CLOSED LOOP IDENTIFICATION OF UNSTABLE POLES AND NON-MINIMUM PHASE ZEROS

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Abstract: This paper addresses estimation of poles and zeros in closed loop systems. For many quantities of interest, e.g. frequency function estimates, overparameterization results in a large increase of the variance but this is not the case for estimates of non-minimum phase zeros and unstable poles. Variance expressions that are asymptotic in model order and sample size are derived and for some systems it is found that open loop and closed loop experiments give the same accuracy.

Keywords: Closed-loop identification, Non-minimum phase systems, Analysis of variance.

1. INTRODUCTION

Quantification of the variance of estimated poles and zeros is closely related to variance quantification for estimated frequency functions as all these problems deal with quadratic forms based on the covariance matrix of the underlying parameter estimate. The latter problem has received significant interest. In the mid-eighties, an expression for the asymptotic (as the sample size grows) variance of estimated frequency functions was presented in (Ljung and Yuan, 1985; Ljung, 1985). It showed that the variance increases proportionally to the model order regardless of model structure as the model order becomes large. An alternative asymptotic variance expression with, for many model structures, improved accuracy was proposed in (Ninness *et al.*, 1999). In (Xie and Ljung, 2001) and (Ninness and Hjalmarsson, 2004) expressions that are exact for finite model orders were derived for the variance of estimated frequency functions. Closed loop estimation is treated in the same framework in (Ninness and Hjalmarsson, 2003).

Parallel to this, there has been a series of results regarding the accuracy of non-minimum phase zeros estimated in open loop. As mentioned above the variance of an estimate usually increases proportionally with the model order, but estimates of non-minimum phase zeros only suffer from a moderate increase in the variance. This was shown for FIR-models in (Lindqvist, 2001) and ARX-models in (Hjalmarsson and Lindqvist, 2002). More general models, such as output-error and Box-Jenkins, were treated in (Mårtensson and Hjalmarsson, 2003).

In this paper these results are extended to *closed loop identification* of non-minimum phase zeros and unstable poles. The main result still holds for closed loop identification: using high order models when estimating non-minimum phase zeros and unstable poles only gives a small increase in the variance.

In Section 2 the parameter estimation method is described. Estimation of poles and zeros is introduced in Section 3 and closed loop identification of ARX-systems is covered in Section 4. Direct and indirect identification methods are compared in Section 5. Section 6 describes a method that can be applied to some model structures other than ARX and the main conclusions are summarized in Section 7.

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2. PARAMETER ESTIMATION

The setup is the standard prediction error method, see e.g. (Ljung, 1999), which is briefly outlined in the following. Assume a model structure parameterized by the vector θ,

$$y_t = G(q,\theta)u_t + H(q,\theta)e_t, \tag{1}$$

where $G(q,\theta)$ and $H(q,\theta)$ are rational transfer functions, $H(q,\theta)$ is monic and $\{e_t\}$ is a zero-mean white noise sequence. The parameter vector is estimated by minimizing the sum of squared prediction errors,

$$\hat{\theta}_N = \arg\min_{\theta} \frac{1}{N}\sum_{t=1}^{N} \varepsilon_t^2(\theta), \tag{2}$$

where the prediction error is given by

$$\varepsilon_t(\theta) = \frac{1}{H(q,\theta)}\left(y_t - G(q,\theta)u_t\right). \tag{3}$$

Assume that there exists a description of the underlying true system within the model structure, i.e. there is a parameter vector θ^o and a zero-mean white noise sequence $\{e_t^o\}$ with variance λ_0 such that

$$y_t = G(q,\theta^o)u_t + H(q,\theta^o)e_t^o. \tag{4}$$

Then, under some mild conditions, the parameter estimate has the statistical properties

$$\lim_{N\to\infty} \hat{\theta}_N = \theta^o, \quad \text{w.p.1}, \tag{5}$$

$$\lim_{N\to\infty} N \operatorname{Cov}\hat{\theta}_N = \lambda_0\left(\mathbf{E}\psi_t(\theta^o)\psi_t^T(\theta^o)\right)^{-1}, \tag{6}$$

$$\psi_t(\theta^o) = -\frac{\partial}{\partial\theta}\varepsilon_t(\theta)\Big|_{\theta=\theta^o}. \tag{7}$$

In this paper, the notation $\overline{\operatorname{Cov}}\,\hat{\theta} = \lim_{N\to\infty} N\operatorname{Cov}\hat{\theta}_N$ and $\overline{var}\ \hat{\theta} = \lim_{N\to\infty} N \operatorname{var}\ \hat{\theta}_N$ is used.

