A SWITCHING SCHEME FOR THE ROBUST STABILIZATION OF DISCRETE TIME SYSTEMS WITH UNMATCHED UNCERTAINTIES

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Abstract: In this paper robust control of discrete time linear systems in presence of unmatched uncertainties is considered. Discrete time Variable Structure Control is used, and the presence of the sector, always affecting this control approach, is avoided by means of Switching Control. This latter is based on a suitable partition of the parameter space, while each single controller is based on Discrete time Variable Structure Control. It is worth noticing that the proposed approach does not depend on the duration of the sampling interval, hence it works both for sampled data system and for "pure" discrete time systems

Keywords: Discrete Time Sliding Mode Control, Switching Control, Robust Control, Discrete Time systems.

1. INTRODUCTION

Continuous time Variable Structure Control (VSC), or Sliding Mode (SM) control, has been extensively studied since the early 1950's (Utkin, 1992) (De Carlo *et al.*, 1988) (Edwards and Spurgeon, 1998). As well known, the most significant aspect of VSC is its robustness to parameter variations and external disturbances. However, implementation problems, always present in real applications of VSC, can lead to unacceptable performances.

An answer to these problems can be the use of Discrete time Variable Structure Control (DVSC), as recognized in (Young *et al.*, 1999). A number of approaches have been presented in the last years, based on DVSC (Furuta, 1990) (Furuta, 1993) (Gao *et al.*, 1995) (Corradini and Orlando, 1997*a*) (Corradini and Orlando, 1997*b*) (Bartoszewicz, 1998) (Emelyanov *et al.*, 1995) (Guo and Zhang, 2002) (Yu and Chen, 2003). However, the main hindrance of DVSC is that a perfect invariance to uncertainties cannot be achieved as in the ideal continuous-time sliding mode: in fact, when uncertainties are present, a "quasi"-sliding mode can be achieved outside of a region (sector) whose width can depend on parameter variations or disturbances (Gao *et al.*, 1995). Inside the sector the discrete time sliding mode can be imposed only approximately: in (Furuta, 1993) adaptive control is used; in (Corradini and Orlando, 1997*b*) uncertainties are approximated by using the concept of Time Delay Control; in (Bartoszewicz, 1998) a reduction of the sector is obtained with respect to (Gao *et al.*, 1995). In (Emelyanov *et al.*, 1995) the presence of the sector is avoided for sampled linear systems, provided that short sampling intervals are considered.

In this paper, the approach described in (Emelyanov *et al.*, 1995) is modified and coupled with switching control, in order to guarantee robust stability in the presence of unmatched parameter variations and nonlinear disturbances. It is worth noticing that the proposed control law does not depend on the sampling interval duration, i.e. it works also for inherently discrete time systems, and ensures the asymptotic stability of the considered plant with no approximation and without sector. As in (Emelyanov *et al.*, 1995), an asymptotically stable subspace of the state space is made invariant by the control law, which guarantees also that the trajectories starting outside the invariant subspace are asymptotically stable. The proposed DVSC controller has been inserted in a switching control architecture, in order to be able to impose the conditions ensuring the existence of the invariant subspace regardless of the uncertainties. The parameter space is partitioned in different hyperspheres, centered in different nominal values. A supervisor switches among a number of fixed controllers, one for each hypersphere.

The paper is organized as follows. In Section 2 some preliminaries and the problem statement are reported. The robust control law for each hypersphere of the partition is presented in Section 3, while Section 4 contains the switching logic. Simulation results have not been reported for the sake of brevity.

2. PROBLEM STATEMENT

An uncertain discrete time single input linear system is given:

$$\mathbf{x}(k+1) = \hat{\mathbf{A}}(\mathbf{p})\mathbf{x}(k) + \hat{\mathbf{B}}(\mathbf{p})u(k) + \hat{\mathbf{z}}(\mathbf{x}(k), \mathbf{p}, k) \tag{1}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}$ is the input vector, $\hat{\mathbf{B}}(\mathbf{p}) : \mathbb{R}^l \to \mathbb{R}^n$ is the input to state vector, $\hat{\mathbf{A}}(\mathbf{p}) : \mathbb{R}^l \to \mathbb{R}^{n\times n}$ is the state matrix, and $\hat{\mathbf{z}}(\mathbf{x}(k), \mathbf{p}, k) : \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{Z} \to \mathbb{R}$ represents nonlinear uncertainties of the system. The l-vector of uncertain parameters $\mathbf{p} = [p_1, \dots, p_l]^T$ takes values in the Q-box $\subset \mathbb{R}^l$, defined by $P = \{\mathbf{p} : p_i \in [p_i^-, p_i^+], \ i = 1, \dots, l\}$, p_i^-, p_i^+ being known bounds. It is assumed that:

Assumption 2.1. The linear system $\left(\hat{\mathbf{A}}(\mathbf{p}), \hat{\mathbf{B}}(\mathbf{p})\right)$ is controllable $\forall \mathbf{p} \in P$.

