PERTURBATION ANALYSIS FOR THE COMPLEX MATRIX EQUATION $\mathbf{X} - \mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\mathbf{A} = \mathbf{I}$

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Abstract: Condition numbers, local and non-local perturbation bounds are obtained for the complex matrix equation $X - A^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}A = I$, which arises in a number of problems of linear control systems theory. The technique used is based on Lyapunov majorants and fixed point principles. An illustrative numerical example is given.

Keywords: Perturbation analysis, Matrix equations

1. INTRODUCTION

In this paper a complete perturbation analysis of the complex matrix equation

$$F(\mathbf{X}, \mathbf{A}) := \mathbf{X} - \mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\mathbf{A} - \mathbf{I} = \mathbf{0}, \qquad (1)$$

with data matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, solution $\mathbf{X} \in \mathbb{C}^{n\times n}$ and $\mathbf{I}$ - the identity matrix is presented. This equation arises in control theory when solving systems of linear equations by LU decomposition. The sufficient conditions for existence of positive definite solution of (1) are proved in (Ivanov *et al.*, 1999). An iterative method for obtaining a positive definite solution is also proposed.

The more general non-linear matrix equation $\mathbf{X} + \mathbf{A}^*\mathcal{F}(\mathbf{X})\mathbf{A} = \mathbf{Q}$ is investigated in (El-Sayed and Ran, 2001; Ran and Reurings, 2002). The Hermitian positive definite solution of the equation and its properties are studied. Theorems of necessary and sufficient conditions for the existence of solution are proved. Perturbation upper bounds are obtained. Iterative methods for computing a positive definite solution are proposed. Particular cases of $\mathcal{F}(\mathbf{X})$ are discussed in (W. Anderson *et al.*, 1990; Engwerda *et al.*, 1993; Engwerda, 1993; Guo, 2001; Konstantinov, 2002; Konstantinov *et al.*, 2002*a*; Meini, 2001; Sun and Xu, 2003; Xu, 2001; Zhan, 1996) for $\mathcal{F}(\mathbf{X}) = \pm\mathbf{X}^{-1}$; in (Angelova, 2003; Ivanov *et al.*, 2001; Ivanov and El-Sayed, 1998; Zhang, 2003) for $\mathcal{F}(\mathbf{X}) = \pm\mathbf{X}^{-2}$; in (Hassanov and Ivanov, 2001; Hassanov and Ivanov, 2003; Ivanov and Georgieva, 2003; Ivanov and Hassanov, 2003) for $\mathcal{F}(\mathbf{X}) = \pm\mathbf{X}^{-n}$, $n = 2, 3, \ldots$

In practice the elements of the matrix $\mathbf{A}$ are contaminated with measurement errors. Also, when a numerically stable algorithm (Higham, 2002; Petkov *et al.*, 1991) is applied to solve (1) then the solution, computed in finite arithmetics, will be close to the solution of a near equation. Obtaining condition and accuracy estimates is important from both theoretical and computational point of view.

In this paper the conditioning of (1) and the sensitivity of its solution to perturbation in the data are studied. Local and non-local perturbation analysis are made. Perturbation bounds of the error in the solution are proposed.

The paper is organized as follows. In Section 2 a local perturbation analysis is presented. The perturbations $\delta\mathbf{A}$ in the data $\mathbf{A}$ and $\delta\mathbf{X}$ in the solution $\mathbf{X}$ are estimated in terms of the Frobenius matrix norm $\|\cdot\|_{\mathrm{F}}$. Explicit expression for the condition number of $\mathbf{X}$ relative to perturbation in $\mathbf{A}$ is obtained. Rewriting (1) as an equivalent matrix equation for the perturbation in the solution, non-linear non-local bound is obtained in Section 3. The technique used is based on Lyapunov majorants and fixed point principles (Konstantinov *et al.*, 2002*b*). In Section 4 a numerical example demonstrates the effectiveness of the bounds proposed.