3. POLES AND ZEROS

In this section the variances of estimated poles and zeros are related to the covariance matrix of the parameter estimates. For non-minimum phase zeros of ARX-systems, identified in open loop, there are asymptotic (in model order) results that relate the variance directly to the true system.

An ARX-system is described as

$$y_t = \frac{B(q,\theta)}{A(q,\theta)}u_t + \frac{1}{A(q,\theta)}e_t, \tag{8}$$

where

$$A(q,\theta) = A(q,\theta_a) = 1 + a_1 q^{-1} + \cdots + a_{n_a}q^{-n_a}, \tag{9a}$$

$$B(q,\theta) = B(q,\theta_b) = b_1 q^{-1} + \cdots + b_{n_b}q^{-n_b}. \tag{9b}$$

The parameter vectors are

$$\theta_a = \begin{pmatrix} a_1 & \cdots & a_{n_a}\end{pmatrix}^T, \quad \theta_b = \begin{pmatrix} b_1 & \cdots & b_{n_b}\end{pmatrix}^T, \tag{10}$$

$$\theta^T = \begin{pmatrix}\theta_a^T & \theta_b^T\end{pmatrix}. \tag{11}$$

The poles p_k and zeros z_k are defined as the roots of the polynomials

$$p_k:\quad z^n + a_1 z^{n_a-1} + \cdots + a_{n_a} = 0, \tag{12a}$$

$$z_k:\quad b_1 z^{n_b-1} + \cdots + b_{n_b} = 0. \tag{12b}$$

Let the zeros be denoted by $z_k^o = z_k(\theta^o)$, $\hat{z}_k = z_k(\hat{\theta})$ and similarly for the poles. The variances of the estimated poles and zeros can be calculated using a first order Taylor approximation, see (Lindqvist, 2001). For a discussion on the accuracy of these Taylor approximations, see (Vuerinckx *et al.*, 2001). The asymptotic variances of the estimated poles and zeros are

$$\overline{var}\ \hat{p}_k = \frac{|p_k^o|^2}{|\widetilde{A}_0(p_k^o)|^2}\Gamma_n^T(p_k^o)P_a\Gamma_n(p_k^o), \tag{13a}$$

$$\overline{var}\ \hat{z}_k = \frac{|z_k^o|^2}{|\widetilde{B}_0(z_k^o)|^2}\Gamma_m^T(z_k^o)P_b\Gamma_m(z_k^o), \tag{13b}$$

where the following notation is used

$$\Gamma_n(z) = \begin{pmatrix} z^{-1} & \cdots & z^{-n}\end{pmatrix}^T, \tag{14}$$

$$\widetilde{A}_0(q) = \frac{A(q,\theta_a^o)}{1-p_k^o q^{-1}}, \quad \widetilde{B}_0(q) = \frac{B(q,\theta_b^o)}{1-z_k^o q^{-1}}, \tag{15}$$

$$P = \overline{\operatorname{Cov}}\,\hat{\theta} = \begin{pmatrix} P_a & P_{ab} \\ P_{ba} & P_b \end{pmatrix}. \tag{16}$$

The parameter covariance $\overline{\operatorname{Cov}}\,\hat{\theta}$ can now be calculated, see (6) and (7). The ARX-model (8) can be expressed in predictor form

$$\hat{y}_t = \theta^T\psi_t(\theta) \tag{17}$$

where the gradient of the prediction error is

$$\psi_t(\theta) = \begin{pmatrix} -\Gamma_{n_a}(q)y_t \\ \Gamma_{n_b}(q)u_t \end{pmatrix} \tag{18}$$

and the parameter covariance is

$$P = \lambda_0 \begin{pmatrix} R_{yy} & -R_{yu} \\ -R_{uy} & R_{uu}\end{pmatrix}^{-1}, [R_{uv}]_{(i,j)} = \mathbf{E}u_{t-i}v_{t-j}. \tag{19}$$

The following result will be useful in the sequel.

