DECENTRALIZED $\mathcal{H}_2$ CONTROLLER DESIGN FOR DESCRIPTOR SYSTEMS: AN LMI APPROACH

Guisheng Zhai [*] **Masaharu Yoshida** [**] **Joe Imae** [*] **Tomoaki Kobayashi** [*]

[*] *Dept. Mechanical Engineering, Osaka Prefecture Univ.*
1-1 Gakuen-cho, Sakai, Osaka 599-8531, Japan
Email: zhai@mecha.osakafu-u.ac.jp

[**] *Dept. Opto-Mechatronics, Wakayama Univ.*
930 Sakaedani, Wakayama 640-8510, Japan

Abstract: This paper considers a decentralized $\mathcal{H}_2$ control problem for multi-channel linear time-invariant (LTI) descriptor systems. Our interest is to design a *low order* dynamic output feedback controller. The control problem is reduced to a feasibility problem of a bilinear matrix inequality (BMI) with respect to variables of a coefficient matrix defining the controller, a Lyapunov matrix and a matrix related to the descriptor matrix. Under a matching condition between the descriptor matrix and the measurement output matrix (or the control input matrix), we propose to set the Lyapunov matrix in the BMI as block diagonal appropriately so that the BMI is reduced to LMIs.

Keywords: multi-channel descriptor system, $\mathcal{H}_2$ control, decentralized control, bilinear matrix inequality (BMI), linear matrix inequality (LMI)

1. INTRODUCTION

It is well known that descriptor systems (also known as singular systems or implicit systems) have high abilities in representing dynamical systems. They can preserve physical parameters in the coefficient matrices, and describe the dynamic part, static part, and even improper part of the system in the same form. In this sense, descriptor systems are much superior to systems represented by state-space models.

There have been reported many works on descriptor systems, e.g., (Cobb, 1983; Takaba, Morihara & Katayama, 1995; Masubuchi, Kamitane, Ohara & Suda, 1997). Among these works, (Masubuchi *et al.*, 1997) applied the LMI approach (e.g., (Boyd, El Ghaoui, Feron & Balakrishnan, 1994)) to stabilization and $\mathcal{H}_\infty$ control problems for descriptor systems. Since the LMI-type conditions proposed there contain equality constraints, which are not desirable in real applications, (Uezato & Ikeda, 1999) derived strict LMI conditions for stability, robust stabilization and $\mathcal{H}_\infty$ control of linear descriptor systems. Since the strict LMIs are definite ones without equality constraints, they are highly tractable and reliable when we use recent popular softwares for solving LMIs. Later, (Ikeda, Lee & Uezato, 2000) extended the consideration to $\mathcal{H}_2$ control problem for descriptor systems and derived a strict LMI condition which is necessary and sufficient for $\mathcal{H}_2$ control.

Concerning decentralized control of descriptor systems, (Ikeda, Zhai & Uezato, 2001) considered a decentralized stabilization problem for large-scale interconnected descriptor systems, which are special cases of multi-channel descriptor systems. In that context, the design problem was reduced to feasibility of a BMI, and to solve the BMI, a homotopy-based method was proposed, where the interconnections between subsystems are increased gradually from zeros to the given magnitudes. Recently, (Zhai, Koyama & Yoshida, 2004) extended the results in (Uezato & Ikeda, 1999) to decentralized $\mathcal{H}_\infty$ control for multi-channel descriptor systems and proposed strict LMI conditions for designing low order decentralized controller. However, to the best of our knowledge, there is very few existing result considering decentralized $\mathcal{H}_2$ controller design for multi-channel descriptor systems.

Motivated by the above observations, we consider a low order decentralized $\mathcal{H}_2$ controller design for multi-channel descriptor systems in this paper. More precisely, for the multi-channel descriptor systems under consideration, in addition to the requirement that the controller should be decentralized (composed of local controllers), we require that the sum of the orders of local controllers should be smaller than the order of the system to be controlled. As pointed out in many references (Fu & Luo, 1997; Grigoriadis & Skelton, 1996), the problem of computing a low order controller is quite difficult. In (Zhai, Ikeda & Fujisaki, 2001), the homotopy-based algorithm was also extended to low order decentralized $\mathcal{H}_\infty$ controller design for multi-channel LTI systems, by augmenting the matrix variable defining the decentralized controller of desired low order to a matrix variable defining a full order decentralized controller. Although the homotopy-based method in (Zhai *et al.*, 2001) can also be applied for the present problem by some modifications, the convergence of the algorithm depends on how to choose the initial full order centralized controller, and the random search of such a centralized controller introduced in (Zhai *et al.*, 2001) needs huge computational efforts in general.