Partition the parameter space P in ν hyperspheres $P^{(j)}$, $j \in \mathbb{S} \overset{\text{def}}{=} \{1 \dots \nu\}$, with ν arbitrary finite integer. Each hypersphere $P^{(j)}$ is centered in a nominal value $\mathbf{p}^{(j)}$. Defining $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}^{(j)}$, the j-th hypersphere is given by:

$$P^{(j)} = \{\mathbf{p} \in P : ||\Delta\mathbf{p}|| \le r^{(j)}, \ \ r^{(j)} > 0\}, \ j \in \mathbb{S}. \tag{2}$$

Parameters vector $\mathbf{p}^{(j)}$ and scalars $r^{(j)}$ are such that the ν hyperspheres $P^{(j)}$ constitute a complete covering of P. Moreover, as will be explained in the following, suitable values of $r^{(j)}$ and ν will be chosen in order to satisfy conditions necessary for synthesizing the control law.

As in (Emelyanov *et al.*, 1995), for each $\mathbf{p}^{(j)} \in P$, a change of coordinates $\mathbf{x}(k) = \mathbf{M}^{(j)}\mathbf{s}(k)$ exists such that system (1) is transformed as follows:

$$\begin{cases} \mathbf{s_1}(k+1) = \left[\bar{\mathbf{A}}_{11}^{(j)} + \mathbf{G}_{11}^{(j)}(\Delta\mathbf{p})\right]\mathbf{s_1}(k) + \left[\bar{\mathbf{A}}_{12}^{(j)} + \right. \\ \left. + \mathbf{G}_{12}^{(j)}(\Delta\mathbf{p})\right] s_2(k) + \bar{\mathbf{z}}_1(\mathbf{s}(k), \mathbf{p}, k) \\ s_2(k+1) = \left[\bar{\mathbf{A}}_{21}^{(j)} + \mathbf{G}_{21}^{(j)}(\Delta\mathbf{p})\right]\mathbf{s_1}(k) + \left[\bar{A}_{22}^{(j)} + \right. \\ \left. + G_{22}^{(j)}(\Delta\mathbf{p})\right] s_2(k) + \bar{d}^{(j)}(\mathbf{p})u(k) + \bar{z}_2(\mathbf{s}(k), \mathbf{p}, k) \end{cases} \tag{3}$$

where:

- $\mathbf{s}(k) \overset{\text{def}}{=} [\mathbf{s_1}(k)^T s_2(k)]^T$, $\mathbf{s_1}(k) \in \mathbb{R}^{n-1}$, $s_2(k) \in \mathbb{R}$;
- $\bar{\mathbf{A}}_{11}^{(j)}$, $\bar{\mathbf{A}}_{12}^{(j)}$, $\bar{\mathbf{A}}_{21}^{(j)}$, $\bar{A}_{22}^{(j)}$ are blocks of $\left[\mathbf{M}^{(j)}\right]^{-1}$ $\hat{\mathbf{A}}(\mathbf{p}^{(j)})\ \mathbf{M}^{(j)}$;
- $\left[\mathbf{0}^T \bar{d}^{(j)}(\mathbf{p})\right]^T = \left[\mathbf{M}^{(j)}\right]^{-1}\ \hat{\mathbf{B}}(\mathbf{p}^{(j)})$, $\bar{d}^{(j)}(\mathbf{p}) \in \mathbb{R}$;
- $\mathbf{G}_{11}^{(j)}(\Delta\mathbf{p})$, $\mathbf{G}_{12}^{(j)}(\Delta\mathbf{p})$ are the unmatched linear uncertainties while $\mathbf{G}_{21}^{(j)}(\Delta\mathbf{p})$, $G_{22}^{(j)}(\Delta\mathbf{p})$ are the matched linear uncertainties ;
- $\bar{z}_2(\mathbf{s}(k), \mathbf{p}, k)$, $\bar{\mathbf{z}}_1(\mathbf{s}(k), \mathbf{p}, k)$ are the nonlinear matched and unmatched uncertainties, respectively.

The following assumptions hold $\forall \mathbf{p} \in P^{(j)}$, $j \in \mathbb{S}$:

Assumption 2.2. Variable $\bar{d}^{(j)}(\mathbf{p})$ is given by: $\bar{d}^{(j)}(\mathbf{p}) = d^{(j)}\left[1 + h^{(j)}(\Delta\mathbf{p})\right]$, $d^{(j)} \overset{\text{def}}{=} \bar{d}(\mathbf{p}^{(j)})$ being the nominal value and $h^{(j)}(\Delta\mathbf{p})$ satisfying:

$$\lim_{\mathbf{p}\to\mathbf{p}^{(j)}} h^{(j)}(\Delta\mathbf{p}) = h^{(j)}(\mathbf{0}) = 0, \ \ 1 + h^{(j)}(\Delta\mathbf{p}) \neq 0. \tag{4}$$

Without loss of generality, it can be assumed: $1 + h^{(j)}(\Delta\mathbf{p}) > 0 \ \ \forall \mathbf{p} \in P^{(j)}$.