Throughout the paper the following notations are used: $\mathbb{R}^{n\times n}$ – the space of $n\times n$ real matrices; $\mathbb{C}^{n\times n}$ – the space of $n\times n$ complex matrices; $\mathbb{R}_+ = [0,\infty)$; $\mathbf{A}^\top$ – the transpose of the matrix $\mathbf{A}$; $\bar{\mathbf{A}}$ - the complex conjugate of $\mathbf{A}$; $\mathbf{A}^{\mathrm{H}} = \bar{\mathbf{A}}^\top$; $\mathrm{vec}(\mathbf{A}) \in \mathbb{C}^{n^2}$ – the column-wise vector representation of the matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$; $\mathrm{Mat}(L) \in \mathbb{C}^{n^2\times n^2}$ – the matrix representation of the linear matrix operator $L : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$; $\mathbf{I}_n$ – the identity $n\times n$ matrix; $\mathbf{A}\otimes\mathbf{B} = [a_{pq}\mathbf{B}]$ – the Kronecker product of the matrices $\mathbf{A} = [a_{pq}]$ and $\mathbf{B}$; $\|\cdot\|_2$ – the Euclidean norm in $\mathbb{C}^n$ or the spectral (or 2-) norm in $\mathbb{C}^{n\times n}$; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius (or F-) norm in $\mathbb{C}^{n\times n}$; $\|\cdot\|$ – a replacement of either $\|\cdot\|_2$ or $\|\cdot\|_{\mathrm{F}}$; $\mathbf{z}^{\mathcal{R}} = \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \in \mathbb{R}^{2n}$ - the real version of $\mathbf{z} = \begin{bmatrix}\mathbf{x}+\mathbf{i}\mathbf{y}\end{bmatrix} \in \mathbb{C}^n$, with $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$; $\mathbf{\Gamma}^{\mathcal{R}} \in \mathbb{R}^{2n\times 2n}$ - the real version of $\mathbf{\Gamma}\in\mathbb{C}^{n\times n}$; $\mathcal{L}_n(\mathbb{C})$ – the space of linear operators $\mathbb{C}^{n\times n}\to\mathbb{C}^{n\times n}$.

The notation ':=' stands for 'equal by definition'.

2. LOCAL PERTURBATION ANALYSIS

Suppose that (1) has a positive definite solution $\mathbf{X}$. Let $\mathbf{A}$ and $\mathbf{X}$ be slightly perturbed to $\mathbf{A}+\delta\mathbf{A}$, $\mathbf{X}+\delta\mathbf{X}$, where $\delta\mathbf{A},\delta\mathbf{X}\in\mathbb{C}^{n\times n}$. Let the number $\alpha \geq 0$ be given and suppose that $\|\delta\mathbf{A}\|_{\mathrm{F}} = \alpha$. The perturbed equation is obtained from (1) by replacing the nominal value $\mathbf{A}$ with $\mathbf{A}+\delta\mathbf{A}$ and $\mathbf{X}$ with $\mathbf{X}+\delta\mathbf{X}$

$$F(\mathbf{X}+\delta\mathbf{X},\mathbf{A}+\delta\mathbf{A}) = 0. \qquad (2)$$

For α sufficiently small equation (2) has a solution $\mathbf{X}+\delta\mathbf{X}$, depending on $\delta\mathbf{A}$.

Rewrite (2) as an equivalent equation for the perturbation $\delta\mathbf{X}$ in $\mathbf{X}$

$$\begin{aligned} F(\mathbf{X}+\delta\mathbf{X},\mathbf{A}+\delta\mathbf{A}) &= F(\mathbf{X},\mathbf{A}) \qquad (3)\\ &+ F_X(\mathbf{X},\mathbf{A})(\delta\mathbf{X}) + F_{\mathbf{A}}(\mathbf{X},\mathbf{A})(\delta\mathbf{A})\\ &+ F_{\bar{\mathbf{A}}}(\mathbf{X},\mathbf{A})(\overline{\delta\mathbf{A}}) + G(\mathbf{X},\mathbf{A})(\delta\mathbf{X},\delta\mathbf{A}), \end{aligned}$$