Lemma 3.1. Suppose that $|z| > 1$ is a zero of $B(q)$ and that $y_t = \frac{B(q)}{A(q)}u_t + \frac{1}{A(q)}e_t$ is a stable system. Let e_t be a zero-mean white noise sequence with variance λ, independent of u_t and let $u_t = Q(q)v_t$ where $Q(q)$ is a minimum phase transfer function and v_t is a zero-mean white noise sequence with variance 1. Further, let $n_a = n_b = n$. Then it holds that

$$\lim_{n\to\infty}\Gamma_n^T(z)P_b\Gamma_n(z) = \frac{\lambda|z|^{-2}}{(1-|z|^{-2})|Q(z)|^2} \tag{20}$$

Proof: See (Hjalmarsson and Lindqvist, 2002). □

Now, by combining (13b) and (20) the asymptotic (in model order) variance of a non-minimum phase zero estimated in open loop can be expressed in terms of the true system and the input filter.

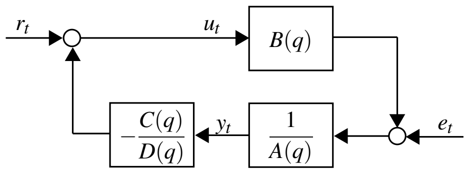

Fig. 1. Closed loop system.

4. CLOSED LOOP IDENTIFICATION

Closed loop identification of ARX-systems is addressed in this section. Lemma 3.1 can not be applied directly in this scenario since the input is correlated with the output. A special parametrization of the closed loop system will circumvent this problem.

Assume that the ARX-system (8) (with $\theta = \theta^0$) is operating in a stable closed loop where the linear time-invariant (LTI) feedback controller is assumed to be known and minimum phase. The input $\{u_t\}$ is generated as

$$u_t = r_t - F(q)y_t \tag{21}$$

where $F(q) = \frac{C(q)}{D(q)}$ and

$$\begin{aligned} C(q) &= c_0 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}, \\ D(q) &= 1 + d_1 q^{-1} + \cdots + d_{n_d} q^{-n_d}. \end{aligned} \tag{22}$$

The reference signal r_t is assumed to be a known sequence of filtered white noise. The closed loop system is depicted in Figure 1.

The closed loop system can be written in the following form, (where the argument q is dropped)

$$\begin{aligned} u_t &= \frac{A(\theta_a)D}{A(\theta_a)D + B(\theta_b)C} r_t - \frac{C}{A(\theta_a)D + B(\theta_b)C} e_t, \\ y_t &= \frac{B(\theta_b)D}{A(\theta_a)D + B(\theta_b)C} r_t + \frac{D}{A(\theta_a)D + B(\theta_b)C} e_t. \end{aligned} \tag{23}$$

Here the closed loop system is parameterized in the open loop parameters θ. Since the controller is assumed to be known and LTI, there is no difference between using direct identification with the open loop model (8) and using indirect identification with either of the closed loop models (23), see (Forssell and Ljung, 1999). The covariance matrix of the parameter estimate is given by (19).

In open loop where u_t and e_t are independent, Lemma 3.1 can be applied to get an expression for the asymptotic variance of the poles and zeros (if they are non-minimum phase). In closed loop there is feedback from e_t to u_t and Lemma 3.1 can not be used. However, this problem can be avoided by parameterizing the closed loop system in another way. Consider indirect identification in the closed loop model structure

$$u_t = \frac{A(\theta_a)D}{X(\theta_x)} r_t - \frac{C}{X(\theta_x)} e_t, \tag{24a}$$

$$y_t = \frac{B(\theta_b)D}{X(\theta_x)} r_t + \frac{D}{X(\theta_x)} e_t, \tag{24b}$$

where

$$X(\theta_x) = 1 + x_1 q^{-1} + \cdots + x_{n_x} q^{-n_x} \tag{25}$$

and

$$n_x \geq \max\{n_a + n_d, n_b + n_c\}. \tag{26}$$

Notice that the parametrizations above are of ARX-type where r_t takes the role of the input (cf (8)). The reason for introducing these parametrizations is that r and e are independent in closed loop operation. This will allow the application of Lemma 3.1.

