mathcal{P}(\tau)\Bigg\},$$

and the sign "$\dot{-}$" denotes the geometric (Minkowski) difference of convex sets.

Note that the union in (6), (7) may be actually taken over the smaller interval $[0\vee k-\delta, k]$, where

$$\delta = \int_t^{t_1}\left|R^{\frac{1}{2}}(\tau)\mathcal{P}(\tau)\right|\,\mathrm{d}\tau = O(t_1-t).$$

Now define the *alternated sums* (6), (7). Let $\mathcal{T}$ be a partition of $[t,t_1]$: $t=\tau_0<\tau_1<\ldots<\tau_m=t_1$, $\sigma_i=\tau_i-\tau_{i-1}>0$. At time t_1 set

$$W^+_{\mathcal{T}}[k,\tau_m] = W^-_{\mathcal{T}}[k,\tau_m] = \mathcal{M}(k),$$

and then in each previous instant

$$W^+_{\mathcal{T}}[k,\tau_{i-1}] = W^+(k,\tau_{i-1};\tau_i,W^+_{\mathcal{T}}[\cdot,\tau_i]),$$
$$W^-_{\mathcal{T}}[k,\tau_{i-1}] = W^-(k,\tau_{i-1};\tau_i,W^-_{\mathcal{T}}[\cdot,\tau_i]).$$

The sets at time t,

$$W^+_{\mathcal{T}}[k,\tau_0] = \mathcal{I}^+_{\mathcal{T}}(k,t;t_1,\mathcal{M}(\cdot)) = \mathcal{I}^+_{\mathcal{T}}[k,t],$$
$$W^-_{\mathcal{T}}[k,\tau_0] = \mathcal{I}^-_{\mathcal{T}}(k,t;t_1,\mathcal{M}(\cdot)) = \mathcal{I}^-_{\mathcal{T}}[k,t]$$

are the *upper* and *lower alternated sums.*

In general, the sets (6), (7) (and therefore the alternated sums) may be not convex. To overcome this difficulty the following assumption is introduced:

- A_4 For each partition $\mathcal{T}$ the mappings $\mathcal{I}^+_{\mathcal{T}}[\cdot,t]$ and $\mathcal{I}^-_{\mathcal{T}}[\cdot,t]$ belong to class $\mathfrak{M}$.

This assumption allows to consider the alternated sums as mappings $\mathfrak{M}\to\mathfrak{M}$, which further enables to state the principle of optimality in form of the semigroup property for the solvability domain.

Note that if $\mathcal{M}$ is convex, i.e. it belongs to $\mathfrak{M}'$, then the assumption A_4 is always true and the alternated sums are mappings $\mathfrak{M}'\to\mathfrak{M}'$.

Definition 5. If for some $k\geqslant 0$ the following Hausdorff limit exists:

$$\lim_{\operatorname{diam}\mathcal{T}\to 0} h(\mathcal{I}^+_{\mathcal{T}}[k,t],\mathcal{I}^+[k,t]) = 0,$$

and does not depend on the sequence of partitions, then $\mathcal{I}^+[k,t] = \mathcal{I}^+(k,t;t_1,\mathcal{M}(\cdot))$ is called the *upper alternated integral.*

Definition 6. The *lower alternated integral* $\mathcal{I}^-[k,t] = \mathcal{I}^-(k,t;t_1,\mathcal{M}(\cdot))$ is defined similarly to the Definition 5 as the limit of lower alternated sums.

Lemma 7. The following inclusions are true:

$$\mathcal{I}^-[k,t]\subseteq\mathcal{W}[k,t]\subseteq\mathcal{I}^+[k,t],\quad t\in T,\quad k\geqslant 0.$$

Definition 8. If the upper and lower alternated integrals do coincide, then the set $\mathcal{I}[k,t] = \mathcal{I}^+[k,t] = \mathcal{I}^-[k,t]$ is said to be the *alternated integral.*

An important property of the constructed set-valued integrals is the validity of the *principle of optimality* (Bellman, 1957) expressed in the form of the *semigroup property.* To formulate the latter, it is necessary to consider the alternated integral as an operator on $\mathfrak{M}$, parameterized by initial and terminal times, t_0 and t_1.