In this paper, we first apply the existing results in (Ikeda *et al.*, 2000) for $\mathcal{H}_2$ control of linear descriptor systems, to express the existence condition of decentralized $\mathcal{H}_2$ controllers with desired orders as a BMI with respect to variables of a coefficient matrix defining the controller, a Lyapunov matrix and a matrix related to the descriptor matrix. As also pointed out in (Zhai *et al.*, 2001), although it is not difficult to obtain such a BMI, there has been no guaranteed method for solving general BMIs, especially of large size (Goh, Safonov & Papavassilopoulos, 1994; Liu & Papavassilopoulos, 1996). Here, under a matching condition between the descriptor matrix and the measurement output matrix (or the control input matrix), we apply and modify the method developed in (Mattei, 2000; Zhai, Tamaoki & Murao, 2002; Murao, Zhai, Ikeda & Tamaoki, 2002) so that the BMI on hand is reduced to LMIs (Boyd *et al.*, 1994) which is sufficient to the BMI but much more tractable. More precisely, we propose to set the Lyapunov matrix variable in the BMI as block diagonal appropriately corresponding to the controller's desired order. Because the structure of the block diagonal matrix variable can be set freely, we can consider the controller's order arbitrarily.

The remainder of this paper is organized as follows. In Section 2 we formulate our control problem and rewrite compactly the closed-loop decentralized control system composed of the original descriptor system and the local controllers, by defining some notations. In Section 3, under a matching condition between the descriptor matrix and the measurement output matrix, we derive the first LMI condition for existence of a desired controller by setting the Lyapunov matrix variable in the BMI as block diagonal appropriately. In Section 4, under a matching condition between the descriptor matrix and the control input matrix, we derive the second LMI condition.

2. PROBLEM FORMULATION

We consider the N-channel LTI descriptor system described by

$$\begin{cases} E\dot{x} = Ax + B_1 w + \sum_{i=1}^{N} B_{2i} u_i \\ z = C_1 x \\ y_i = C_{2i} x\,, \quad i = 1\,,2\,,\cdots,N\,, \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$ is the descriptor variable, $w \in \mathcal{R}^h$ is the disturbance input, $z \in \mathcal{R}^p$ is the controlled output, $u_i \in \mathcal{R}^{m_i}$ and $y_i \in \mathcal{R}^{q_i}$ are the control input and the measurement output of channel i ($i = 1, 2, \cdots, N$). The matrices $E, A, B_1, B_{2i}, C_1, C_{2i}$ are constant and of appropriate dimension, $N > 1$ is the number of subsystems. The matrix E may be singular and we denote its rank by $r = \operatorname{rank} E \leq n$. Without loss of generality, we assume that for every i, B_{2i} is of full column rank, and C_{2i} is of full row rank. Furthermore, to ensure fitness of the $\mathcal{H}_2$ control problem, we assume that the system (1) satisfies the following condition (Takaba & Katayama, 1997; Ikeda *et al.*, 2000).

$$\ker E \subset \ker C_1 \tag{2}$$

For the system (1), we consider a decentralized output feedback controller

$$\begin{cases} \dot{x}_{ci} = A_{ci} x_{ci} + B_{ci} y_i \\ u_i = C_{ci} x_{ci} + D_{ci} y_i \end{cases} \tag{3}$$

where $x_{ci} \in \mathcal{R}^{n_{ci}}$ is the state of the ith local controller, n_{ci} is a *specified dimension*, and $A_{ci}, B_{ci}, C_{ci}, D_{ci}$, $i = 1, 2, \cdots, N$, are constant matrices to be determined. Since we are interested in designing a low order decentralized controller, we require that $n_c = \sum_{i=1}^{N} n_{ci} < \bar{n} \leq n$, where $\bar{n}$ is the order of the system described by the transfer function $C_1(sE - A)^{-1}B_1$.