By introducing $y^a = \frac{1}{C}u - \frac{D}{C}r$ and $y^b = \frac{1}{D}y$ the closed loop models (24) can be written in predictor form as linear regressions

$$\hat{y}_t^a = \eta_a^T \psi_t(\eta_a), \quad \hat{y}_t^b = \eta_b^T \psi_t(\eta_b), \tag{27}$$

where the parameter vectors and regression vectors are defined as

$$\begin{aligned} \eta_a &= \begin{pmatrix} \theta_a \\ \theta_x \end{pmatrix}, \quad \psi_t(\eta_a) = \begin{pmatrix} \Gamma_{n_a}(q)\frac{D(q)}{C(q)} r_t \\ -\Gamma_{n_x}(q)\frac{1}{C(q)} u_t \end{pmatrix}, \\ \eta_b &= \begin{pmatrix} \theta_x \\ \theta_b \end{pmatrix}, \quad \psi_t(\eta_b) = \begin{pmatrix} -\Gamma_{n_x}(q)\frac{1}{D(q)} y_t \\ \Gamma_{n_b}(q) r_t \end{pmatrix}. \end{aligned} \tag{28}$$

In this treatment the ARX-system (8) is assumed to have either one unstable pole p_k or one non-minimum phase zero z_k. The system is estimated in either the model structure (24a) or (24b), depending on whether the system has a non-minimum phase zero or unstable pole. The poles or zeros are calculated from the estimates of θ_a in (24a) or θ_b in (24b) respectively. If the model order is increased, the accuracy of the estimated poles and zeros can be evaluated using Lemma 3.1. The results are presented in Theorem 4.1 and Theorem 4.2 below.

Theorem 4.1. Let $r_t = Q(q)v_t$, where v_t is a zero-mean white noise sequence with variance 1 and $Q(q)$ is a minimum phase transfer function. Suppose that an unstable ARX-system (8) with stabilizing feedback (21) is identified in the model structure (24a) with $n_x = n_a = n$. Let p_k^o be the unstable pole. Then, for high model orders, the variance of the estimated pole is

$$\boxed{\lim_{n\to\infty} \overline{var}\ \hat{p}_k = \frac{\lambda_0}{(1-|p_k^o|^{-2})} \frac{|F(p_k^o)|^2}{|\widetilde{A}(p_k^o)|^2 |Q(p_k^o)|^2}.} \tag{29}$$

Proof: Let $\tilde{r}_t = \frac{D}{C} r_t$ and $\tilde{e}_t = -e_t$. Then the gradient of the prediction error is, see (28),

$$\psi_t(\eta_a) = \begin{pmatrix} \Gamma_{n_a}\tilde{r}_t \\ -\Gamma_{n_x}\tilde{y}_t \end{pmatrix}, \quad \tilde{y}_t = \frac{A}{X}\tilde{r}_t + \frac{1}{X}\tilde{e}_t. \tag{30}$$

Now the parameter covariance is

$$P = \begin{pmatrix} P_a & P_{ax} \\ P_{xa} & P_x \end{pmatrix} = \lambda_0 \begin{pmatrix} R_{\tilde{r}\tilde{r}} & -R_{\tilde{r}\tilde{y}} \\ -R_{\tilde{y}\tilde{r}} & R_{\tilde{y}\tilde{y}} \end{pmatrix}^{-1} \tag{31}$$

and according to Lemma 3.1 this gives

$$\lim_{n_a\to\infty} \Gamma_{n_a}^T(p_k^o) P_a \Gamma_{n_a}(p_k^o) = \frac{\lambda_0 |p_k^o|^{-2} |F(p_k^o)|^2}{(1-|p_k^o|^{-2})|Q(p_k^o)|^2} \tag{32}$$

and together with (13a) we get the result (29). (Note that $\tilde{r}_t = \frac{Q}{F} v_t$.) □

Theorem 4.2. Let $r_t = Q(q)v_t$, where v_t is a zero-mean white noise sequence with variance 1 and $Q(q)$ is a minimum phase transfer function. Suppose that an ARX-system (8) with stabilizing feedback (21) is identified in the model structure (24b) with $n_x = n_b = n$. Let z_k^o be a non-minimum phase zero of the system. Then, for high model orders, the variance of the estimated zero is

$$\boxed{\lim_{n\to\infty} \overline{var}\ \hat{z}_k = \frac{\lambda_0}{(1-|z_k^o|^{-2})} \frac{1}{|\widetilde{B}(z_k^o)|^2 |Q(z_k^o)|^2}.} \tag{33}$$