Lemma 9. For the set of mappings $\mathcal{M}(\cdot)\to\mathcal{I}(\cdot,t;\tau,\mathcal{M}(\cdot))$ the semigroup property is true:

$$\mathcal{I}(k,t;t_1,\mathcal{M}(\cdot)) = \mathcal{I}(k,t;\tau,\mathcal{I}(\cdot,\tau;t_1,\mathcal{M}(\cdot))),$$

for $t\in T$, $\tau\in[t,t_1]$, $k\geqslant 0$. The same is also true for both upper and lower alternated integrals $\mathcal{I}^+$ and $\mathcal{I}^-$, in case they do not coincide.

Now consider the case when the target set $\mathcal{M}$ is convex. Then the open-loop solvability domains are also convex, and in space $\mathbb{R}^{n+1}$ they may be found due to the following formulae (Pontryagin, 1967; Pontryagin, 1980):

$$W^+(t;t_1,\mathcal{M}) = \left(\mathcal{M} - \int_t^{t_1}\mathcal{B}(\tau,\mathcal{P}(\tau))\,\mathrm{d}\tau\right)\dot{-}\,\dot{-}\int_t^{t_1}\mathcal{Q}(\tau)\,\mathrm{d}\tau,$$

$$W^-(t;t_1,\mathcal{M}) = \left(\mathcal{M}\,\dot{-}\int_t^{t_1}\mathcal{Q}(\tau)\,\mathrm{d}\tau\right) - -\int_t^{t_1}\mathcal{B}(\tau,\mathcal{P}(\tau))\,\mathrm{d}\tau,$$

$$\mathcal{B}(t,\mathcal{U}) = \left\{\begin{pmatrix} B(t)u \\ -\|u\|^2_{R(t)}\end{pmatrix}\;\middle|\; u\in\mathcal{U}\right\}.$$

The following assumptions are used to prove the convergence of alternated sums in this case (cf. Kurzhanski and Melnikov, 2000; Ponomarev and Rozov, 1978):

- A_5 There exist continuous positive functions $\varkappa(t)$ and $r(t)$, $t_0\leqslant t\leqslant t_1$, such that

$\mathcal{I}^+_{\mathcal{T}}[\varkappa(\tau_i), \tau_i] \supseteq \mathcal{B}_{r(\tau_i)}$ for any partition $\mathcal{T} = \{\tau_0, \ldots, \tau_m\}$ and $i = 0, \ldots, m$.

A_6 There exist continuous positive functions $\varkappa(t)$ and $r(t)$, $t_0 \leqslant t \leqslant t_1$, and a number $\varepsilon > 0$, such that $\mathcal{I}^-_{\mathcal{T}}[\varkappa(\tau_i), \tau_i] \supseteq \mathcal{B}_{r(\tau_i)}$ for any partition $\mathcal{T} = \{\tau_0, \ldots, \tau_m\}$ of diameter less than ε, $i = 0, \ldots, m$.

Theorem 10. Let $\mathcal{M}(\cdot) \in \mathfrak{M}'$. Denote

$$k_0^+(t) = \inf \{ k \mid \forall \mathcal{T} \quad \mathcal{I}^+_{\mathcal{T}}(k, t; t_1, \mathcal{M}(\cdot)) \neq \varnothing \},$$
$$k_0^-(t) = \inf \{ k \mid \exists \mathcal{T} \quad \mathcal{I}^-_{\mathcal{T}}(k, t; t_1, \mathcal{M}(\cdot)) \neq \varnothing \}.$$

Then

(1) under assumption A_5 for any $k \geqslant k_0^+(t)$ the upper alternated integral exists;
(2) under assumption A_6 for any $k \geqslant k_0^-(t)$ the lower alternated integral exists;
(3) under both assumptions $k_0^+(t) \equiv k_0^-(t) = k_0(t)$, and
  (a) when $k > k_0(t)$, upper and lower alternated integrals coincide, and are equal to the solvability domain:
  $$\mathcal{I}^+[k, t] = \mathcal{I}^-[k, t] = \mathcal{W}[k, t], \quad t \in T;$$
  (b) when $k < k_0(t)$, upper and lower alternated integrals are empty;
  (c) when $k = k_0(t)$, the following inclusion holds: $\mathcal{I}^-[k_0(t), t] \subseteq \mathcal{I}^+[k_0(t), t]$, $t \in T$;
(4) the upper alternated integral coincides with the solvability domain: $\mathcal{I}^+[k, t] = \mathcal{W}[k, t]$, $k \geqslant 0$, $t \in T$.