Assumption 2.3. Uncertainties $\mathbf{G}_{11}^{(j)}(\Delta\mathbf{p})$, $\mathbf{G}_{12}^{(j)}(\Delta\mathbf{p})$, $\mathbf{G}_{21}^{(j)}(\Delta\mathbf{p})$, $G_{22}^{(j)}(\Delta\mathbf{p})$ are continuous functions such that:

$$\lim_{\mathbf{p}\to\mathbf{p}^{(j)}} ||\mathbf{G}_{kl}^{(j)}(\boldsymbol{\Delta}\mathbf{p})|| = ||\mathbf{G}_{kl}^{(j)}(\mathbf{0})|| = 0, \ k, l = 1, 2 \tag{5}$$

The problem considered is to find a switching control law stabilizing plant (3) (hence, plant (1)) $\forall \mathbf{p} \in P$. The control scheme is composed of: i) a family of ν controllers, corresponding to the partition of P in ν hyperspheres, such that the j-th controller can stabilize plant (3) $\forall \mathbf{p} \in P^{(j)}$; ii) a suitable switching logic able to choose the right controller, i.e. to identify the hypersphere the true parameter $\mathbf{p}$ belongs to. Hence, the above problem can be formalized as follows. For system (3), in each hypersphere $P^{(j)}$, $j \in \mathbb{S}$, the control law $u^{(j)}(\mathbf{s}(k))$ is introduced (Emelyanov *et al.*, 1995):

$$u(k) = u^{(j)}(\mathbf{s}(k)) \stackrel{\text{def}}{=} -\frac{1}{d^{(j)}}\mathbf{K}^{(j)}\mathbf{s}(k) + \bar{u}^{(j)}(k) \quad (6)$$

where $\bar{u}^{(j)}(k)$ will be determined in the following, and $\mathbf{K}^{(j)}$ is given by:

$$\mathbf{K}^{(j)} = \left[\ \bar{\mathbf{A}}_{21}^{(j)} + \mathbf{K}_1^{(j)}\bar{\mathbf{A}}_{11}^{(j)} \mid \bar{A}_{22}^{(j)} + \mathbf{K}_1^{(j)}\bar{\mathbf{A}}_{12}^{(j)}\ \right] \quad (7)$$

being $\mathbf{K}_1^{(j)} \in \mathbb{R}^{(n-1)\times(n-1)}$ a design matrix to be determined in the following, too. Considering a further change of coordinates: $s_2(k) = w_2(k) - \mathbf{K}_1^{(j)}\mathbf{s_1}(k)$, and introducing (6) in (3), system (3) becomes (Emelyanov *et al.*, 1995):

$$\begin{cases} \mathbf{s_1}(k+1) = \left[\bar{\mathbf{A}}_{11}^{(j)} - \bar{\mathbf{A}}_{12}^{(j)}\mathbf{K}_1^{(j)} + \mathbf{\Delta}\bar{\mathbf{A}}_{11}^{(j)}(\Delta\mathbf{p})\right]\cdot \\ \qquad \mathbf{s_1}(k) + \left[\bar{\mathbf{A}}_{12}^{(j)} + \mathbf{\Delta}\bar{\mathbf{A}}_{12}^{(j)}(\Delta\mathbf{p})\right]\cdot \\ \qquad w_2(k) + \bar{\mathbf{z}}_1(\mathbf{s}(k),\mathbf{p},k) \\ w_2(k+1) = \mathbf{\Delta}\bar{\mathbf{A}}_{21}^{(j)}(\Delta\mathbf{p})\mathbf{s_1}(k) + \Delta\bar{A}_{22}^{(j)}(\Delta\mathbf{p})\cdot \\ \qquad w_2(k) + d^{(j)}\left[1 + h^{(j)}(\Delta\mathbf{p})\right]\bar{u}^{(j)}(k) \\ \qquad + \bar{z}_2(\mathbf{s}(k),\mathbf{p},k) + \mathbf{K}_1^{(j)}\bar{\mathbf{z}}_1(\mathbf{s}(k),\mathbf{p},k) \end{cases} \quad (8)$$

where $\mathbf{\Delta}\bar{\mathbf{A}}_{kl}^{(j)}(\Delta\mathbf{p})$, $k,l = 1,2$, are the transformed uncertainties.

Control law (6) will be shown to stabilize system (3) $\forall\mathbf{p} \in P^{(j)}$. In order to stabilize (3) $\forall\mathbf{p} \in P$, the 'right controller', i.e. the controller built for the hypersphere corresponding to the 'true parameter' $\mathbf{p}$, needs to be chosen among the ν elements of the family. This choice will be achieved by the switching law $j = j(k)$, taking values on $\mathbb{S}$. The time evolution of the signal $j(k)$, i.e. the transitions among the different controllers in the family, is determined by:

$$j(k) = \phi(\mathbf{s}(k), j(k-1)), \quad k \geq 0 \quad (9)$$

where $\phi : \mathbb{R} \times \mathbb{S} \to \mathbb{S}$ is a transition function, determined by a suitable switching logic. The control problem is summarized below:

Problem 1. Find the state feedback controller $\bar{u}^{(j)}(k)$ in (6) and the transition function (9) ensuring the global asymptotic stabilization of (3), i.e. of (1), independently on $\mathbf{p} \in P$.

