A GENERALISED MINIMUM VARIANCE CONTROLLER FOR TIME-VARYING SYSTEMS

Zheng Li[1] **and Christian Schmid**[2]

[1]*School of Electrical, Computer and Telecomunication Engineering*
University of Wollongong
Wollongong NSW 2522, Australia
zheng@elec.uow.edu.au
[2]*Institute of Automation and Computer Control*
Ruhr-Universität Bochum
D-44801 Bochum, Germany
cs@atp.rub.de

Abstract: In this paper, the problem of generalised minimum variance control of linear time-varying systems is studied. A time-varying optimal controller is developed for a standard controlled autoregressive moving average model using a cost function that is the sum of the plant output tracking error variance plus a penalty term of the squared manipulated variable.

Keywords: Minimum variance control, Time-varying systems, Linear systems, Discrete-time systems, Adaptive control.

1. INTRODUCTION

The generalised minimum variance controller (GMVC) was developed by Clarke and Gawthrop (1975, 1979) for linear time-invariant (LTI) systems in order to remove the requirement of the stable plant inverse for the minimum variance controller (MVC) of Aström (1970). The LTI GMVC is very popular in stochastic adaptive control and has seen many applications. For adaptive control of rapidly time-varying systems the LTI GMVC was extended for linear time-varying (LTV) systems by Li and Evans (2002). This GMVC was based on a pseudocommutation technique that requires the LTV plants to satisfy pseudocommutability defined by a time-varying Sylvester matrix. This requirement can lead to difficulties when the LTV GMVC is applied to adaptive control, because even when a plant is actually pseudocommutable its estimate can be non-pseudo-commutable. The purpose of this paper is to remove the additional requirement for pseudocommutability and to extend the previous result of LTV MVC from Li (1997) for LTV GMVC such that the requirement of the stable plant inverse will be removed.

2. CONTROL OBJECTIVE

The standard LTV controlled autoregressive moving average (CARMA) model given by

$$\begin{aligned} &A(k,q^{-1})\,y(k+d) \\ &\qquad = B(k,q^{-1})u(k) + C(k,q^{-1})w(k+d) \end{aligned} \tag{1}$$

is considered, where $u(k)$ and $y(k)$ are the plant input and output, d>0 is the time delay between the input and output in terms of the discrete sampling time k, $w(k)$ is a zero mean, independent and possibly non-stationary Gaussian sequence with unknown and uniformly bounded variance. The LTV moving average operators (MAOs) are defined by

$$\begin{aligned} A(k,q^{-1}) &= 1 + a_1(k)q^{-1} + \ldots + a_n(k)q^{-n} \\ B(k,q^{-1}) &= b_0(k) + b_1(k)q^{-1} + \ldots + b_m(k)q^{-m}\,, \\ C(k,q^{-1}) &= 1 + c_1(k)q^{-1} + \ldots + c_h(k)q^{-h} \end{aligned} \tag{2}$$

where q^{-1} is the one-step-delay operator. It is assumed that

a) $C^{-1}(k,q^{-1})$ is exponentially stable and

b) the coefficients of $A(k,q^{-1})$, $B(k,q^{-1})$ and $C(k,q^{-1})$ are all uniformly bounded away from infinite, and $b_0(k)$ is uniformly bounded away from zero.

All these assumptions are natural extensions of the assumptions made by the LTI GMVC from LTI to LTV plants.

Given a uniformly bounded reference $s(k)$, the objective is to design a controller which minimises the conditional expectation of a generalised output tracking cost functional

$$J(k+d) = \tfrac{1}{2}E\left\{\left(y(k+d)-s(k)\right)^2 + \left(\lambda(k)u(k)\right)^2 \middle| D(k)\right\}, \tag{3}$$

where $D(k)=\{y(k), y(k\text{-}1), ..., u(k), u(k\text{-}1) , ...\}$ is the set of all the available input and output data up to and including time k for determination of the optimal control, and $\lambda(k)$ is a weighting function that is chosen to be uniformly bounded away from both zero and infinite. This weighting function is introduced to restrict the magnitude of the control variable $u(k)$ and, thus, remove the requirement of the stable plant inverse required for the MVCs.

3. GENERALISED MINIMUM VARIANCE CONTROL

Left dividing $C(k,q^{-1})$ by $A(k,q^{-1})$ one obtains

$$\begin{aligned} &A^{-1}(k,q^{-1})C(k,q^{-1}) \\ &\quad = F(k,q^{-1}) + A^{-1}(k,q^{-1})G(k,q^{-1})q^{-d}, \end{aligned} \tag{4}$$

where

$$F(k,q^{-1}) = 1 + f_1(k)q^{-1} + \ldots + f_{d-1}(k)q^{-d+1}. \tag{5}$$

Substituting (4) into the CARMA model (1) one obtains

$$\begin{aligned} &A(k,q^{-1})y(k+d) = B(k,q^{-1})u(k) \\ &\quad + A(k,q^{-1})F(k,q^{-1})w(k+d) + G(k,q^{-1})w(k). \end{aligned} \tag{6}$$

Thus

$$\begin{aligned} &y(k+d) - F(k,q^{-1})w(k+d) \\ &\quad = A^{-1}(k,q^{-1})\left[B(k,q^{-1})u(k) + G(k,q^{-1})w(k)\right], \end{aligned} \tag{7}$$

where the right-hand side depends on the input and output data up to and including time k only, and the left-hand side can be used as a prediction of the plant output.

