ON LIAPUNOV-KRASOVSKII FUNCTIONALS UNDER CARATHEODORY CONDITIONS

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Abstract In this paper, a basic property that a Liapunov-Krasovskii functional must have when used for studying the stability of time delay systems under Carathéodory conditions is studied. This property concerns the equality, almost everywhere, of the upper right-hand Dini derivative of the Liapunov-Krasovskii functional evaluated at the solution, and of the Liapunov-Krasovskii derivative (computed by using everywhere the system dynamics) evaluated at the solution and at the forcing input. Sufficient conditions for that property are provided.

Keywords: Time Delay, Stability Analysis, Liapunov Function, Liapunov Stability, Stability Criteria, Liapunov Methods.

1. INTRODUCTION

In this paper, time invariant functional differential equations, forced by measurable, locally essentially bounded inputs, are considered (the parallel delayed case of system 5 in Angeli et. al., 2000). The importance in applications of the equations here considered is well known (see Burton, 1985, Hale and Lunel, 1993, Kolmanovskii and Myshkis, 1999, Niculescu, 2001, Gu et al., 2003). It is assumed that the function involved in the dynamics and the input are such that the Carathéodory conditions are verified. As well known, a functional differential equation, under Carathéodory conditions, admits an absolutely continuous solution which satisfies such an equation almost everywhere on a maximal right-open time interval (see Kolmanovskii and Myshkis, 1999, Section 2.4, pp. 100).

When using Liapunov-Krasovskii functionals for studying the stability of functional differential equations in this general case by the Liapunov's second method, the following two assumptions must be introduced:

A1) the Liapunov-Krasovskii functional, evaluated at the solution of the system, returns a locally absolutely continuous time function;

A2) the upper right-hand Dini derivative of such absolutely continuous time function is equal, almost everywhere, to the Liapunov-

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Krasovskii functional derivative (computed by using everywhere the functional differential equation, without knowing the solution) evaluated at the solution and at the forcing input.

The assumption A1 must be introduced because otherwise the non positivity, almost everywhere, of the upper right-hand Dini derivative of the time function obtained by evaluating the Liapunov-Krasovskii functional at the solution is not sufficient for such time function to be non-increasing (as proved by the famous counter example of the devil's staircase). The assumption A2 must be introduced in order to get the utility of the Liapunov's second method for practical use, which stands upon the fact that it does not require the knowledge of the solution.

As far as the assumption A1 is concerned, it can be taken care of in the choice of Liapunov-Krasovskii functionals by the facts that a locally Lipschitz function of an absolutely continuous function returns an absolutely continuous function, and that the integral of a continuous function (the initial conditions of the functional differential equation are assumed continuous) returns an absolutely continuous function. In the delayless case, since euclidean spaces are considered, the locally Lipschitz property of the Liapunov function is sufficient for the assumption A1 to hold.

As far as the assumption A2 is concerned, the Theorem 4.2.3, pp. 258 in (Burton, 1985) (due to Driver, 1962), states that, when the function describing the (time varying) functional differential equation is continuous and locally Lipschitz with respect to the state argument, and the Liapunov-Krasovskii functional is locally Lipschitz, then such assumption holds (moreover the equality claimed in A2 holds everywhere). Note that the hypothesis of continuity of the function describing the delay differential equation would involve, for the systems studied here, the hypothesis of continuity of the input, which in general is not possible to introduce when disturbances are considered.

On the basis of the Theorem 4.2.3 in (Burton, 1985), conditions for the assumption A2 to hold without assuming continuity of functions involved in the system equations are studied here. The Carathéodory conditions are instead assumed.

It is shown here that the locally Lipschitz condition of the Liapunov-Krasovskii functional, and a further condition, besides the Carathéodory and locally Lipschitz ones, on the function describing the system dynamics and on the input function are sufficient for A2 to hold.

Such further condition, which appears to be very simple and natural, is the following: almost everywhere, the Lebesgue mean of suitable functions of the input converges to zero as the interval of integration decreases to zero.

Moreover it is proved that: if for any function describing the dynamics of the system, any input function and any initial state, the upper right-hand Dini derivative at zero of the Liapunov-Krasovskii functional evaluated at the solution of the system is given by a function (which is defined uniquely by the functional) of the initial state and of the right-hand Dini derivative at zero of the solution, then no further condition, besides Carathéodory ones, is needed for the system in order to have the assumption A2 to hold.