where

$$F_X(\mathbf{X},\mathbf{A})(\mathbf{Y}) := \mathbf{Y} + \frac{1}{2}\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\mathbf{Y}\mathbf{X}^{-1}\mathbf{A}$$

$$F_X(\mathbf{X},\mathbf{A})(\mathbf{Y}) \in \mathcal{L}_n(\mathbb{C})$$

is the partial Fréchet derivative of F in $\mathbf{X}$ calculated at the point $(\mathbf{X},\mathbf{A})$, and

$$F_{\mathbf{A}}(\mathbf{X},\mathbf{A})(\delta\mathbf{A}) + F_{\bar{\mathbf{A}}}(\mathbf{X},\mathbf{A})(\overline{\delta\mathbf{A}}) \in \mathcal{L}_n(\mathbb{C}),$$

with

$$F_{\mathbf{A}}(\mathbf{X},\mathbf{A})(\mathbf{Y}) := \mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\mathbf{Y},$$
$$F_{\bar{\mathbf{A}}}(\mathbf{X},\mathbf{A})(\mathbf{Y}) := \mathbf{Y}\sqrt{\mathbf{X}^{-1}}\mathbf{A}$$

is the Fréchet pseudo-derivative in $\mathbf{A}$, computed at the point $(\mathbf{X},\mathbf{A})$. The term $G(\mathbf{X},\mathbf{A})(\delta\mathbf{X},\delta\mathbf{A})$ contains second and higher order terms in $\delta\mathbf{X}$, $\delta\mathbf{A}$.

The matrix of the operator $F_X(\mathbf{X},\mathbf{A})$ is

$$\mathbf{L} := I_{n^2} + \frac{1}{2}(\mathbf{X}^{-1}\mathbf{A})^\top \otimes (\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}). \qquad (4)$$

The eigenvalues of $F_X(\mathbf{X},\mathbf{A})$ are the eigenvalues of its matrix $\mathbf{L}$ and are equal to

$$1+\lambda_i\left(\mathbf{X}^{-1}\mathbf{A}\right)\mu_j\left(\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\right), \quad i,j = 1,2,\ldots,n.$$

Here $\lambda_i(\mathbf{Z})$, $\mu_j(\mathbf{Z})$ are the eigenvalue of the matrix $\mathbf{Z}$.

The operator $F_X(\mathbf{X},\mathbf{A})$ and its matrix $\mathbf{L}$ are invertible iff

$$\lambda_i\left(\mathbf{X}^{-1}\mathbf{A}\right)\mu_j\left(\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\right) \neq -1. \qquad (5)$$

In what follows it is assumed that the inequalities (5) hold true.

Substracting $F(\mathbf{X},\mathbf{A}) = 0$ from

$$F(\mathbf{X}+\delta\mathbf{X},\mathbf{A}+\delta\mathbf{A}) = 0$$

and assuming that the operator $F_X(\mathbf{X},\mathbf{A})$ is invertible, from (3) one has

$$\delta\mathbf{X} = -F_X^{-1}\circ F_A(\delta\mathbf{A}) - F_X^{-1}\circ F_{\bar{\mathbf{A}}}(\overline{\delta\mathbf{A}}) + O(\alpha^2),$$
$$\alpha\to 0$$

and

$$\mathbf{x} = \mathbf{M}_1\mathbf{a} + \mathbf{M}_2\mathbf{a} + O(\|\mathbf{a}\|^2),\ \mathbf{a}\to 0. \qquad (6)$$