The closed-loop system obtained by applying the controller (3) to the system (1) is

$$\begin{cases} E\dot{x} = (A + \sum_{i=1}^{N} B_{2i} D_{ci} C_{2i}) x \\ \qquad + \sum_{i=1}^{N} B_{2i} C_{ci} x_{ci} + B_1 w \\ \dot{x}_{ci} = B_{ci} C_{2i} x + A_{ci} x_{ci} \\ z = C_1 x \,. \end{cases} \tag{4}$$

By $T_{zw}(s)$, we denote the transfer function from w to z in the above closed-loop system. Then, the control problem of this paper is stated as follows:

Decentralized $\mathcal{H}_2$ control problem. *For a specified scalar $\gamma > 0$, design a low order decentralized controller (3) for the system (1) so that the resultant closed-loop system (4) is stable and* $\|T_{zw}(s)\|_2 < \gamma$. If such a decentralized controller exists, we say the descriptor system (1) is stabilizable with $\mathcal{H}_2$ norm γ via a decentralized controller (3).

We collect the controller's state x_{ci} and the coefficient matrices $A_{ci}, B_{ci}, C_{ci}, D_{ci}$ as

$$\begin{aligned} x_c &= \left[\, x_{c1}^T \; x_{c2}^T \; \cdots \; x_{cN}^T \,\right]^T \\ A_{cD} &= \operatorname{diag}\{\, A_{c1},\, A_{c2},\, \cdots,\, A_{cN} \,\} \\ B_{cD} &= \operatorname{diag}\{\, B_{c1},\, B_{c2},\, \cdots,\, B_{cN} \,\} \\ C_{cD} &= \operatorname{diag}\{\, C_{c1},\, C_{c2},\, \cdots,\, C_{cN} \,\} \\ D_{cD} &= \operatorname{diag}\{\, D_{c1},\, D_{c2},\, \cdots,\, D_{cN} \,\}, \end{aligned} \tag{5}$$

and define the matrices

$$\begin{aligned} B_2 &= \left[\, B_{21} \; B_{22} \; \cdots \; B_{2N} \,\right] \\ C_2 &= \left[\, C_{21}^T \; C_{22}^T \; \cdots \; C_{2N}^T \,\right]^T \end{aligned} \tag{6}$$

to describe the closed-loop system (4) as

$$\begin{cases} E\dot{x} = (A + B_2 D_{cD} C_2) x + B_2 C_{cD} x_c + B_1 w \\ \dot{x}_c = B_{cD} C_2 x + A_{cD} x_c \\ z = C_1 x \,. \end{cases} \tag{7}$$

Since it is reasonable to consider the case where all the input/output channels are independent and there is no redundant input/output, we assume that B_2 is of full column rank and C_2 is of full row rank.

We further write the matrices A_{cD}, B_{cD}, C_{cD} and D_{cD} in a single matrix

$$G_D = \begin{bmatrix} A_{cD} & B_{cD} \\ C_{cD} & D_{cD} \end{bmatrix} \tag{8}$$

and introduce the notations

$$\begin{aligned} \left[\, \tilde{E} \;\; \tilde{A} \,\right] &= \left[\begin{array}{cc|cc} E & 0 & A & 0_{n\times n_c} \\ 0 & I_{n_c} & 0_{n_c\times n} & 0_{n_c\times n_c} \end{array}\right] \\ \left[\, \tilde{B}_1 \;\; \tilde{B}_2 \,\right] &= \left[\begin{array}{c|cc} B_1 & 0_{n\times n_c} & B_2 \\ 0_{n_c\times h} & I_{n_c} & 0_{n_c\times m} \end{array}\right] \\ \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} &= \left[\begin{array}{cc} C_1 & 0_{p\times n_c} \\ \hline 0_{n_c\times n} & I_{n_c} \\ C_2 & 0_{q\times n_c} \end{array}\right] \end{aligned} \tag{9}$$

where $m = \sum_{i=1}^{N} m_i, q = \sum_{i=1}^{N} q_i$. Then, the system (7) is written in a compact form as

$$\begin{cases} \tilde{E}\dot{\tilde{x}} = (\tilde{A} + \tilde{B}_2 G_D \tilde{C}_2)\tilde{x} + \tilde{B}_1 w \\ z = \tilde{C}_1 \tilde{x} \end{cases} \tag{10}$$

where $\tilde{x} = [x^T \;\; x_c^T]^T \in \mathcal{R}^{n+n_c}$. In this description, only the controller coefficient matrix G_D is unknown, while all the other matrices are given by the system (1) and the specified orders of local controllers.