3. THE CONTROLLER IN $P^{(J)}$

In this section the control law (6) will be determined, in order to stabilize system (3) $\forall\mathbf{p} \in P^{(j)}$. Since plant (1) is assumed to be controllable, couple $\left(\bar{\mathbf{A}}_{11}^{(j)}, \bar{\mathbf{A}}_{12}^{(j)}\right)$ is controllable, too, as shown in (Emelyanov *et al.*, 1995). Therefore, the $n-1$ eigenvalues of $\bar{\mathbf{A}}_{11}^{(j)} - \bar{\mathbf{A}}_{12}^{(j)}\mathbf{K}_1^{(j)}$ can be assigned by a suitable choice of $\mathbf{K}_1^{(j)}$. Assume that they have been assigned distinctly in the interval $(0.5, 1)$. A non singular matrix $\mathbf{N}^{(j)} \in \mathbb{R}^{(n-1)\times(n-1)}$ can be found, defining the change of coordinates: $\mathbf{s_1}(k) = \mathbf{N}^{(j)}\mathbf{w}_1(k)$, such that, in the new coordinates matrix, $\mathbf{\Lambda}^{(j)} \stackrel{\text{def}}{=} \left[\mathbf{N}^{(j)}\right]^{-1}\left[\bar{\mathbf{A}}_{11}^{(j)} - \bar{\mathbf{A}}_{12}^{(j)}\mathbf{K}_1^{(j)}\right]\mathbf{N}^{(j)}$ is a diagonal matrix, whose entries are the assigned eigenvalues. Plant (8) is transformed as follows:

$$\begin{cases} \mathbf{w}_1(k+1) = \left[\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})\right]\mathbf{w}_1(k) + \\ + \left[\mathbf{A}_{12}^{(j)} + \mathbf{\Delta A}_{12}^{(j)}(\Delta\mathbf{p})\right]w_2(k) + \mathbf{z}_1(\mathbf{w}(k),\mathbf{p},k) \\ w_2(k+1) = \mathbf{\Delta A}_{21}^{(j)}(\Delta\mathbf{p})\mathbf{w}_1(k) + \Delta A_{22}^{(j)}(\Delta\mathbf{p})\cdot \\ w_2(k) + d^{(j)}\left[1 + h^{(j)}(\Delta\mathbf{p})\right]\cdot\bar{u}^{(j)}(k) + \\ + z_2(\mathbf{w}(k),\mathbf{p},k) \end{cases} \quad (10)$$

where: $\mathbf{\Lambda}^{(j)} = diag\left\{\lambda_l^{(j)}\right\}$, $\lambda_l^{(j)} \in (0.5, 1)$, $l = 1 \ldots (n-1)$; $\mathbf{A}_{12}^{(j)}$ is the nominal block remaining after transformation, $\mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})$, $\mathbf{\Delta A}_{12}^{(j)}(\Delta\mathbf{p})$, $\mathbf{\Delta A}_{21}^{(j)}(\Delta\mathbf{p})$, $\Delta A_{22}^{(j)}(\Delta\mathbf{p})$ are the transformed uncertainties; $\mathbf{z}_1(\mathbf{w}(k),\mathbf{p},k)$ and $z_2(\mathbf{w}(k),\mathbf{p},k)$ represent nonlinear uncertainties after transformation. Assume that:

Assumption 3.1. $\mathbf{z}_1(\mathbf{w}(k),\mathbf{p},k)$ and $z_2(\mathbf{w}(k),\mathbf{p},k)$ are continuous functions, and :

$$\begin{array}{c} ||\mathbf{z}_1(\mathbf{w}(k),\mathbf{p},k)|| \leq |w_2(k)|\zeta_1(\Delta\mathbf{p}), \\ |z_2(\mathbf{w}(k),\mathbf{p},k)| \leq ||\mathbf{w}(k)||\zeta_2(\Delta\mathbf{p}) \end{array} \quad (11)$$

with $\zeta_1(\Delta\mathbf{p})$, $\zeta_2(\Delta\mathbf{p})$ continuous too, and:

$$\lim_{\mathbf{p}\to\mathbf{p}^{(j)}} \zeta_1(\Delta\mathbf{p}) = 0, \quad \lim_{\mathbf{p}\to\mathbf{p}^{(j)}} \zeta_2(\Delta\mathbf{p}) = 0 \quad (12)$$

To explain the rationale followed during controller design, a short description of the steps to be performed will be given below.

I First, conditions on the partition of P will be given, in order to synthesize the control law. In other words, it will be shown that it is possible to choose the partition, i.e. $r^{(j)}$ and ν, such that some conditions, necessary for controller synthesis, are satisfied in each $P^{(j)}$, $j \in \mathbb{S}$.

II Then, a particular stable subspace will be introduced for system (10).

III Finally, stabilization will be guaranteed making the above stable subspace invariant for each $\mathbf{p} \in P^{(j)}$, $j \in \mathbb{S}$, and ensuring that trajectories starting outside the invariant subspace are steered to zero.

The following considerations hold for step I.