4. THE DYNAMIC PROGRAMMING EQUATION

The problem of guaranteed control synthesis (2) may be cast as an optimization one if the control is required to minimize the distance to the target set regardless of disturbance. This is formalized by introducing the following *value function*

$$V(t, x, k) = \inf_{\mathcal{U} \in \mathcal{U}_{\mathrm{CL}}} \sup_{z(\cdot) \in \mathcal{Z}} d(x(t_1), \mathcal{M}(k(t_1))), \quad (8)$$

where $z(t) = (x(t), k(t))$, $\mathcal{Z}$ is the assembly of solutions to the differential inclusion (5) with control $\mathcal{U}$.

Problem 11. Calculate the value function $V(t, x, k)$ and find a minimizing feedback strategy $\mathcal{U}(t, x, k) \in \mathcal{U}_{\mathrm{CL}}$ in (8).

It is clear that the solvability domain is the level set of the value function:

$$\mathcal{W}[k, t] = \{ x \in \mathbb{R}^n \mid V(t, x, k) \leqslant 0 \}.$$

The next result follows from the theory of minimax and viscosity generalized solutions to Hamilton–Jacobi equations (Subbotin, 1995; Fleming and Soner, 1993):

Theorem 12. The value function (8) is the minimax (viscosity) solution to the Hamilton–Jacobi–Bellman–Isaacs equation

$$\frac{\partial V}{\partial t} + \min_{u \in \mathcal{P}(t)} \max_{v \in \mathcal{Q}(t)} \left\{ \left\langle \frac{\partial V}{\partial x}, B(t)u + v \right\rangle - \frac{\partial V}{\partial k} \|u\|^2_{R(t)} \right\} = 0, \quad (9)$$

$t_0 \leqslant t < t_1$, $k > 0$, $x \in \mathbb{R}^n$, with boundary condition

$$\frac{\partial V}{\partial t} + \max_{v \in \mathcal{Q}(t)} \left\langle \frac{\partial V}{\partial x}, v \right\rangle \bigg|_{k=0} = 0, \quad (10)$$

$t_0 \leqslant t \leqslant t_1$, $x \in \mathbb{R}^n$, and initial condition

$$V(t_1, x, k) = d(x, \mathcal{M}(k)), \ k \geqslant 0, \ x \in \mathbb{R}^n. \quad (11)$$

The boundary condition (10) arises here in a natural way and does not cause the overdetermination of the system. It may be written in an explicit form:

$$V(t, x, 0) = \max_{v(\tau) \in \mathcal{Q}(\tau)} d\left(x + \int_t^{t_1} v(\tau) \, \mathrm{d}\tau, \mathcal{M}(0) \right).$$

In general case this becomes non-smooth when $t < t_1$, thus the value function would not be a classical solution to (9) even if $V(t_1, x, k)$ were smooth.

Theorem 13. Let $\mathcal{M}_\mu(k) = \mathcal{M}(k) + \mu \mathcal{B}_1$, where $\mathcal{B}_1$ is a unit ball in $\mathbb{R}^n$. Then

$$V(t, x, k) = \inf \{ \mu \geqslant 0 \mid x \in \mathcal{W}(k, t; t_1, \mathcal{M}_\mu(\cdot)) \}.$$

If the function function is calculated, the optimal feedback strategy is the minimizer in (9):

$$\mathcal{U}^*(t, x, k) = \operatorname*{Arg\,min}_{u \in \mathcal{P}(t)} \left\{ \left\langle \frac{\partial V}{\partial x}, B(t)u \right\rangle - \frac{\partial V}{\partial k} \|u\|^2_{R(t)} \right\}. \quad (12)$$

Since $\frac{\partial V}{\partial k} \leqslant 0$, the function under the minimum sign is convex, thus $\mathcal{U}^*(t, x, k)$ is a convex set. Continuity of the value function ensures that the mapping $\mathcal{U}^*(t, x, k)$ is upper semicontinuous in x and k. Therefore this feedback strategy guarantees existence of solutions to differential inclusion (5) on the interval $[t_0, t_1]$.