3. CONTROLLER DESIGN I

We first recall an existing result for $\mathcal{H}_2$ control of linear descriptor systems.

Lemma 1.(Ikeda *et al.*, 2000) *Consider the linear descriptor system described by*

$$\begin{cases} E\dot{x} = Ax + Bw \\ z = Cx \,, \end{cases} \tag{11}$$

where x, w, z are the same as in (1), and E, A, B, C are constant matrices of appropriate dimension. The matrix E may be singular and rank$E = r \le n$*. Let matrices $V, U \in \mathcal{R}^{n\times(n-r)}$ be of full column rank and composed of bases of Null$E$ and NullE^T, respectively. Assume that the fitness condition (2) is true between E and C. Then, for a given positive scalar γ, the system (11) is stable and $\|C(sE-A)^{-1}B\|_2 < \gamma$ if and only if there exist $P > 0$ and S satisfying the LMIs*

$$\begin{aligned} A(PE^T + VSU^T) + (PE^T + VSU^T)^T A^T \\ + BB^T < 0 \end{aligned} \tag{12}$$

$$\operatorname{trace}[CPC^T] < \gamma^2 \,. \tag{13}$$

Translating Lemma 1 in terms of the closed-loop system (10), we see that the decentralized $\mathcal{H}_2$ control problem is reduced to solving the matrix inequalities

$$\begin{aligned} &(\tilde{A} + \tilde{B}_2 G_D \tilde{C}_2)(\tilde{P}\tilde{E}^T + \tilde{V}\tilde{S}\tilde{U}^T) \\ &\quad + (\tilde{P}\tilde{E}^T + \tilde{V}\tilde{S}\tilde{U}^T)^T (\tilde{A} + \tilde{B}_2 G_D \tilde{C}_2)^T \\ &\quad + \tilde{B}_1 \tilde{B}_1^T < 0 \end{aligned} \tag{14}$$

$$\operatorname{trace}[\tilde{C}_1 \tilde{P} \tilde{C}_1^T] < \gamma^2 \tag{15}$$

with respect to G_D, $\tilde{P} > 0$ and $\tilde{S}$, where

$$\tilde{V} = \begin{bmatrix} V \\ 0_{n_c\times(n-r)} \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} U \\ 0_{n_c\times(n-r)} \end{bmatrix}. \tag{16}$$

It is observed from the above that the existence condition (14) for a desired decentralized $\mathcal{H}_2$ controller is a BMI with respect to $(\tilde{P}, \tilde{S})$ and G_D, and at present there is no globally effective method to solve general BMI problems. Although global optimization approaches using branch and bound methods for general BMIs have been proposed (Goh *et al.*, 1994; Liu & Papavassilopoulos, 1996), the necessary computational efforts would be prohibitive when their methods are applied to solve our BMI for systems of high dimensions in unlimited regions of the matrix variables in (14). Another algorithm has been proposed in (Zhai *et al.*, 2001) for solving the BMI (14) by using the idea of the homotopy method, where the controller's coefficient matrices are deformed from full matrices defined by a centralized controller, to block diagonal matrices of specified dimensions which describe a decentralized controller. Since the convergence of the algorithm in (Zhai *et al.*, 2001) depends on the choice of the initial centralized controller, a random search has been proposed for such centralized controller. However, for large scale problems, the computation efforts for such random search is still very large. For this reason, we propose to set the Lyapunov matrix variable in (14) as block diagonal appropriately so that the BMI (14) is reduced to an LMI, which is easy to solve by using the existing softwares (for example, the LMI Control Toolbox of MATLAB (Gahinet, Nemirovskii, Laub & Chilali, 1994)).

Throughout this section, we assume:

Assumption 1. There exists a matrix C_{2e} such that $C_2 = C_{2e}E$.

This assumption requires a *matching condition* between the descriptor matrix E and the measurement output matrix C_2, which implies that the null space of E is included in that of C_2. We note that the measurement output in control systems is the quantity that we can adjust in real implementation, and thus Assumption 1 is not an unrealistic condition.