GMVC Theorem

Consider the LTV CARMA model (1) where $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable. The GMVC law has the form

$$u(k) = T^{-1}(k,q^{-1})\left[A(k,q^{-1})s(k) - G(k,q^{-1})\hat{w}(k)\right] \tag{8a}$$

$$\begin{aligned} \hat{w}(k) &= C^{-1}(k-d,q^{-1}) \\ &\quad \left[A(k-d,q^{-1})y(k) - B(k-d,q^{-1})u(k-d)\right] \end{aligned} \tag{8b}$$

$$T(k,q^{-1}) = B(k,q^{-1}) + A(k,q^{-1})\frac{\lambda^2(k)}{b_0(k)}. \tag{8c}$$

Proof

Letting

$$\hat{y}(k+d) = y(k+d) - F(k,q^{-1})w(k+d). \tag{9}$$

Substitute it into the cost functional (3) one obtains

$$\begin{aligned} J(k+d) &= \left(\hat{y}(k+d)-s(k)\right)^2 + \left(\lambda(k)u(k)\right)^2 \\ &\quad + E\left\{\left(F(k,q^{-1})w(k+d)\right)^2 \middle| D(k)\right\}. \end{aligned} \tag{10}$$

From (1) and (9) one knows that

$$\frac{\partial \hat{y}(k+d)}{\partial u(k)} = \frac{\partial y(k+d)}{\partial u(k)} = b_0(k). \tag{11}$$

Thus

$$\frac{\partial J(k+d)}{\partial u(k)} = b_0(k)\hat{y}(k+d) - b_0(k)s(k) + \lambda^2(k)u(k) \tag{12}$$

and

$$\frac{\partial^2 J(k+d)}{\partial u^2(k)} = b_0^2(k) + \lambda^2(k) \tag{13}$$

follows. Equations (12) and (13) indicate that the control signal $u(k)$, which minimises the cost functional (3), must satisfy

$$\frac{\lambda^2(k)}{b_0(k)}u(k) = s(k) - \hat{y}(k+d). \tag{14}$$

Noting (7) and (9) and the exponential stability of $A^{-1}(k,q^{-1})$, one obtains

$$\begin{aligned} &\frac{\lambda^2(k)}{b_0(k)}u(k) = s(k) \\ &\quad - A^{-1}(k,q^{-1})[B(k,q^{-1})u(k) + G(k,q^{-1})w(k)]. \end{aligned} \tag{15}$$

Left multiplying $A(k,q^{-1})$ on both sides of the above equation, one obtains using (8c)

$$T(k,q^{-1})u(k) = A(k,q^{-1})s(k) - G(k,q^{-1})w(k) . \quad (16)$$

Subtracting (8b) from (1), one gets

$$C(k-d,q^{-1})[w(k) - \hat{w}(k)] = 0 . \quad (17)$$

Thus $\hat{w}(k)$ will exponentially converge to $w(k)$ because of the exponential stability of $C^{-1}(k,q^{-1})$. Replacing $w(k)$ in (16) using its estimate $\hat{w}(k)$ one obtains (8a). ■

The LTV GMVC calculates the control signal in two steps. First, the noise is estimated using (8b). Second, the control signal is obtained using (8a).

In order to derive the closed-loop equation for the LTV GMVC (9) is substituted into (14), which gives

$$\frac{\lambda^2(k)}{b_0(k)} u(k) = s(k) - y(k+d) + F(k,q^{-1})w(k+d) . \quad (18)$$

Left multiplying (18) using $A(k,q^{-1})$ it becomes

$$\begin{aligned} A(k,q^{-1})\frac{\lambda^2(k)}{b_0(k)} u(k) &= A(k,q^{-1})s(k) \\ - A(k,q^{-1})y(k+d) &+ A(k,q^{-1})F(k,q^{-1})w(k+d), \end{aligned} \quad (19)$$

and substituting (1) and (8c) into (19) one obtains

$$\begin{aligned} T(k,q^{-1})u(k) &= A(k,q^{-1})s(k) - C(k,q^{-1})w(k+d) \\ &+ A(k,q^{-1})F(k,q^{-1})w(k+d). \end{aligned} \quad (20)$$

Substituting (4) it becomes

$$T(k,q^{-1})u(k) = A(k,q^{-1})s(k) - G(k,q^{-1})w(k) . \quad (21)$$

Writing (1), (8b) and (21) in matrix form, one obtains the closed-loop system

$$\begin{bmatrix} 0 & 0 & T(k,q^{-1}) \\ 0 & A(k-d,q^{-1}) & -B(k-d,q^{-1})q^{-d} \\ C(k-d,q^{-1}) & -A(k-d,q^{-1}) & B(k-d,q^{-1})q^{-d} \end{bmatrix} \begin{bmatrix} \hat{w}(k) \\ y(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} -G(k,q^{-1}) & A(k,q^{-1}) \\ C(k-d,q^{-1}) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w(k) \\ s(k) \end{bmatrix}. \quad (22)$$