The theorems here developed are useful for studying the Input-to-State Stability (ISS) of time delay systems (see Teel, 1998) on a Liapunov-Krasovskii framework. The Carathéodory environment here proposed is common, for instance, when considering the ISS with respect to measurable, locally essentially bounded disturbances.

2. PRELIMINARIES

Let us consider the following nonlinear time-delay system

$$\begin{aligned} &\dot{x}(t) = f(x_t, u(t)), \qquad t \geq 0, \qquad a.e., \\ &x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ is the input function, n, m are positive integers, for $t \geq 0$ $x_t : [-\Delta, 0] \to R^n$ is the standard function (see Section 2.1, pp. 38 in Hale and Lunel, 1993) given by $x_t(\tau) = x(t+\tau)$, Δ is the maximum involved delay, f is a function from $C([-\Delta, 0]; R^n) \times R^m$ to R^n, $C([-\Delta, 0], R^n)$ denotes the set of the continuous functions which are defined on $[-\Delta, 0]$ and take values in R^n, $\xi_0 \in C([-\Delta, 0]; R^n)$. Without loss of generality we suppose that $f(0,0) = 0$, thus ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(x_t, 0)$ with zero initial conditions. Multiple discrete non commensurate as well as distributed delays can appear in (1).

The symbol $|\cdot|$ stands for the Euclidean norm of a real vector, or the induced Euclidean norm of a matrix. A function u is said to be *essentially bounded* if $ess\sup_{t\geq 0} |u(t)| < \infty$. We indicate the essential supremum norm of an essentially bounded function with the symbol $\|\cdot\|_\infty$. For given times $0 \leq T_1 < T_2$, we indicate with $u_{[T_1, T_2)} : [0, +\infty) \to R^m$ the function given by $u_{[T_1,T_2)}(t) = u(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input u is said to be *locally essentially bounded* if, for any $T > 0$, $u_{[0,T)}$ is essentially bounded. A function $w : [0, b) \to R$, $0 < b \leq +\infty$, is said to be locally absolutely continuous if it is absolutely continuous in any interval $[0, c]$, $0 < c < b$. $L_2([-\Delta, 0]; R^n)$ is the space of square Lebesgue integrable functions from $[-\Delta, 0]$ to R^n.

The following standard hypothesis (see Hale and Lunel, 1993, Kolmanovskii and Myshkis, 1999) is assumed throughout the paper:

Hp_0) The function $f : C([-\Delta, 0]; R^n) \times R^m \to R^n$ and the input function $u : R^+ \to R^m$ are such that the function

$$g : C([-\Delta, 0]; R^n) \times R^+ \to R^n \tag{2}$$

given, for $(\phi, t) \in C([-\Delta, 0]; R^n) \times R^+$, by $g(\phi, t) = f(\phi, u(t))$, is bounded on any bounded set $U \subset C([-\Delta, 0]; R^n) \times R^+$ (the set $C([-\Delta, 0]; R^n)$ being endowed with the supremum norm), and satisfies the Carathéodory conditions in $C([-\Delta, 0]; R^n) \times R^+$. ■

Remark 1. As is well known, from the hypothesis Hp_0 it follows that the system (1) admits a unique solution on a maximal interval $[0, b)$, $0 < b \leq +\infty$, which is (componentwise) locally absolutely continuous and, if $b < +\infty$, is unbounded in $[0, b)$ (see Section 2.6, pp. 58 in Hale and Lunel, 1993, and Sections 2.2 and 2.4, pp. 96, 100 in Kolmanovskii and Myshkis, 1999). ■

In the following, the continuity of a functional $V : C([-\Delta, 0]; R^n) \to R^+$ is intended with respect to the supremum norm. Given a continuous functional $V : C([-\Delta, 0]; R^n) \to R^+$, the derivative D^+V of the functional V is defined by (see Burton, 1985, Definition 4.2.4, pp. 258)

$$D^+V(\phi, v) = \limsup_{h \to 0^+} \frac{1}{h} \left(V(\phi_h^\star) - V(\phi) \right), \tag{3}$$

where $\phi_h^\star \in C([-\Delta, 0]; R^n)$ is given by

$$\phi_h^\star(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h], \\ \phi(0) + f(\phi, v)(h+s), & s \in (-h, 0], \end{cases} \tag{4}$$

The functional D^+V is generalized because it can take infinite values (see Kolmanovskii and Myshkis, 1999, pp. 205), it is computed by using the system equations without knowing the solution.