Theorem 1. *The system* (1) *under Assumption 1 is stabilizable with* $\mathcal{H}_2$ *norm* γ *via a decentralized controller* (3) *if there exist a matrix* $\tilde{S} \in \mathcal{R}^{(n-r)\times(n-r)}$*, a positive definite matrix* $\hat{P}$ *structured as*

$$\hat{P} = \begin{bmatrix} \hat{P}_1 & 0 \\ 0 & \hat{P}_2 \end{bmatrix}, \quad \hat{P}_1 = \begin{bmatrix} \hat{P}_A & \hat{P}_B \\ \hat{P}_B^T & \hat{P}_D \end{bmatrix} \tag{17}$$

$$\hat{P}_A = \mathrm{diag}\{\hat{P}_{A1}, \hat{P}_{A2}, \cdots, \hat{P}_{AN}\}$$
$$\hat{P}_B = \mathrm{diag}\{\hat{P}_{B1}, \hat{P}_{B2}, \cdots, \hat{P}_{BN}\}$$
$$\hat{P}_D = \mathrm{diag}\{\hat{P}_{D1}, \hat{P}_{D2}, \cdots, \hat{P}_{DN}\}$$

with $\hat{P}_{Ai} \in \mathcal{R}^{n_{ci}\times n_{ci}}$, $\hat{P}_{Bi} \in \mathcal{R}^{n_{ci}\times q_i}$, $\hat{P}_{Di} \in \mathcal{R}^{q_i\times q_i}$*, and a matrix* W *structured as*

$$W = \begin{bmatrix} W_A & W_B \\ W_C & W_D \end{bmatrix} \tag{18}$$

$$W_A = \mathrm{diag}\{W_{A1}, W_{A2}, \cdots, W_{AN}\}$$
$$W_B = \mathrm{diag}\{W_{B1}, W_{B2}, \cdots, W_{BN}\}$$
$$W_C = \mathrm{diag}\{W_{C1}, W_{C2}, \cdots, W_{CN}\}$$
$$W_D = \mathrm{diag}\{W_{D1}, W_{D2}, \cdots, W_{DN}\}$$

with $W_{Ai} \in \mathcal{R}^{n_{ci}\times n_{ci}}$, $W_{Bi} \in \mathcal{R}^{n_{ci}\times q_i}$, $W_{Ci} \in \mathcal{R}^{m_i\times n_{ci}}$, $W_{Di} \in \mathcal{R}^{m_i\times q_i}$*, such that the LMIs*

$$\Phi_1 + \Phi_1^T + \hat{B}_1\hat{B}_1^T < 0 \tag{19}$$

$$\Phi_1 = \hat{A}(\hat{P}\hat{E}^T + \hat{V}\tilde{S}\hat{U}^T) + \hat{B}_2 \begin{bmatrix} W & 0 \end{bmatrix} \hat{E}^T$$

$$\mathrm{trace}[\hat{C}_1\hat{P}\hat{C}_1^T] < \gamma^2 \tag{20}$$

hold. Here, $\hat{E} = T^{-1}\tilde{E}T$, $\hat{A} = T^{-1}\tilde{A}T$, $\hat{B}_1 = T^{-1}\tilde{B}_1$, $\hat{B}_2 = T^{-1}\tilde{B}_2$, $\hat{C}_1 = \tilde{C}_1T$, $\hat{V} = T^{-1}\tilde{V}$, $\hat{U} = T^{-1}\tilde{U}$*, and* $T \in \mathcal{R}^{(n+n_c)\times(n+n_c)}$ *is a nonsingular matrix satisfying*

$$\tilde{C}_2T = \begin{bmatrix} I_{n_c+q} & 0 \end{bmatrix}. \tag{21}$$

When the LMIs (19)-(20) *are feasible, one desired controller is computed as*

$$G_D = W\hat{P}_1^{-1}. \tag{22}$$

Proof. We first note that since we have assumed in the previous section that C_2 is of full row rank, $\tilde{C}_2$ is also of full row rank, and thus there always exists a nonsingular matrix T such that (21) is satisfied. Although such a matrix is not unique, we note that the choice of T does not affect the feasibility of the LMIs (19)-(20).