Here $\mathbf{x} := \mathrm{vec}(\delta\mathbf{X})$, $\mathbf{a} := \mathrm{vec}(\delta\mathbf{A})$ are n^2-vectors,

$$\mathbf{M}_1 = -\mathbf{L}^{-1}\mathbf{L}_{\mathbf{A}} \in \mathbb{C}^{n^2\times n^2}$$

is the matrix of the operator $-F_X^{-1}\circ F_A$,

$$\mathbf{M}_2 = -\mathbf{L}^{-1}\mathbf{L}_{\bar{\mathbf{A}}} \in \mathbb{C}^{n^2\times n^2}$$

is the matrix of the operator $-F_X^{-1}\circ F_{\bar{\mathbf{A}}}$ with

$$\mathbf{L_A} = -\mathbf{I}_n \otimes (\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}) \in \mathbb{C}^{n^2\times n^2}$$
$$\mathbf{L_{\bar{A}}} = -((\sqrt{\mathbf{X}^{-1}}\mathbf{A})^{\top} \otimes \mathbf{I}_n)\mathbf{P}_{n^2} \in \mathbb{C}^{n^2\times n^2},$$

where $\mathbf{P}_{n^2} \in \mathbb{C}^{n^2\times n^2}$ is the so called vec-permutation matrix such that $\mathrm{vec}(\mathbf{Y}^{\top}) = \mathbf{P}_{n^2}\mathrm{vec}(\mathbf{Y})$ for each $\mathbf{Y} \in \mathbb{C}^{n\times n}$.

For the real version of $\mathbf{x}$ it is fulfilled (Konstantinov *et al.*, 2001; Konstantinov *et al.*, 2002*b*)

$$\mathbf{x}^{\mathcal{R}} = \mathbf{\Theta}(\mathbf{M}_1, \mathbf{M}_2)\mathbf{a}^{\mathcal{R}} + O(\|\mathbf{a}^{\mathcal{R}}\|^2),\ \mathbf{a}^{\mathcal{R}} \to 0,$$

where for

$$\mathbf{M}_1 = \mathbf{M}_{10} + i\mathbf{M}_{11} \quad \text{and} \quad \mathbf{M}_2 = \mathbf{M}_{20} + i\mathbf{M}_{21}$$

(with $\mathbf{M}_{10}$, $\mathbf{M}_{11}$, $\mathbf{M}_{20}$, $\mathbf{M}_{21}$ real) the matrix $\mathbf{\Theta}$ is

$$\mathbf{\Theta}(\mathbf{M}_1, \mathbf{M}_2) := \begin{bmatrix} \mathbf{M}_{10} + \mathbf{M}_{20} & \mathbf{M}_{21} - \mathbf{M}_{11} \\ \mathbf{M}_{11} + \mathbf{M}_{21} & \mathbf{M}_{10} - \mathbf{M}_{20} \end{bmatrix} \quad (7)$$

$$\mathbf{\Theta}(\mathbf{M}_1, \mathbf{M}_2) \in \mathbb{R}^{n^2\times n^2}$$

The real version $\mathbf{a}^{\mathcal{R}}$ of the complex vector $\mathbf{a} = a_1 + ia_2 \in \mathbb{C}^n$ (with a_1, a_2 real) is

$$\mathbf{a}^{\mathcal{R}} := \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

The matrices $\mathbf{M}_1$, $\mathbf{M}_2$ are

$$\mathbf{M}_1 = \mathbf{L}^{-1}(\mathbf{I}_n \otimes (\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}})),$$
$$\mathbf{M}_2 = \mathbf{L}^{-1}(((\sqrt{\mathbf{X}^{-1}}\mathbf{A})^{\top} \otimes \mathbf{I}_n)\mathbf{P}_{n^2}).$$

Denote

$$\alpha := \|\delta\mathbf{A}\|_{\mathrm{F}} \in \mathbb{R}_+,$$
$$\xi := \|\delta\mathbf{X}\|_{\mathrm{F}} = \|\mathrm{vec}(\delta\mathbf{X})\|_2 = \|\mathbf{x}\|_2$$

and since $\|\mathbf{x}\|_2 = \|\mathbf{x}^{\mathcal{R}}\|_2$ the following local estimate is obtained

$$\xi \leq \mathrm{est}(\alpha) + O(\alpha^2),\ \alpha \to 0, \quad (8)$$
$$\mathrm{est}(\alpha) := c\alpha,\ \ c := \|\mathbf{\Theta}(\mathbf{M}_1, \mathbf{M}_2)\|_2 \in \mathbb{R}_+,$$

where the matrix $\mathbf{\Theta}(\mathbf{M}_1, \mathbf{M}_2)$ is given by (7). Here c is the absolute condition number. The relative condition number is then computed from $\gamma = c\|\mathbf{A}\|_{\mathrm{F}}/\|\mathbf{X}\|_{\mathrm{F}}$.