3. MAIN RESULTS

Theorem 2. *Let the function u in (1) be measurable and locally essentially bounded. Let $V : C([-\Delta, 0]; R^n) \to R^+$ be a continuous functional. Let the function f and the function u in system (1) and the functional V satisfy the following hypotheses:*

HS_1) $\forall \psi \in C([-\Delta, 0]; R^n)$, there exist a neighborhood of ψ and a locally bounded function $K_\psi : R^m \to R^+$, such that, $\forall \phi_1, \phi_2$ in that neighborhood, $\forall v \in R^m$, the inequality holds

$$|f(\phi_1, v) - f(\phi_2, v)| \leq K_\psi(v) \|\phi_1 - \phi_2\|_\infty \ ;$$

HS_2) there exists a function

$$P : R^m \times R^m \to R^+ \tag{5}$$

such that:

i) $\forall \psi \in C([-\Delta, 0]; R^n)$, there exists a positive real M_ψ such that, $\forall v_1, v_2 \in R^m$, the inequality holds

$$|f(\psi, v_1) - f(\psi, v_2)| \leq M_\psi P(v_1, v_2); \tag{6}$$

ii) the input function u satisfies, for almost all $t \in R^+$,

$$\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} P\left(u(s), u(t)\right) ds = 0; \tag{7}$$

HS_3) $\forall \psi \in C([-\Delta, 0]$, there exist a neighborhood of ψ and a function L_ψ, such that, $\forall \phi_1, \phi_2$ in that neighborhood, the inequality holds

$$|V(\phi_1) - V(\phi_2)| \leq L_\psi \|\phi_1 - \phi_2\|_\infty \ . \tag{8}$$

Let $x(t)$ be the locally absolutely continuous solution of system (1) in a maximal interval $[0, b)$. Let $w : R^+ \to R$ be the function defined by $w(t) = V(x_t)$. Let D^+w be the upper right-hand Dini derivative of w, that is

$$D^+w(t) = \limsup_{h \to 0^+} \frac{w(t+h) - w(t)}{h} \ . \tag{9}$$

Then, almost everywhere,

$$D^+w(t) = D^+V(x_t, u(t)), \qquad t \in [0, b). \tag{10}$$

Proof. Let x_t be the function obtained from the solution $x(t)$ of (1) as usual. Let $t^\star \in [0, b)$. Let us consider the following system with unknown variable $\hat{x}$

$$\begin{aligned} \dot{\hat{x}}(t) &= f(x_{t^\star}, u(t^\star)), \qquad t \geq t^\star, \\ \hat{x}_{t^\star}(\tau) &= x_{t^\star}(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{11}$$

Let $\hat{x}(t)$, $t \geq t^\star$, be the solution of system (11) and let $\hat{x}_t$ be the function obtained from $\hat{x}(t)$ as usual. From (3) (4) it follows that

$$D^+V(x_{t^\star}, u(t^\star)) = \limsup_{h \to 0^+} \frac{V(\hat{x}_{t^\star+h}) - V(x_{t^\star})}{h} \tag{12}$$

From (12), from (*9*) i.e. from

$$D^+w(t^\star) = \limsup_{h \to 0^+} \frac{V(x_{t^\star+h}) - V(x_t)}{h} \tag{13}$$

and from the equalities

$$\begin{aligned} &\frac{V(x_{t^\star+h}) - V(x_t)}{h} = \\ &\qquad \frac{V(x_{t^\star+h}) - V(\hat{x}_{t^\star+h})}{h} + \\ &\qquad + \frac{V(\hat{x}_{t^\star+h}) - V(x_{t^\star})}{h}, \end{aligned} \tag{14}$$

it follows right by the definition of *limsup* that, in order to prove the theorem, it is sufficient to prove that, for almost all $t^\star \in [0, b)$,

$$\limsup_{h \to 0^+} \frac{|V(x_{t^\star+h}) - V(\hat{x}_{t^\star+h})|}{h} = 0. \tag{15}$$