Pre-multiplying the first LMI (19) by T and post-multiplying it by T^T, and then substituting all the notations we defined together with $\tilde{P} = T\hat{P}T^T$, we obtain

$$\tilde{\Phi}_1 + \tilde{\Phi}_1^T + \tilde{B}_1\tilde{B}_1^T < 0 \tag{23}$$

$$\tilde{\Phi}_1 = \tilde{A}(\tilde{P}\tilde{E}^T + \tilde{V}\tilde{S}\tilde{U}^T) + \tilde{B}_2 \begin{bmatrix} W & 0 \end{bmatrix} T^T\tilde{E}^T.$$

It is easy to confirm from (21) and (22) that

$$\begin{bmatrix} W & 0 \end{bmatrix} = G_D\tilde{C}_2T\hat{P}, \tag{24}$$

and that

$$\begin{aligned} \tilde{C}_2\tilde{V} &= \begin{bmatrix} 0_{n_c\times n} & I_{n_c} \\ C_2 & 0_{q\times n_c} \end{bmatrix} \begin{bmatrix} V \\ 0_{n_c\times(n-r)} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ C_2V \end{bmatrix} = \begin{bmatrix} 0 \\ C_{2e}EV \end{bmatrix} = 0. \end{aligned} \tag{25}$$

Thus, we obtain from (23) that

$$\begin{aligned} &\tilde{A}(\tilde{P}\tilde{E}^T + \tilde{V}\tilde{S}\tilde{U}^T) + (\tilde{P}\tilde{E}^T + \tilde{V}\tilde{S}\tilde{U}^T)^T\tilde{A}^T \\ &\quad + \tilde{B}_2G_D\tilde{C}_2\tilde{V}\tilde{S}\tilde{U}^T + \left(\tilde{B}_2G_D\tilde{C}_2\tilde{V}\tilde{S}\tilde{U}^T\right)^T \\ &\quad + \tilde{B}_2G_D\tilde{C}_2\tilde{P}\tilde{E}^T + \left(\tilde{B}_2G_D\tilde{C}_2\tilde{P}\tilde{E}^T\right)^T \\ &\quad + \tilde{B}_1\tilde{B}_1^T < 0 \end{aligned} \tag{26}$$

which is exactly the matrix inequality (14). Since the second LMI (20) is the same as (15), we declare that the closed-loop system (10) with (22) is stable with $\mathcal{H}_2$ norm γ.

What we have to do next is to prove that the controller coefficient matrix G_D given by (22) has the decentralized structure defined in (8). Since we required $\hat{P} > 0$ in the theorem, we get $\hat{P}_A > 0$ and $\hat{P}_D > 0$. Then, it is not difficult to obtain that

$$\hat{P}_1^{-1} = \begin{bmatrix} \bar{P}_A & \bar{P}_B \\ \bar{P}_B^T & \bar{P}_D \end{bmatrix} \tag{27}$$

where

$$\begin{aligned} \bar{P}_A &= \hat{P}_A^{-1} + \hat{P}_A^{-1}\hat{P}_B(\hat{P}_D - \hat{P}_B^T\hat{P}_A^T\hat{P}_B)^{-1}\hat{P}_B^T\hat{P}_A^{-1} \\ \bar{P}_B &= -\hat{P}_A^{-1}\hat{P}_B(\hat{P}_D - \hat{P}_B^T\hat{P}_A^{-1}\hat{P}_B)^{-1} \\ \bar{P}_D &= (\hat{P}_D - \hat{P}_B^T\hat{P}_A^{-1}\hat{P}_B)^{-1}\,. \end{aligned} \tag{28}$$

Since $\hat{P}_A, \hat{P}_B, \hat{P}_D$ are block diagonal, $\bar{P}_A, \bar{P}_B$ and $\bar{P}_D$ are block diagonal too. Then, we obtain from (22) that

$$\begin{aligned} G_D &= W\hat{P}_1^{-1} \\ &= \begin{bmatrix} W_A\bar{P}_A + W_B\bar{P}_B^T & W_A\bar{P}_B + W_B\bar{P}_D \\ W_C\bar{P}_A + W_D\bar{P}_B^T & W_C\bar{P}_B + W_D\bar{P}_D \end{bmatrix}. \end{aligned} \tag{29}$$

Since W_A, W_B, W_C, W_D are block diagonal, we see that all the four elements in (29) are block diagonal and thus the above G_D has the decentralized structure specified in (8). ∎

Remark 1. It is understood from the above proof that the block diagonal structures of W and $\hat{P}_1$ are designed so that a decentralized controller is obtained, and the block diagonal structure of $\hat{P}$ is assumed so that the coupling between G_D and $\tilde{P}$ can be removed by using some equivalent transformation. Although the structures of the variables are complicated at a first glimpse, the matrix inequalities (19)-(20) are linear with respect to $\tilde{S}$, $\hat{P}$, W, and thus are very easy to solve by using the existing software LMI Control Toolbox (Gahinet *et al.*, 1994). ∎

4. CONTROLLER DESIGN II

In this section, similarly to Assumption 1, we assume:

Assumption 2. There exists a matrix B_{2e} such that $B_2 = EB_{2e}$.