In general case calculating the value function exactly is computationally hard. However, the following upper bound may be useful:

Theorem 14. If the alternated integral exists, then the following estimate holds for those k and t where $\mathcal{I}[k, t] \neq \varnothing$:

$$V(t, x, k) \leqslant d(x, \mathcal{I}[k, t]). \quad (13)$$

5. THE EVOLUTION EQUATION

Definition 15. A set-valued mapping $(k,t) \to \mathcal{Z}[k,t]$ is called *weakly invariant*, if for $t_0 \leqslant t < t+\sigma \leqslant t_1$ the following inclusion holds:

$$\mathcal{Z}[k,t] \subseteq W^{+}(k,t;t+\sigma,\mathcal{Z}[\cdot,t+\sigma]) = \\ = \bigcup_{0\leqslant\gamma\leqslant k} (\mathcal{Z}[\gamma,t+\sigma] - \\ - \mathcal{X}_{\mathrm{GI}}(t,t+\sigma,k-\gamma)) \dot{-} \int_t^{t+\sigma} \mathcal{Q}(\tau)\,\mathrm{d}\tau. \quad (14)$$

Remark 16. The notion of weak invariance is in fact equivalent to the u-stability property in the theory of N. N. Krasovski (Krasovski, 1971; Krasovskii and Subbotin, 1988).

The solvability tube $\mathcal{W}[k,t]$ is weakly invariant, because its values are certainly contained in the max-min open-loop solvability set. This is also the maximal weakly invariant set-valued mapping with respect to inclusion satisfying $\mathcal{Z}[k,t_1] \subseteq \mathcal{M}(k)$.

Replacing in (14) the integral of $\mathcal{Q}(\tau)$ by the set $\sigma\mathcal{Q}(t)$ (with $O(\sigma^2)$ accuracy) and proceeding to the limit, the following *evolution equation* (Panasyuk and Panasyuk, 1980; Kurzhanski and Nikonov, 1993) for $\mathcal{Z}[k,t]$ is derived:

$$\lim_{\sigma\downarrow 0} \sigma^{-1} h_{+}\Bigg(\mathcal{Z}[k,t]+\sigma\mathcal{Q}(t), \bigcup_{0\leqslant\gamma\leqslant k} \mathcal{Z}[\gamma,t+\sigma] - \\ - \mathcal{X}_{\mathrm{GI}}(t,t+\sigma,k-\gamma)\Bigg) = 0. \quad (15)$$

The evolution equation (15) may be simplified. For sufficiently small values of σ the reachability set under double constraint $\mathcal{X}_{\mathrm{GI}}$ may be approximated by the intersection of reachability sets under single constraint (one for hard bound and one for soft bound).

Lemma 17. Let the following conditions hold:

(1) $0 \in \operatorname{int} B(t)\mathcal{P}(t)$;
(2) the support function $\rho(\,\ell \mid \mathcal{P}(t))$ and function $R(t)$ are Lipschitz continuous in t.

Then

$$h(\mathcal{X}_{\mathrm{GI}}(t,t+\sigma,\delta), \\ \mathcal{X}_{\mathrm{G}}(t,t+\sigma) \cap \mathcal{X}_{\mathrm{I}}(t,t+\sigma,\delta)) = O(\sigma^2), \quad (16)$$

where $\mathcal{X}_{\mathrm{G}}$ and $\mathcal{X}_{\mathrm{I}}$ are the reachability sets under hard and soft bound respectively:

$$\mathcal{X}_{\mathrm{G}}(t,t+\sigma) = \int_t^{t+\sigma} B(\tau)\mathcal{P}(\tau)\,\mathrm{d}\tau,$$

$$\mathcal{X}_{\mathrm{I}}(t,t+\sigma,\delta) = \mathcal{E}\left(0, \delta\int_t^{t+\sigma} R^{-1}(\tau)\,\mathrm{d}\tau\right).$$

By applying (16) one comes to the following statement.

Theorem 18. The evolution equation (15) is equivalent to

$$\lim_{\sigma\downarrow 0} \sigma^{-1} h_{+}\Bigg(\mathcal{Z}[k,t]+\sigma\mathcal{Q}(t), \bigcup_{0\leqslant\gamma\leqslant k} \mathcal{Z}[\gamma,t+\sigma] - \\ - \sigma B(t)\mathcal{P}(t) \cap \mathcal{E}\big(0,(k-\gamma)\sigma R^{-1}(t)\big)\Bigg) = 0.$$

6. THE SYNTHESIZING CONTROL

Theorem 19. Let $\mathcal{Z}[k,t]$ be a weakly invariant set-valued mapping with support function continuously differentiable in t and k. Then the function $G(t,x,k) = d^2(x,\mathcal{Z}[k,t])$ satisfies the following differential inequality:

$$\min_{u\in\mathcal{P}(t)} \max_{v\in\mathcal{Q}(t)} \frac{\mathrm{d}G}{\mathrm{d}t}(t,x(t),k(t)) \leqslant 0. \quad (17)$$

The latter means that the function $G(t,x,k)$ is the upper viscosity solution to (9).