Proof: The proof is equivalent to the proof of Theorem 4.1. Here we have

$$\psi_t(\eta_b) = \begin{pmatrix} \Gamma_{n_x}\tilde{y}_t \\ -\Gamma_{n_b} r_t \end{pmatrix}, \quad \tilde{y}_t = \frac{B}{X} r_t + \frac{1}{X} e_t, \tag{34}$$

which gives

$$\lim_{n_b\to\infty} \Gamma_{n_b}^T(z_k^o) P_b \Gamma_{n_b}(z_k^o) = \frac{\lambda_0 |z_k^o|^{-2}}{(1-|z_k^o|^{-2})|Q(z_k^o)|^2} \tag{35}$$

and together with (13) we get the result (33). □

Remark 1: In the asymptotic expressions (29) and (33) the variance is normalized with N. The convergence in model order is exponential and for finite sample sizes and model orders the variance is approximately (29) and (33) divided by N.

Remark 2: It is worth noting that the variance for the estimated non-minimum phase zero given in (33) is exactly the same as when the system is identified in open loop, see (Hjalmarsson and Lindqvist, 2002). This means that for high model orders the variance is *independent of the controller*. The same can be noted for unstable poles (29) if the reference signal is chosen as $r_t = Q(q)F(q)v_t$.

5. DIRECT VS. INDIRECT IDENTIFICATION

In the previous section the asymptotic variance for poles and zeros estimated with the parametrizations (24) were derived. We now shift the attention to the direct paramerization (8). We will relate the corresponding pole/zero estimates to those of (24). It is shown that direct identification gives better or equal variance compared to the indirect method but if the model order is increased sufficiently they perform equally well.

There is a direct relation between the parameters η_i in the closed loop model (24) and θ in the open loop model (8). The relation is given by

$$\eta_i = \Pi_i \theta + \pi_i, \quad i \in \{a, b\}. \tag{36}$$

The matrices Π_a and Π_b and the vectors π_a and π_b are defined in Appendix A. The following relation holds for the parameter covariances. This is similar to the results on ARMAX modelling in (Forssell and Ljung, 1999).

Lemma 5.1. The relation between the covariance of $\hat{\eta}_i$ and $\hat{\theta}$ is

$$\overline{\mathrm{Cov}}\,\hat{\theta} = \left[\Pi_i^T (\overline{\mathrm{Cov}}\,\hat{\eta}_i)^{-1} \Pi_i\right]^{-1}. \tag{37}$$

Proof: It is straightforward to establish that

$$\Pi_i^T \psi_t(\eta_i) = \psi_t(\theta) \tag{38}$$

where $\psi_t(\theta)$ and $\psi_t(\eta_i)$ are given in (18) and (28). Now this gives the relation

$$\overline{\mathrm{Cov}}\,\hat{\theta} = \lambda_0 \left(\mathbf{E}\psi_t(\theta)\psi_t^T(\theta)\right)^{-1} = \lambda_0 \left(\Pi_i^T \mathbf{E}\psi_t^T(\eta_i)\psi_t(\eta_i)\Pi_i\right)^{-1} = \left[\Pi_i^T (\overline{\mathrm{Cov}}\,\hat{\eta}_i)^{-1}\Pi_i\right]^{-1} \tag{39}$$

which concludes the proof. □

Now we return to the expression for the variance of estimated poles and zeros (13a) and (13b). Introduce the vectors

$$\Gamma_a = \begin{pmatrix} \Gamma_{n_a} \\ 0 \end{pmatrix} \text{ and } \Gamma_b = \begin{pmatrix} 0 \\ \Gamma_{n_b} \end{pmatrix}, \tag{40}$$

where the 0's represent zero vectors of appropriate lengths. Then the following relations hold for estimated poles and zeros

$$\overline{var}\ \hat{p}_k \propto \Gamma_a^T(p_k^o)(\overline{\mathrm{Cov}}\,\hat{\theta})\Gamma_a(p_k^o) \tag{41}$$
$$\overline{var}\ \hat{z}_k \propto \Gamma_b^T(z_k^o)(\overline{\mathrm{Cov}}\,\hat{\theta})\Gamma_b(z_k^o). \tag{42}$$

Now the arguments of Γ_i are dropped and Lemma 5.1 gives

$$\Gamma_i^T(\overline{\mathrm{Cov}}\,\hat{\theta})\Gamma_i = \Gamma_i^T \left[\Pi_i^T (\overline{\mathrm{Cov}}\,\hat{\eta}_i)^{-1}\Pi_i\right]^{-1} \Gamma_i. \tag{43}$$