- Due to (12), $r^{(j)}$ can be found, such that:

$$|\zeta_1(\Delta\mathbf{p})| \leq \bar{\zeta}_1^{(j)}, \ |\zeta_2(\Delta\mathbf{p})| \leq \bar{\zeta}_2^{(j)} \quad (13)$$

  $\forall\mathbf{p} \in P^{(j)}$, with $\bar{\zeta}_1^{(j)}$, $\bar{\zeta}_2^{(j)}$ arbitrarily small.
- Since $\mathbf{\Lambda}^{(j)} = diag\left\{\lambda_l^{(j)}\right\}$, $\lambda_l^{(j)} \in (0.5, 1)$, also matrix $\mathbf{\Lambda}^{(j)} - \mathbf{I}_{n-1}$ is stable, with $||\mathbf{\Lambda}^{(j)} -$

$\mathbf{I}_{n-1}|| < 0.5$. Due to Assumption 2.3, $r^{(j)}$ can be found such that $\forall \mathbf{p} \in P^{(j)}$:

$$\begin{aligned} &||\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})|| < 1 \\ &||\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p}) - \mathbf{I}_{n-1}|| < 0.5 \end{aligned} \quad (14)$$

As a consequence:

$$\rho_1^{(j)} \stackrel{\text{def}}{=} \max_{\mathbf{p}\in P^{(j)}} \left\{||\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})||\right\} < 1 \quad (15)$$

$$\rho_2^{(j)} \stackrel{\text{def}}{=} \max_{\mathbf{p}\in P^{(j)}} \left\{||\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p}) + -\mathbf{I}_{n-1}||\right\} < 0.5 \quad (16)$$

- Due to Assumption 2.2, it is possible to define $\psi_{min}^{(j)} \stackrel{\text{def}}{=} \min_{\mathbf{p}\in P^{(j)}} \left[1 + h^{(j)}(\Delta\mathbf{p})\right]$, $\psi_{max}^{(j)} \stackrel{\text{def}}{=} \max_{\mathbf{p}\in P^{(j)}} \left[1 + h^{(j)}(\Delta\mathbf{p})\right]$, such that:

$$0 < \psi_{min}^{(j)} \leq (1+h^{(j)}(\Delta\mathbf{p})) \leq \psi_{max}^{(j)} \quad \forall\mathbf{p} \in P^{(j)} \quad (17)$$

Define also: $\bar{\psi} \stackrel{\text{def}}{=} \dfrac{\psi_{max}^{(j)} - \psi_{min}^{(j)}}{\psi_{max}^{(j)}}$. From (4) and (17), it follows:

$$0 \leq \bar{\psi} < 1, \quad \lim_{\mathbf{\Delta p}\to\mathbf{0}} \bar{\psi} = 0 \quad (18)$$

- Since $\rho_2^{(j)} < 0.5$, a design parameter $\gamma^{(j)}$, $0 < \gamma^{(j)} \leq 0.5$, can be introduced such that: $\gamma^{(j)} - \rho_2^{(j)} > 0$. Moreover, due to (18), a suitable partition can be chosen such that: $\bar{\psi} < 0.5$. Summing up, it can be assumed that the following inequalities hold:

$$\begin{cases} 0 < \rho_2^{(j)} < \gamma^{(j)} \leq 0.5 \\ 0 < \bar{\psi} < \gamma^{(j)} \leq 0.5 \end{cases} \quad (19)$$

- Because of (15), quantity $1 - \rho_1^{(j)}$ is strictly positive. Define the positive scalar $\bar{\alpha}^{(j)}$:

$$\bar{\alpha}^{(j)} \stackrel{\text{def}}{=} \min\left\{\frac{1-\rho_1^{(j)}}{\left(A_{12,max}^{(j)} + \bar{\zeta}_1^{(j)}\right)}, \frac{\gamma^{(j)} - \rho_2^{(j)}}{\left(A_{12,max}^{(j)} + \bar{\zeta}_1^{(j)}\right)}\right\} \quad (20)$$

with:

$$A_{12,max}^{(j)} \stackrel{\text{def}}{=} \max_{\mathbf{p}\in P_j} ||\mathbf{A}_{12}^{(j)} + \mathbf{\Delta A}_{12}^{(j)}(\Delta\mathbf{p})|| \quad (21)$$

Introduce the design parameter $\alpha^{(j)} \leq \bar{\alpha}^{(j)}$.

- Due to (12) and (13), it is always possible to choose $r^{(j)}$ such that:

$$\left[\alpha^{(j)}\gamma^{(j)} + A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + \bar{\zeta}_1^{(j)}(1+\alpha^{(j)})\right]\left[q\cdot\psi_{max}^{(j)} + 1\right] \leq (1-\gamma^{(j)})\alpha^{(j)} \quad (22)$$

with $q > \dfrac{1}{\psi_{min}^{(j)}}$ and:

$$A_{21,max}^{(j)} \stackrel{\text{def}}{=} \max_{\mathbf{p}\in P_j} ||\mathbf{\Delta A}_{21}^{(j)}(\Delta\mathbf{p})||,$$

$$A_{22,max}^{(j)} \stackrel{\text{def}}{=} \max_{\mathbf{p}\in P_j} ||\Delta A_{22}^{(j)}(\Delta\mathbf{p})||.$$

As far as step II is concerned, consider the following subsystem, describing the dynamics of $\mathbf{w}_1(k)$:

$$\mathbf{w}_1(k+1) = \left[\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})\right]\mathbf{w}_1(k) + \\ + \left[\mathbf{A}_{12}^{(j)} + \mathbf{\Delta A}_{12}^{(j)}(\Delta\mathbf{p})\right] w_2(k) + \mathbf{z}_1(\mathbf{w}(k), \mathbf{p}, k) \quad (23)$$