Definition 20. The feedback strategy $\mathcal{U}_{\mathcal{Z}}$, which is the minimizer in (17), is called the *extremal strategy* for $\mathcal{Z}[k,t]$.

It consists of all the vectors $u^* \in \mathcal{P}(t)$ satisfying

$$\langle \ell_0, B(t)u^*\rangle + \|u^*\|^2_{R(t)}\rho_k(t,k,\ell_0) = \\ \min_{u\in\mathcal{P}(t)} \Big\{\langle \ell_0, B(t)u\rangle + \|u\|^2_{R(t)}\rho_k(t,k,\ell_0)\Big\}, \quad (18)$$

where $\rho(t,k,\ell) = \rho(\,\ell \mid \mathcal{Z}[k,t])$, and $\ell_0 = \ell_0(t,x,k)$ is the maximizer in

$$\langle \ell, x\rangle - \rho(\,\ell \mid \mathcal{Z}[k,t]) - \tfrac{1}{4}\|\ell\|^2 \to \max.$$

Theorem 21. Let $(x(t),k(t))$ be a solution to the differential inclusion (5) under control $\mathcal{U}_{\mathcal{Z}}$. If $x(t) \in \mathcal{Z}[k(t),t]$, then $x(\tau) \in \mathcal{Z}[k(\tau),\tau]$ for $t \leqslant \tau \leqslant t_1$.

When $\mathcal{Z}[k,t_1] \subseteq \mathcal{M}$, the corresponding extremal strategy $\mathcal{U}_{\mathcal{Z}}$ solves the problem 2 of guaranteed control synthesis. Thus choosing $\mathcal{Z}[k,t] = \mathcal{W}[k,t]$ one has the solution to problem 2, namely the feedback strategy $\mathcal{U}_{\mathcal{W}}$.

To construct the feedback strategy $\mathcal{U}_{\mathcal{W}}$ it is not necessary to calculate the value function itself, but only to find its zero level set. (Note that the optimal feedback strategy $\mathcal{U}^*$ in (12) is expressed in terms of the gradient of the value function, i.e. in also terms of its level sets.) Comparing (13) and (17) shows that $\mathcal{U}^*$ and $\mathcal{U}_{\mathcal{W}}$ both guarantee the same thing: if a trajectory is started on some distance from $\mathcal{W}[k,t]$, it will be on the same or smaller distance from $\mathcal{M}$ at the terminal time.

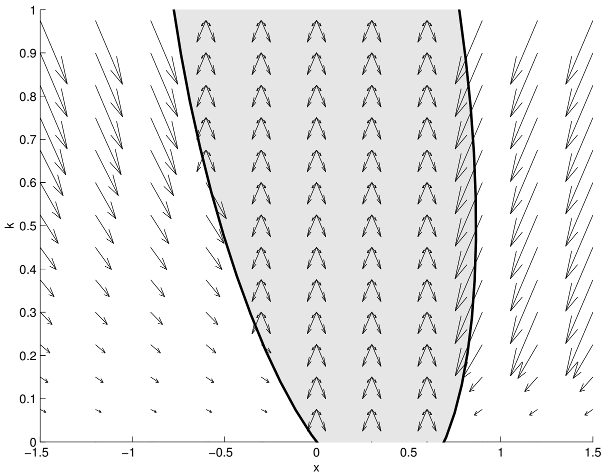

Fig. 1. An Example of Control Synthesis

Example 22. An example feedback strategy is depicted in figure 1. This feedback strategy is constructed for a problem with $x \in \mathbb{R}^1$. The gray-shaded area is the solvability domain at the start time, presented in the coordinates (x, k). The arrows indicate the directions of the vector field $(u, -\|u\|_R^2)$. The bundles of arrows inside the solvability domain mean that the control may choose any direction inside $\mathcal{P}(t)$.

7. CONCLUSION

The solution to the control synthesis problem is expressed in terms of the solvability tube, which describes the feasibility of the online position to be steered to the terminal target set. The solvability domain is described in terms of a set-valued integral which is a modification of L. S. Pontryagin's scheme to the case of double constraints. The dynamic programming equation here allows a viscosity solution, through which the required control strategy may be derived. Numerical schemes may be based on approximating the set-valued solutions to the evolution equation for the solvability tube.

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