From the hypotheses HS_1, HS_2, HS_3, since the functions $t \to x_t$, $t \to \hat{x}_t$ are continuous (see Lemma 2.1, pp. 40, in Hale and Lunel, 1993), it follows that, for sufficiently small $h < \Delta$,

$$\begin{aligned} &|V(x_{t^\star+h}) - V(\hat{x}_{t^\star+h})| \leq \\ &\leq L_{x_{t^\star}} \|x_{t^\star+h} - \hat{x}_{t^\star+h}\|_\infty \leq \\ &\leq L_{x_{t^\star}} \sup_{\theta \in [-\Delta, 0]} |x_{t^\star+h}(\theta) - \hat{x}_{t^\star+h}(\theta)| = \\ &= L_{x_{t^\star}} \sup_{\theta \in [-h, 0]} \Big| x_{t^\star+h}(-h) + \\ &+ \int_{t^\star}^{t^\star-\theta} f(x_s, u(s)) ds - x_{t^\star+h}(-h) - \\ &- \int_{t^\star}^{t^\star-\theta} f(x_{t^\star}, u(t^\star)) ds \Big| \leq \\ &\leq L_{x_{t^\star}} \sup_{\theta \in [-h, 0]} \int_{t^\star}^{t^\star-\theta} |f(x_s, u(s)) - \\ &\qquad - f(x_{t^\star}, u(t^\star))| ds \leq \\ &\leq L_{x_{t^\star}} \int_{t^\star}^{t^\star+h} |f(x_s, u(s)) - f(x_{t^\star}, u(s)) + \\ &\qquad + f(x_{t^\star}, u(s)) - f(x_{t^\star}, u(t^\star))| ds \leq \\ &\leq L_{x_{t^\star}} \int_{t^\star}^{t^\star+h} K_{x_{t^\star}}(u(s)) \|x_s - x_{t^\star}\|_\infty ds + \\ &+ L_{x_{t^\star}} \int_{t^\star}^{t^\star+h} M_{x_{t^\star}} P(u(s), u(t^\star)) ds \end{aligned} \tag{16}$$

Since the function u is locally essentially bounded, the function $K_{x_{t^\star}}$ is locally bounded, the function $t \to x_t$ is continuous, it follows that

$$\limsup_{h \to 0^+} \frac{1}{h} \int_{t^\star}^{t^\star+h} K_{x_{t^\star}}(u(s)) \|x_s - x_{t^\star}\|_\infty ds = 0. \tag{17}$$

From the hypothesis HS_2 it follows that, for almost all $t^\star \in [0, b)$,

$$\limsup_{h \to 0^+} \frac{1}{h} \int_{t^\star}^{t^\star+h} M_{x_{t^\star}} P(u(s), u(t^\star)) ds = 0, \tag{18}$$

and the theorem is proved. ■

Remark 3. As far as the hypothesis HS_2 is concerned, note that, if

$$P(v_1, v_2) = |v_1 - v_2|, \tag{19}$$

then (*7*) becomes

$$\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |u(s) - u(t)| ds = 0, \qquad a.e. \tag{20}$$

■

Theorem 4. *Let $V : C([-\Delta, 0]; R^n) \to R^+$ be a continuous functional. Let there exist a functional $D_V : C([-\Delta, 0]; R^n) \times R^n \to R$ such that, for any function f, input u and initial condition ξ_0 in (1), the following proposition holds true:* if the solution $x(t)$ of system (1) admits finite right-hand derivative at zero, then the following equality holds

$$\begin{aligned} &\limsup_{h \to 0^+} \frac{V(x_h) - V(\xi_0)}{h} = \\ &\qquad D_V\left(\xi_0, \lim_{h \to 0^+} \frac{x(h) - x(0)}{h}\right). \end{aligned} \tag{21}$$

Let $w : R^+ \to R$ be the function defined by $w(t) = V(x_t)$. Let D^+w be the upper right-hand Dini derivative of w (see (9)).