The left most matrix in the above equation is a lower triangular matrix with two of the inverses of its diagonal elements being exponentially stable. The closed-loop stability is therefore determined by the diagonal MAO $T(k,q^{-1})$ on the upright corner. If $B^{-1}(k,q^{-1})$ is exponentially stable then the weighting factor $\lambda(k)$ can be set to zero and the LTV GMVC will reduce to an LTV MVC (Li, 1997). If $B^{-1}(k,q^{-1})$ is unstable a proper $\lambda(k)$ has to be chosen such that $T^{-1}(k,q^{-1})$ is exponentially stable in order to ensure an exponentially stable closed-loop system.

4. SIMULATION RESULTS

The system to be controlled is a first-order system with a delay d of two sampling intervals as follows:

$$\begin{aligned} &y(k+2) + a(k)y(k+1) \\ &\quad = u(k) + b(k)u(k-1) + w(k+2) + c(k)w(k+1), \end{aligned} \quad (23)$$

where $w(k)$ is an independent, stationary and Gaussian noise with zero mean and unit variance. The plant parameters are as follows:

$$\begin{aligned} a(k) &= 0.7\left(1+\exp(-k-2)\right)(-1)^{k-1} \\ b(k) &= 1.1\left(1+\exp(-k-2)\right)(-1)^{k} \\ c(k) &= 0.9\frac{k+2}{k+3}(-1)^{k-1} \end{aligned} \quad (24)$$

For this plant $B^{-1}(k,q^{-1})$ is exponentially unstable and the plant cannot be controlled by the MVC (Li, 1997). It can be verified that $A^{-1}(k,q^{-1})$ and $C^{-1}(k,q^{-1})$ are exponentially stable. The choice of the weighting factor will effect the performance of the closed-loop system through (22) and (8c). In this example, $\lambda(k)$=0.3 in order to make $T^{-1}(k,q^{-1})$ exponentially stable and, consequently, the closed-loop system (22) exponentially stable. Fig. 1 shows that this LTV GMVC is able to stabilise the system and make it to follow the square wave reference signal $s(k)$.

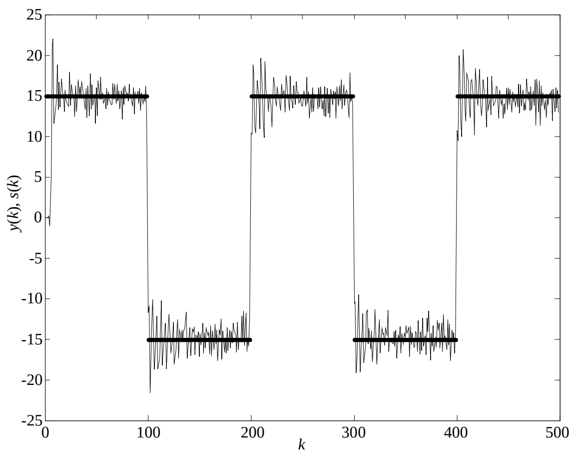

Fig. 1. GMVC simulation results with the controlled variable $y(k)$ and the square wave reference $s(k)$ with an amplitude of 15 and period of 200 samples.

In the above example a fast time-varying system is used in order to demonstrate the potential of our controller. A practical example of time-varying systems is a turbo-generator set (Rachev and Unbehauen, 1993).

5. CONCLUSION

An LTV GMVC has been developed for exponentially stable LTV plants described using a standard

LTV CARMA model. By introducing a time-varying weighting function into an MVC cost functional for the restriction of control magnitude it is shown that the condition of the stable plant inverse required for the LTV MVC (Li, 1997) can be removed. An advantage of this LTV GMVC is that it does not require the pseudocommutativity required by the previous LTV GMVC design approach.

REFERENCES

Aström, K.J. (1970). *Introduction to Stochastic Control Theory*. Academic Press, New York.

Clarke, D.W. and P.J. Gawthrop (1975). Self-tuning controller. *Proc. IEE Part D*, **122**, 929-934

Clarke, D.W. and P.J. Gawthrop (1979). Self-tuning control. *Proc. IEE Part D*, **126**, 633-640

Li, Z. (1997). A minimum variance controller for time-varying systems. *Proc. of the 36th IEEE Conference on Decision and Control*, San Diego, CA, pp. 4554-4555.

Li, Z. and R.J. Evans (2002). Generalised minimum variance control of linear time-varying systems. *IEE Proc.-Control Theory Appl.*, **149**, 111-116.

Rachev, V. and H. Unbehauen (1993). Identification of fast time-varying systems applied to a turbo-generator set. *Proc. of the 12th IFAC World Congress, Sydney, Australia*, ***6***, *187-192.*