If $w_2(k) = 0$, system (23) is stable, since eigenvalues of $\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})$ are stable $\forall\mathbf{p} \in P^{(j)}$. For $w_2(k) \neq 0$, consider the subspace:

$$\mathbb{G}(\alpha^{(j)}) \stackrel{\text{def}}{=} \left\{\mathbf{w}(k) \in \mathbb{R}^n : |w_2(k)| \leq \alpha^{(j)}||\mathbf{w}_1(k)||\right\}$$

and the theorem (Emelyanov *et al.*, 1995):

Theorem 3.1. If $w_2(k) \in \mathbb{G}(\alpha^{(j)})$, with $\alpha^{(j)} < \dfrac{1-\rho_1^{(j)}}{\left(A_{12,max}^{(j)} + \bar{\zeta}_1^{(j)}\right)}$, subsystem (23) is still stable, i.e. $\lim_{k\to\infty} ||\mathbf{w}_1(k)|| = 0 \;\; \forall\mathbf{p} \in P^{(j)}$.

Using Theorem 3.1, the stabilization of (10) (step III) can be split into two different phases:

1) The control law is first synthesized in order to make $\mathbb{G}(\alpha^{(j)})$ an invariant subspace $\forall\mathbf{p} \in P^{(j)}$. In other words, the control law should guarantee that $\mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)}) \Rightarrow \mathbf{w}(k+1) \in \mathbb{G}(\alpha^{(j)})$, $\forall\mathbf{p} \in P^{(j)}$. In this way, for all the initial states belonging to $\mathbb{G}(\alpha^{(j)})$, $\lim_{k\to\infty} ||\mathbf{w}_1(k)|| = 0$, due to Theorem 3.1. Hence, $\lim_{k\to\infty} |w_2(k)| = 0$. In fact, if $\mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)})$, $|w_2(k)| \leq \alpha^{(j)}||\mathbf{w}_1(k)||$. Therefore, system (10) is asymptotically stable $\forall\mathbf{w}(0) \in \mathbb{G}(\alpha^{(j)})$, $\forall\mathbf{p} \in P^{(j)}$.
2) Finally, it is shown that the developed controller also guarantees that $\forall\mathbf{w}(0) \notin \mathbb{G}(\alpha^{(j)})$, $\mathbf{w}(k)$ is asymptotically stable $\forall\mathbf{p} \in P_j$.

For phase 1), consider the two Theorems below:

Theorem 3.2. (Emelyanov *et al.*, 1995) Consider system (10), with $\mathbf{p} \in P^{(j)}$. Suppose that $\mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)})$, and define $\omega(k) = \dfrac{w_2(k)}{||\mathbf{w}_1(k)||}$, $\Delta\omega(k) = \omega(k+1) - \omega(k)$. If:

a) $sgn(\omega(k)) \cdot \Delta\omega(k) \leq 0$;
b) $||\Delta\mathbf{w}_1(k)|| \leq \gamma||\mathbf{w}_1(k)||$;
c) $|\Delta w_2(k)| \leq (1-\gamma)\alpha^{(j)}||\mathbf{w}_1(k)||$;

where $\gamma \in (0,1)$, then $\mathbb{G}(\alpha^{(j)})$ is an invariant subspace for system (3.1), i.e. $\mathbf{w}(k+1) \in \mathbb{G}(\alpha^{(j)})$.

Theorem 3.3. Consider system (10) and subsystem (23). If $\mathbf{p} \in P^{(j)}$, the control law:

$$\bar{u}^{(j)}(k) = \frac{w_2(k)}{\psi_{max}^{(j)} d^{(j)}} - \frac{q\cdot sgn\left[w_2(k)\right]}{d^{(j)}} \cdot \left[\alpha^{(j)}\gamma^{(j)} + \\ + A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + \bar{\zeta}_2^{(j)}(1+\alpha^{(j)})\right] \cdot ||\mathbf{w}_1(k)|| \quad (24)$$

with $\alpha^{(j)} \leq \bar{\alpha}^{(j)}$, $\gamma^{(j)}$ satisfying (19), $q > \dfrac{1}{\psi_{min}^{(j)}}$, ensures that $\mathbb{G}(\alpha^{(j)})$ is an invariant subspace.

Proof. It will be shown that (24) satisfies conditions a)-c) of Theorem 3.2 $\forall \mathbf{p} \in P^{(j)}$. Consider first condition b), with $\gamma = \gamma^{(j)}$:

$$||\mathbf{w}_1(k+1) - \mathbf{w}_1(k)|| \leq \gamma^{(j)} ||\mathbf{w}_1(k)||$$

i.e.:

$$|| \left[\mathbf{A}_{11}^{(j)} + \mathbf{G}_{11}^{(j)}(\Delta\mathbf{p}) - \mathbf{I}_{n-1}\right] \mathbf{w}_1(k) + \left[\mathbf{A}_{12}^{(j)} + \mathbf{G}_{12}^{(j)}(\Delta\mathbf{p})\right] w_2(k) + \mathbf{z}_1(\mathbf{w}(k), \mathbf{p}, k)|| \leq \gamma^{(j)} ||\mathbf{w}_1(k)|| \quad (25)$$