Then, almost everywhere,

$$D^+w(t) = D^+V(x_t, u(t)), \qquad t \in [0, b). \tag{22}$$

Proof. Let $\phi \in C([-\Delta, 0]; R^n)$, $v \in R^m$. Let us consider the following system with unknown variable $\hat{x}$

$$\begin{aligned} \dot{\hat{x}}(t) &= f(\phi, v), \qquad t \geq 0, \\ \hat{x}(\tau) &= \phi(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{23}$$

Let $\hat{x}(t)$, $t \geq 0$, be the solution of system (23) and let $\hat{x}_t$ be the function obtained from $\hat{x}(t)$ as usual. From (3) (4) it follows that

$$D^+V(\phi, v) = \limsup_{h \to 0^+} \frac{V(\hat{x}_h) - V(\phi)}{h} \qquad (24)$$

Since the equality (21) holds for any f, u, ξ_0, from (24) and (21) the equalities follow

$$D^+V(\phi, v) = D_V\left(\phi, \lim_{h \to 0^+} \frac{\hat{x}(h) - \hat{x}(0)}{h}\right) = D_V(\phi, f(\phi, v)) \qquad (25)$$

Since the system (1) and the functional V are time invariant, the equality (21) holds for any f, u, ξ_0, and the solution $x(\cdot)$ admits almost everywhere in $[0, b)$ finite right-hand derivative, the equality

$$D^+w(t) = D_V\left(x_t, \lim_{h \to 0^+} \frac{x(t+h) - x(t)}{h}\right) \qquad (26)$$

holds almost everywhere in $[0, b)$. Since, for almost all $t \in [0, b)$,

$$\lim_{h \to 0^+} \frac{x(t+h) - x(t)}{h} = f(x_t, u(t)) \qquad (27)$$

it follows that, for almost all $t \in [0, b)$,

$$D^+w(t) = D_V\left(x_t, f(x_t, u(t))\right) = D^+V(x_t, u(t)) \qquad (28)$$

■

Remark 5. In order to check the hypotheses of Theorem 4 it is not necessary to check whether the solution admits finite right-hand derivative at zero. For instance, the well known Liapunov-Krasovskii functional

$$V(\phi) = \phi^T(0)Q\phi(0) + \int_{-\Delta}^{0} \phi^T(\tau)S\phi(\tau), \qquad (29)$$

with Q, S symmetric positive matrices, satisfies the hypotheses of Theorem 4. In this case

$$\begin{aligned} D_V\left(\xi_0, \lim_{h \to 0^+} \frac{x(h) - x(0)}{h}\right) = \\ \xi_0^T(0)S\xi_0(0) - \xi_0^T(-\Delta)S\xi_0(-\Delta) + \\ 2\xi_0^T(0)Q \lim_{h \to 0^+} \frac{x(h) - x(0)}{h} \end{aligned} \qquad (30)$$

■

Remark 6. As far as the delayless case

$$\dot{x}(t) = f(x(t), u(t)), \qquad t \geq 0, \qquad a.e. \qquad (31)$$

is concerned, when f is continuous and the input u is continuous, then the Theorem 4.3, App. I, in (Rouche et al., 1977) (due to Yoshizawa, 1966), states that locally Lipschitz Liapunov functions yield that the assumption A2 (in a form equivalent for the delayless case) holds. In the case that only Carathéodory conditions are assumed, results parallel to the ones shown here for the delayed case hold true. If locally Lipschitz Liapunov functions are used, then the assumptions $A1$ and $A2$ hold true, provided that the hypotheses HS_1, HS_2 (rewritten substituting $C([-\Delta, 0]; R^n)$ with R^n) are satisfied. However, when the Liapunov functions are chosen continuously differentiable (see Sontag 1989, Lin et al., 1996, Sontag, 1998, Sontag, 2000, Angeli et al., 2000), no further hypothesis, besides Hp_0 (rewritten in the equivalent form for the delayless case), is needed for system (31), in order to have $A1$ and $A2$ to hold. ■

4. CONCLUSIONS

In this paper, a basic property that a Liapunov-Krasovskii functional must have when used for studying the stability of time delay systems under Carathéodory conditions is studied. Time invariant time delay systems forced by measurable, locally essentially bounded inputs are considered. This property consists of the equality, almost everywhere, between the upper right-hand Dini derivative of the time function obtained by evaluating the Liapunov-Krasovskii functional at the solution, and of the Liapunov-Krasovskii derivative (computed by using everywhere the system dynamics), evaluated at the solution and at the input. Sufficient conditions on the Liapunov-Krasovskii functional, on the function involved in the system dynamics and on the input are given such that this property holds.

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