It is easily verified that $\Pi_i^T \Gamma_i = \Gamma_i$ and from (Wahlberg and Ljung, 1992) we get the following inequality

$$\Gamma_i^T \left[\Pi_i^T (\overline{\mathrm{Cov}}\,\hat{\eta}_i)^{-1}\Pi_i\right]^{-1} \Gamma_i \le \Gamma_i^T (\overline{\mathrm{Cov}}\,\hat{\eta}_i)\Gamma_i. \tag{44}$$

This means that indirect identification with the models (24) gives larger (or equal) variance than indirect identification with the model (8), i.e.

$$\begin{aligned} \overline{var}\ \hat{p}_k(\hat{\theta}) &\le \overline{var}\ \hat{p}_k(\hat{\eta}_a) \\ \overline{var}\ \hat{z}_k(\hat{\theta}) &\le \overline{var}\ \hat{z}_k(\hat{\eta}_b). \end{aligned} \tag{45}$$

Numerical calculations and simulations suggests that when the model orders are increased the inequalities in (45) becomes equalities, i.e. direct and indirect identification give the same asymptotic variance.

Conjecture 5.1. Let the ARX-system (8) operate in closed loop with the feedback (21). Let the open loop system be identified directly from the signals u_t and y_t. For high model orders the variance of estimated non-minimum phase zeros and unstable poles will be the same as if the systems were identified in the closed loop models (24). This means that the asymptotic variance will be given by (29) and (33). □

5.1 Simulations

Conjecture 5.1 is supported by Monte Carlo simulations of an ARX-system with a non-minimum phase

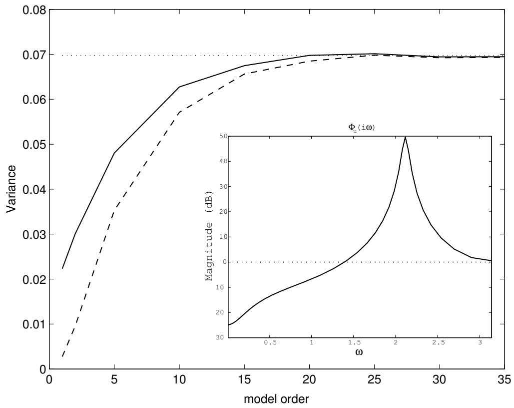

Fig. 2. Simulations of ARX-system in open and closed loop. Main figure: Variance of the estimated nmp-zero. Solid - open loop. Dashed - closed loop. Dotted - asymptotic variance. Enclosed figure: Spectrum of the input signals. Dotted - open loop. Solid - closed loop.

zero. This example also exemplifies the case when open loop and closed loop experiments give the same accuracy, see Section 4.

The simulated system is

$$y_t = \frac{q^{-1}+1.1q^{-2}}{1-0.9^{-1}} u_t + \frac{1}{1-0.9^{-1}} e_t \tag{46}$$

where e_t is zero mean white noise with variance 0.01. Both open loop and closed loop operation is considered. The inputs are chosen as

$$\text{Open loop:} \qquad u_t = r_t \tag{47}$$

$$\text{Closed loop:} \qquad u_t = r_t - \frac{1-0.5q^{-1}}{1+0.3q^{-1}} y_t, \tag{48}$$

where r_t is zero mean white noise with variance 1.

The systems are simulated for 5,000 time steps for a number of different model orders. The system is identified with the direct method from u_t and y_t. Each case is evaluated by 5,000 Monte Carlo simulations and the variance of the estimated non-minimum phase zero is calculated. Figure 2 shows the results of the simulations. Although the input spectra are very different, for high model orders the open loop and closed loop experiments give the same accuracy. The simulated variances are also in good agreement with the asymptotic value.