Since $\mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)})$, $|w_2(k)| \leq \alpha^{(j)} ||\mathbf{w}_1(k)||$. Therefore, considering also (21), (15), (11) and (13), inequality (25) is verified if:

$$\left[\rho_2^{(j)} + \alpha^{(j)} \left(A_{12,max}^{(j)} + \bar{\zeta}_1^{(j)}\right)\right] ||\mathbf{w}_1(k)|| \leq \leq \gamma^{(j)} ||\mathbf{w}_1(k)||$$

This latter inequality corresponds to:

$$\alpha^{(j)} \leq \frac{\gamma^{(j)} - \rho_2^{(j)}}{A_{12,max}^{(j)} + \bar{\zeta}_1^{(j)}}$$

which is verified, since $\alpha^{(j)} \leq \bar{\alpha}^{(j)}$.

Condition a), with $\gamma = \gamma^{(j)}$, is equivalent to (Emelyanov *et al.*, 1995):

$$sgn(w_2(k)) \cdot \Delta w_2(k) + \alpha^{(j)} \gamma^{(j)} ||\mathbf{w}_1(k)|| < 0$$

which, replacing $w_2(k+1)$ (see (10)), becomes:

$$sgn(w_2(k)) \cdot \left[\mathbf{\Delta A}_{21}^{(j)}(\Delta\mathbf{p})\mathbf{w}_1(k) + \Delta A_{22}^{(j)}(\Delta\mathbf{p}) \cdot \cdot w_2(k) + d^{(j)}(1 + h^{(j)}(\Delta\mathbf{p}))\bar{u}(k) + + z_2(\mathbf{w}(k), \mathbf{p}, k)\right] - |w_2(k)| + \alpha^{(j)} \gamma^{(j)} ||\mathbf{w}_1(k)|| < 0$$

Taking the worst case condition, and considering that $\mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)})$, one has:

$$sgn(w_2(k)) d^{(j)} (1 + h^{(j)}(\Delta\mathbf{p}))\bar{u}(k) - |w_2(k)| + + \left[A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + \alpha^{(j)}\gamma^{(j)} + + \bar{\zeta}_2^{(j)}(1 + \alpha^{(j)})\right] \cdot ||\mathbf{w}_1(k)|| < 0$$

which, replacing $\bar{u}(k)$ with (24), becomes:

$$|w_2(k)| \left[\frac{1 + h^{(j)}(\Delta\mathbf{p})}{\psi_{max}^{(j)}} - 1\right] + \left[A_{21,max}^{(j)} + \alpha^{(j)} \cdot \cdot A_{22,max}^{(j)} + \alpha^{(j)}\gamma^{(j)} + \bar{\zeta}_2^{(j)}(1 + \alpha^{(j)})\right] \cdot ||\mathbf{w}_1(k)|| \cdot \left\{1 - q \cdot \left[1 + h^{(j)}(\Delta\mathbf{p})\right]\right\} < 0$$

Since $|w_2(k)| \left[\dfrac{1 + h^{(j)}(\Delta\mathbf{p})}{\psi_{max}^{(j)}} - 1\right] \leq 0$, and considering the worst case condition, one has:

$$\left[1 - q\psi_{min}^{(j)}\right] \cdot \left[A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + \alpha^{(j)}\gamma^{(j)} + \bar{\zeta}_2^{(j)}(1 + \alpha^{(j)})\right] < 0$$

Due to the assumption on q, the above inequality is always verified.

Consider now condition c), with $\gamma = \gamma^{(j)}$:

$$\left|\mathbf{\Delta A}_{21}^{(j)}(\Delta\mathbf{p})\mathbf{w}_1(k) + \Delta A_{22}^{(j)}(\Delta\mathbf{p}) w_2(k) + + z_2(\mathbf{w}(k), \mathbf{p}, k) d^{(j)} \left[1 + h^{(j)}(\Delta\mathbf{p})\right] \bar{u}^{(j)}(k) + - w_2(k)\right| \leq (1 - \gamma^{(j)}) |w_2(k)|$$

Replacing control law (24) and considering the worst case condition, one has:

$$q \cdot \psi_{max}^{(j)} \cdot \left[\alpha^{(j)}\gamma^{(j)} + A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + \bar{\zeta}_2^{(j)}(1 + \alpha^{(j)})\right] \cdot ||\mathbf{w}_1(k)|| + \left[\alpha^{(j)}\bar{\psi} + + A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + \bar{\zeta}_2^{(j)}(1 + \alpha^{(j)})\right] \cdot \cdot ||\mathbf{w}_1(k)|| \leq (1 - \gamma^{(j)})\alpha^{(j)} ||\mathbf{w}_1(k)|| \quad (26)$$

According to (19), (26) is satisfied if:

$$\left[\alpha^{(j)}\gamma^{(j)} + A_{21,max}^{(j)} + \alpha^{(j)} A_{22,max}^{(j)} + + \bar{\zeta}_2^{(j)}(1 + \alpha^{(j)})\right] \cdot \left[q \cdot \psi_{max}^{(j)} + 1\right] \leq (1 - \gamma^{(j)})\alpha^{(j)} \quad (27)$$

which is true, due to (22). △

Theorems 3.1 and 3.3 guarantee that plant (10) is asymptotically stable for all the initial states belonging to $\mathbb{G}(\alpha^{(j)})$. The following Theorem states that asymptotic stability is ensured also for initial states not belonging to $\mathbb{G}(\alpha^{(j)})$, i.e. phase 2).