This assumption requires a *matching condition* between the descriptor matrix E and the control input matrix B_2, which implies that the space spanned by B_2 is included in that by E.

To proceed, we first derive another form of Lemma 1 for the benefit of the discussion in this section. To do this, we consider the same system (11) as in Lemma 1. Noticing that $\|C(sE-A)^{-1}B\|_2 < \gamma$ is equivalent to $\|B^T(sE^T - A^T)^{-1}C^T\|_2 < \gamma$ together with the fact

$$(E^T)^TV = 0\,,\ \ (E^T)U = 0\,, \tag{30}$$

we apply Lemma 1 to the dual system of (11), described by (E^T, A^T, C^T, B^T), to obtain the following result. It is noted that the result has also appeared in (Ikeda *et al.*, 2000).

Lemma 2. *For a given positive scalar* γ*, the system* (11) *is stable and* $\|C(sE-A)^{-1}B\|_2 < \gamma$ *if and only if there exist* $Q > 0$ *and* R *satisfying the LMIs*

$$\begin{aligned} A^T(QE + URV^T) + (QE + URV^T)^TA & \\ + C^TC < 0 & \end{aligned} \tag{31}$$

$$\text{trace}[B^TQB] < \gamma^2\,. \tag{32}$$

Translating Lemma 2 in terms of the closed-loop system (10), we see that the decentralized $\mathcal{H}_2$ control problem is reduced to solving the matrix inequalities

$$\begin{aligned} &\left(\tilde{A} + \tilde{B}_2G_D\tilde{C}_2\right)^T(\tilde{Q}\tilde{E} + \tilde{U}\tilde{R}\tilde{V}^T) \\ &\quad + (\tilde{Q}\tilde{E} + \tilde{U}\tilde{R}\tilde{V}^T)^T\left(\tilde{A} + \tilde{B}_2G_D\tilde{C}_2\right) \\ &\quad + \tilde{C}_1^T\tilde{C}_1 < 0 \end{aligned} \tag{33}$$

$$\text{trace}[\tilde{B}_1^T\tilde{Q}\tilde{B}_1] < \gamma^2 \tag{34}$$

with rsepect to G_D, $\tilde{Q} > 0$ and $\tilde{R}$. Same as in the previous section, the matrix inequality (33) is a BMI with respect to $(\tilde{Q}, \tilde{R})$ and G_D, there is no globally effective method for solving it. Here, under Assumption 2, we propose to set the Lyapunov matrix variable $\tilde{Q}$ as block diagonal appropriately so that the BMI (33) is reduced to LMIs.

Theorem 2. *The system* (1) *under Assumption 2 is stabilizable with* $\mathcal{H}_2$ *norm* γ *via a decentralized controller* (3) *if there exist a matrix* $\tilde{R} \in \mathcal{R}^{(n-r)\times(n-r)}$*, a positive definite matrix* $\hat{Q}$ *structured as*

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & 0 \\ 0 & \hat{Q}_2 \end{bmatrix},\quad \hat{Q}_1 = \begin{bmatrix} \hat{Q}_A & \hat{Q}_B \\ \hat{Q}_B^T & \hat{Q}_D \end{bmatrix} \tag{35}$$

$$\begin{aligned} \hat{Q}_A &= \text{diag}\{\hat{Q}_{A1}, \hat{Q}_{A2}, \cdots, \hat{Q}_{AN}\} \\ \hat{Q}_B &= \text{diag}\{\hat{Q}_{B1}, \hat{Q}_{B2}, \cdots, \hat{Q}_{BN}\} \\ \hat{Q}_D &= \text{diag}\{\hat{Q}_{D1}, \hat{Q}_{D2}, \cdots, \hat{Q}_{DN}\} \end{aligned}$$