The estimate (8) allows to define the overall relative condition number as follows. Let $\alpha = \varepsilon\|\mathbf{A}\|_{\mathrm{F}}$, where $\varepsilon > 0$ (in floating point arithmetic the quantity ε may be taken as a multiple of the rounding unit). Then $\mathrm{est}(\alpha) = \varepsilon\,\mathrm{est}(\|\mathbf{A}\|_{\mathrm{F}})$. Hence the relative perturbation in the solution can be estimated as

$$\frac{\|\delta\mathbf{X}\|_{\mathrm{F}}}{\|\mathbf{X}\|_{\mathrm{F}}} \leq \varepsilon\frac{\mathrm{est}(\|\mathbf{A}\|_{\mathrm{F}})}{\|\mathbf{X}\|_{\mathrm{F}}}.$$

The estimate (8) is valid only asymptotically. This means that the perturbations in the data must be small enough to ensure sufficient accuracy of the linear estimate. This disadvantage of the local estimate may be overcome using the techniques of non-local perturbation analysis, presented below.

3. NON-LOCAL PERTURBATION ANALYSIS

The perturbed equation (2) may be written in the form

$$F_X(\delta\mathbf{X}) = \Phi_0(\delta\mathbf{A}) + \Phi_1(\delta\mathbf{X}, \delta\mathbf{A}) + \Phi_2(\delta\mathbf{X}, \delta\mathbf{A}) + O(\|\delta\mathbf{X}\|^3),$$

where

$$\Phi_0(\delta\mathbf{A}) := \mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\delta\mathbf{A} + \delta\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\mathbf{A} + \delta\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\delta\mathbf{A},$$
$$\Phi_1(\delta\mathbf{X}, \delta\mathbf{A}) := -\frac{1}{2}\Big(\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\delta X\mathbf{X}^{-1}\delta\mathbf{A} + \delta\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\delta X\mathbf{X}^{-1}\mathbf{A} + \delta\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\delta X\mathbf{X}^{-1}\delta\mathbf{A}\Big),$$
$$\Phi_2(\delta\mathbf{X}, \delta\mathbf{A}) := \frac{3}{8}(\mathbf{A} + \delta\mathbf{A})^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}(\delta\mathbf{X}\mathbf{X}^{-1})^2 \times (\mathbf{A} + \delta\mathbf{A}).$$

The above relations are obtained using the approximation (according Theorem 5.6.3 in (Lancaster, 1969))

$$\sqrt{(\mathbf{X} + \delta\mathbf{X})^{-1}} = \sqrt{\mathbf{X}^{-1}} - \frac{1}{2}\sqrt{\mathbf{X}^{-1}}\delta\mathbf{X}\mathbf{X}^{-1} + \frac{3}{8}\sqrt{\mathbf{X}^{-1}}(\delta\mathbf{X}\mathbf{X}^{-1})^2 + O(\|\delta\mathbf{X}\|^3).$$

As a result, neglecting the terms of third and higher order in $\delta\mathbf{X}$ the operator equation

$$\delta\mathbf{X} = \Pi(\delta\mathbf{X}, \delta\mathbf{A}) \quad (9)$$
$$\Pi(\delta\mathbf{X}, \delta\mathbf{A}) := \Pi_0(\delta\mathbf{A}) + \Pi_1(\delta\mathbf{X}, \delta\mathbf{A}) + \Pi_2(\delta\mathbf{X}, \delta\mathbf{A})$$

is obtained, where $\Pi_i = F_X^{-1}(\Phi_i)$, $i = 0, 1, 2$.