Theorem 3.4. Consider system (10) when $\mathbf{w}(k) \notin \mathbb{G}(\alpha^{(j)})$, i.e. $||\mathbf{w}_1(k)|| < \dfrac{|w_2(k)|}{\alpha^{(j)}}$. If $\mathbf{p} \in P^{(j)}$, control law (24) ensures that: $\lim_{k\to\infty} |w_2(k)| = 0$

The Proof, omitted for the sake of brevity, can be derived following the lines of an analogous Theorem in (Emelyanov *et al.*, 1995).

When $|w_2(k)|$ vanishes in subplant (23), $||\mathbf{w}_1(k)||$ vanishes too, since matrix $\mathbf{\Lambda}^{(j)} + \mathbf{\Delta\Lambda}^{(j)}(\Delta\mathbf{p})$ has stable eigenvalues, due to the partitioning criterion adopted. Hence: $\lim_{k\to\infty} ||\mathbf{w}(k)|| = 0$.

4. THE TRANSITION FUNCTION

The purpose of this section is the definition of a suitable transition function (9) governing the switching among the finite family of controllers $\mathbb{S}$. The transition function has to ensure both system asymptotic stabilization and that the switching ends after a finite time. Since the controller associated to $P^{(j)}$ has been designed to ensure that at least one of the following conditions is verified:

$$\mathbf{w}(k-1) \in \mathbb{G}(\alpha^{(j)}) \Rightarrow \mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)}) \quad (28)$$

$$|w_2(k)| < |w_2(k-1)| \quad (29)$$

the switching logic will be based on the boolean variable defined below:

$$\mathcal{H} = \left[\left(\mathbf{w}(k-1) \in \mathbb{G}(\alpha^{(j)})\right) AND \left(\mathbf{w}(k) \in \mathbb{G}(\alpha^{(j)})\right)\right] OR \left(|w_2(k)| < |w_2(k-1)|\right) \quad (30)$$

as stated in the following Theorem.

Theorem 4.1. Consider the system described by (10) with $\mathbf{p} \in P$, and assume that a family of controllers has been found according to (6) and (24). Initialize $j = 1$, and consider boolean expression (30), with $j \in \{1 \dots \nu\}$. The following transition function implementing a "prerouted" switching policy:

$$j(k) = \begin{cases} j(k) & if \;\; \mathcal{H} \\ j(k)+1 & otherwise \end{cases} \qquad k = 1 \dots \nu \quad (31)$$

ensures that: i) the switching stops after a finite time interval, and ii) the closed loop system is globally asymptotically stable $\forall \mathbf{p} \in P$.

Proof. The proof directly follows from the results of Sections 2 and 3. Basically, the rationale followed by the switching policy consists in excluding those controllers not satisfying both condition (28) and (29). Consider a time instant k and set $j = 1$. Assume first that $\mathbf{p} \in P_1$. The application of (6) and (24) guarantees that either condition (28) or (29) is verified, due to the fact that the control law $u^{(1)}(s(k))$ has been designed as to satisfy the above conditions $\forall\, \mathbf{p} \in P_1$. It follows that condition $\mathcal{H}$ is always true, no switching can occur, and the global asymptotical stability of the system is guaranteed. Now assume that $\mathbf{p} \in P_l$, $l > 1$, hence $\mathbf{p} \notin P_1$, and that no overlapping occurs between P_1 and P_l. In this case, a violation of both conditions (28), (29) will necessarily occur, otherwise the first configuration would be a stabilizing one. When instability is detected, a switching occurs towards the next configuration $j = 2$, and $u^{(2)}(s(k))$ is applied. Arguments similar to those used before can be applied, producing $l-1$ subsequent switching until the 'correct' controller is found. When finally $u^{(l)}(s(k))$ is applied, it of course ensures the system global asymptotical stabilization by construction.

The above discussion implicitly proves that the switching ends after a finite time. Since the number of the controllers, ν, is finite, the condition $\mathbf{p} \in P_j$ for some $j \in \; 1 \dots \nu$ will be verified in a finite time interval. After such an interval, control law $u^{(j)}(s(k))$ is applied, and one can conclude that $\lim_{k\to\infty} ||\mathbf{w}(k)|| = 0$. △

Remark 1. The transition function consists simply in skipping a controller if none of the conditions (28), (29) is verified, otherwise in maintaining the current configuration. Switching logic given by (31) guarantees that the "right" controller is found in at most ν steps, since the adopted partition is a complete covering of P.

5. CONCLUSIONS

In this note state feedback discrete time VSC is used for the robust stabilization of discrete time linear systems in presence of unmatched uncertainties. The presence of the sector is avoided by means of Switching Control. As a future development, an output feedback controller with the same characteristics could be considered, using e.g. the approach described in (Edwards and Spurgeon, 1998) for continuous time systems.

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