with $\hat{Q}_{Ai} \in \mathcal{R}^{n_{ci}\times n_{ci}}$, $\hat{Q}_{Bi} \in \mathcal{R}^{n_i\times m_i}$, $\hat{Q}_{Di} \in \mathcal{R}^{m_i\times m_i}$*, and a matrix* W *structured as* (18) *such that the LMIs*

$$\Upsilon_1 + \Upsilon_1^T + \breve{C}_1^T\breve{C}_1 < 0 \tag{36}$$

$$\Upsilon_1 = (\breve{E}^T\hat{Q} + \breve{V}\tilde{R}^T\breve{U}^T)\breve{A} + \breve{E}^T\begin{bmatrix} W \\ 0 \end{bmatrix}\breve{C}_2$$

$$\text{trace}[\breve{B}_1^T\hat{Q}\breve{B}_1] < \gamma^2 \tag{37}$$

hold. Here, $\breve{E} = X\tilde{E}X^{-1}$, $\breve{A} = X\tilde{A}X^{-1}$, $\breve{B}_1 = X\tilde{B}_1$, $\breve{C}_1 = \tilde{C}_1X^{-1}$, $\breve{C}_2 = \tilde{C}_2X^{-1}$, $\breve{V} = \left(X^{-1}\right)^T\tilde{V}$, $\breve{U} = \left(X^{-1}\right)^T\tilde{U}$*, and* $X \in \mathcal{R}^{(n+n_c)\times(n+n_c)}$ *is a nonsingular matrix satisfying*

$$X\tilde{B}_2 = \begin{bmatrix} I_{n_c+m} \\ 0 \end{bmatrix}. \tag{38}$$

When the LMIs (36)-(37) *are feasible, one desired controller is computed as*

$$G_D = \hat{Q}_1^{-1} W\,. \tag{39}$$

Proof. Since we have assumed that B_2 is of full column rank, $\tilde{B}_2$ is also of full column rank, and thus there always exists a nonsingular matrix X such that (38) is satisfied. Also, the choice of X does not affect the feasibility of the LMIs (36)-(37). Pre-multiplying the first LMI (36) by X^T and post-multiplying it by X, and then substituting all the notations we defined with $\tilde{Q} = X^T \hat{Q} X$, we obtain

$$\tilde{\Upsilon}_1 + \tilde{\Upsilon}_1^T + \tilde{C}_1^T \tilde{C}_1 < 0 \tag{40}$$

$$\tilde{\Upsilon}_1 = (\tilde{E}^T \tilde{Q} + \tilde{V} \tilde{R}^T \tilde{U}^T)\tilde{A} + \tilde{E}^T X^T \begin{bmatrix} W \\ 0 \end{bmatrix} \tilde{C}_2\,.$$

According to (38) and (39), we compute

$$\hat{Q} X \tilde{B}_2 G_D = \begin{bmatrix} \hat{Q}_1 & 0 \\ 0 & \hat{Q}_2 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \hat{Q}_1^{-1} W = \begin{bmatrix} W \\ 0 \end{bmatrix}. \tag{41}$$

Together with the fact

$$\tilde{B}_2^T \tilde{U} = \begin{bmatrix} 0 & I_{\hat{n}} \\ B_2^T & 0 \end{bmatrix} \begin{bmatrix} U \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ B_{2e}^T E^T U \end{bmatrix} = 0\,, \tag{42}$$

we obtain (33) easily from (40)-(42).

Since the second LMI (37) is the same as (34), and the decentralized structure of $G_D = \hat{Q}_1^{-1} W$ can be proved by using the same technique as used in the proof of Theorem 1, we conclude that the system (1) is stabilized with $\mathcal{H}_2$ norm γ via the decentralized controller (3) given by (39). ∎

Remark 2. Although Theorems 1 and 2 come up with dual forms, they are not equivalent and are supposed to deal with different cases of Assumption 1 or Assumption 2, respectively. Furthermore, the LMI conditions provided by the theorems are sufficient ones. Therefore, even in the case where both Assumption 1 and Assumption 2 hold and thus both theorems can be applied, the LMI conditions of one theorem would be satisfied while the other would not. ∎

Remark 3. When it is necessary, we can try to obtain a tight $\mathcal{H}_2$ norm γ by considering the generalized eigenvalue problem (EVP) (Boyd *et al.*, 1994): "minimize γ^2, s.t. (19)-(20) or (36)-(37), respectively". ∎

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