Suppose that $\xi = \|\delta\mathbf{X}\|_{\mathrm{F}} \leq \rho$, where ρ is a positive quantity. Then, after some calculations, the inequality

$$\|F(\delta\mathbf{X}, \delta\mathbf{A})\|_{\mathrm{F}} \leq h(\rho, \alpha),$$
$$h(\rho, \alpha) := a_0(\alpha) + a_1(\alpha)\rho + a_2(\alpha)\rho^2$$

is obtained.

Here

$$a_0(\alpha) := \mathrm{est}(\alpha) + l\mu\alpha^2,$$
$$a_1(\alpha) := a_{11}\alpha + \frac{1}{2}l\mu\|\mathbf{X}^{-1}\|_2\alpha^2,$$
$$a_2(\alpha) := \frac{3}{8}\mu\|\mathbf{X}^{-1}\|_2^2(\|A\|_2 + \alpha)^2,$$
$$a_{11} := \frac{1}{2}\|\mathbf{X}^{-1}\|_2\left\|\mathbf{L}^{-1}\left(\mathbf{I}_n \otimes \left(\mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\right)\right)\right\|_2$$

$$+\frac{1}{2}\mu\left\|\mathbf{L}^{-1}\left(\left(\mathbf{X}^{-1}\mathbf{A}\right)^{\top}\otimes\mathbf{I_n}\right)\mathbf{P}_{n^2}\right\|_2,$$
$$l := \|\mathbf{L}^{-1}\|_2, \quad \mu := \|\sqrt{\mathbf{X}^{-1}}\|_2.$$

The function h is a Lyapunov majorant for the operator Π, see (Grebenikov and Ryabov, 1979; Konstantinov *et al.*, 2002*b*; Konstantinov *et al.*, 1996). The corresponding majorant equation

$$\rho = h(\rho, \alpha)$$

is equivalent to the quadratic equation

$$a_2(\alpha)\rho^2 - (1 - a_1(\alpha))\rho + a_0(\alpha) = 0.$$

Consider the domain

$$\Omega := \left\{\alpha \in \mathbb{R}_+ \ : \ a_1(\alpha) + 2\sqrt{a_0(\alpha)a_2(\alpha)} < 1\right\}. \quad (10)$$

If $\alpha \in \Omega$ then the majorant equation $\rho = h(\rho, \alpha)$ has a root

$$\rho(\alpha) = f(\alpha) \quad (11)$$

$$f(\alpha) := \frac{2a_0(\alpha)}{1 - a_1(\alpha) + \sqrt{(1 - a_1(\alpha))^2 - 4a_0(\alpha)a_2(\alpha)}}.$$

Hence for $\alpha \in \Omega$ the operator $\Pi(., \delta\mathbf{A})$ maps the set $\mathcal{B}_{f(\alpha)}$ into itself, where

$$\mathcal{B}_r := \left\{\mathbf{x} \in \mathbb{C}^{n^2} \ : \ \|\mathbf{x}\|_2 \le r\right\}$$

is the closed central ball of radius $r \geq 0$. Then according to Schauder fixed point principle (Kantorovich and Akilov, 1964; Ortega and Rheinboldt, 2000) there exists a solution $\delta\mathbf{X} \in \mathcal{B}_{f(\alpha)}$ of equation (9).

Thus the following result is obtained.

Theorem. *Let $\alpha \in \Omega$, where Ω is given in* (10). *Then the non-local perturbation bound $\|\delta\mathbf{X}\|_{\mathrm{F}} \le f(\alpha)$ is valid for equation* (1), *where $f(\alpha)$ is determined by* (11).

4. NUMERICAL EXAMPLE

Consider the complex matrix equation

$$\mathbf{X} - \mathbf{A}^{\mathrm{H}}\sqrt{\mathbf{X}^{-1}}\mathbf{A} = \mathbf{I}$$

with matrices $\mathbf{X} = \mathbf{V} * \mathbf{X_0} * \mathbf{V}$, $\mathbf{A} = \mathbf{V} * \mathbf{A_0} * \mathbf{V}$,

$$\mathbf{A_0} = \mathrm{diag}\left[\,6,\ 0.1 + 0.1i,\ 6\,\right],$$
$$\mathbf{X_0} = \mathrm{diag}\left[\,11.579,\ 1.0198,\ 11.579\,\right],$$

and $\mathbf{V}$ is the elementary reflection

$$\mathbf{V} = \mathbf{I_3} - 2 * \mathbf{v} * \mathbf{v}^{\top}, \quad \mathbf{v} = \left[\,1\ 1\ 1\,\right]^{\top}.$$

The perturbation in the data is taken as

$$\delta\mathbf{A_0} = \mathrm{diag}\left[\,1.8663\ 1.0371\ 1.0026\,\right] * 10^{(-k)}$$

for $k = 10, 9, ..., 1$.

This problem was designed so as to obtain the analytic solution

$$\mathbf{X} = \mathrm{diag}\left[\,x_1\ x_2\ x_3\,\right]$$

and

$$\mathbf{X} + \delta\mathbf{X} = \mathrm{diag}\left[\,x_1 + \delta x_1\ x_2 + \delta x_2\ x_3 + \delta x_3\,\right]$$

of the unperturbed and perturbed equation respectively. According to the physical applications (Ivanov *et al.*, 1999) the positive definite solution of (1) is of interest. One has

$$x_i = y_i^2, \quad \delta x_i = (y_i + \delta y_i)^2 - y_i^2, \quad i = 1, 2, 3,$$
$$y_i = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{1}{27}}},$$
$$y_i + \delta y_i = \sqrt[3]{\frac{q_d}{2} + \sqrt{\frac{q_d^2}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{q_d}{2} - \sqrt{\frac{q_d^2}{4} - \frac{1}{27}}},$$
$$q = \mathrm{Re}(A_0(i,i))^2 + \mathrm{Im}(A_0(i,i))^2,$$
$$q_d = (\mathrm{Re}(A_0(i,i)) + \mathrm{Re}(\delta A_0(i,i)))^2 + (\mathrm{Im}(A_0(i,i)) + \mathrm{Im}(\delta A_0(i,i)))^2.$$

The perturbation $\|\delta\mathbf{X}\|_F$ in the solution is estimated by the local bound $\mathrm{est}(\alpha)$ (8) from Section 2 and the non-local bound $\rho(\alpha)$ (11), (10) from Section 3.

The results obtained for different values of k are shown in Table 1.

Table 1.

k	$\|\delta\mathbf{X}\|_F$	$\mathrm{est}(\alpha)$ (8)	$\rho(\alpha)$ (11)
10	5.1350×10^{-10}	5.7098×10^{-10}	5.7098×10^{-10}
9	5.1327×10^{-9}	5.7098×10^{-9}	5.7098×10^{-9}
8	5.1324×10^{-8}	5.7098×10^{-8}	5.7098×10^{-8}
7	501323×10^{-7}	5.7098×10^{-7}	5.7098×10^{-7}
6	5.1323×10^{-6}	5.7098×10^{-6}	5.7102×10^{-6}
5	5.1323×10^{-5}	5.7098×10^{-5}	5.7142×10^{-5}
4	5.1324×10^{-4}	5.7098×10^{-4}	5.7543×10^{-4}
3	5.1326×10^{-3}	5.7098×10^{-3}	6.2299×10^{-3}
2	5.1351×10^{-2}	5.7098×10^{-2}	*
1	5.1611×10^{-1}	5.7098×10^{-1}	*

The case when the non-local estimate is not valid, since the existence condition $\alpha \in \Omega$ is violated, is denoted by asterisk. When k decreases from 10 to 1 the non-local estimate $\rho(\alpha)$ (11) is slightly more pessimistic than the local bound $\mathrm{est}(\alpha)$ (8).

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