5.2 *Some special cases*

There are some special cases where direct and indirect identification are equal for finite model orders. These are the cases when Π_i is invertible and there is a one-to-one relationship between θ and η. Two such situations are described next. Π_a is invertible when $F(q) = \frac{c_0}{D(q)}$ and Π_b is invertible when $F(q) = C(q)$. In these cases it is easy to show that $\Gamma_i^T \Pi_i^{-1} = \Gamma_i^T$ and (43) becomes

$$\Gamma_i^T (\overline{\text{Cov}}\, \hat{\theta})\Gamma_i = \Gamma_i^T (\overline{\text{Cov}}\, \hat{\eta})\Gamma_i \tag{49}$$

which means that direct and indirect identification give the same variance. Actually the two methods give

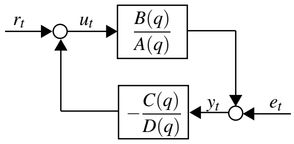

Fig. 3. OE-system with feedback

the exact same estimates of the poles and zeros, i.e. $\hat{\theta}_a(\hat{\theta}) = \hat{\theta}_a(\hat{\eta}_a)$ and $\hat{\theta}_b(\hat{\theta}) = \hat{\theta}_b(\hat{\eta}_b)$.

6. OTHER MODEL STRUCTURES

The results above regard ARX-systems, but there are also some other model structures that can be treated. Only one such example will be presented here but the outlined methodology apply to a variety of different feedback systems. Consider the following closed loop output error (OE) system with a non-minimum phase zero. The system is depicted in Figure 3,

$$y_t = \frac{B(q,\theta_b)}{A(q,\theta_a)} u_t + e_t \tag{50}$$

$$u_t = r_t - \frac{C(q)}{D(q)} y_t. \tag{51}$$

The zeros of $B(q)$ are estimated directly from the signals $\{y_t\}$ and $\{u_t\}$ in the model (50). In (Mårtensson and Hjalmarsson, 2003) it is shown that the asymptotic variance of the zero estimate is

$$\lim_{n_b\to\infty} \overline{var}\; \hat{z}_k = \lim_{n_b\to\infty} \frac{|z_k^o|^2}{|\widetilde{B}(z_k^o)|^2} \Gamma_{n_b}^T(z_k^o) P_b \Gamma_{n_b}(z_k^o), \tag{52}$$

where

$$P_b = \lambda_0 R_{\tilde{u}\tilde{u}}^{-1}, \quad \tilde{u}_t = \frac{1}{A(q,\theta^0)} u_t. \tag{53}$$

The following result was established in (Lindqvist, 2001)

Lemma 6.1. Suppose that $|z| > 1$ and let $u_t = Q(q)v_t$ where $Q(q)$ is a zero mean white noise sequence with variance 1. Then it holds that

$$\lim_{n\to\infty} \Gamma_n^T(z) R_{uu}^{-1} \Gamma_n(z) = \frac{|z|^{-2}}{(1-|z|^{-2})|Q(z)|^2}. \tag{54}$$

Proof: See (Lindqvist, 2001). □

In order to apply Lemma 6.1 to (53) a spectral factorization must be performed. Define the sensitivity function $S = \frac{AD}{AD+BC}$ which is minimum phase since the closed loop system is assumed to be stable and the controller itself is assumed to be minimum phase. The signal $\tilde{u}_t$ can be written

$$\tilde{u}_t = \frac{S(q,\theta^0)}{A(q,\theta^0)} \left(r_t - \frac{C(q)}{D(q)} e_t \right). \tag{55}$$

Now let the reference signal be filtered white noise $r_t = Q(q)v_t$ where v_t is independent of e_t and has zero mean and variance 1. It is possible to find a minimum phase transfer function $K(q)$ such that

$$|K(e^{i\omega})|^2 = |Q(e^{i\omega})|^2 + \frac{|C(e^{i\omega})|^2}{|D(e^{i\omega})|^2} \lambda_0 \tag{56}$$

Table 1. Transfer function parameters.

b_1	b_2	b_3	b_4	b_5	f_0	g_1
1.00	-2.39	1.52	-0.032	-0.13	0.6	-0.8
c_0	c_1	c_2	c_3	d_1	d_2	d_3
1.00	0.51	-0.086	-0.026	-1.72	0.98	-0.185

and the input signal can be represented by

$$u_t = \frac{S(q)}{A(q)} K(q) \upsilon_t \tag{57}$$

where υ_t is a white noise sequence with zero mean and variance 1. Lemma 6.1 can now be applied and since $B(z_k^o) = 0$ we get $S(z_k^o) = 1$. If the model order is increased the variance of an estimated non-minimum phase zero is

$$\boxed{\lim_{n_b \to \infty} \overline{var}\, \hat{z}_k = \frac{\lambda_0 |A(z_k^o)|^2}{(1 - |z_k^o|^{-2}) |K(z_k^o)|^2}.} \tag{58}$$

6.1 Simulations

The relevance of the expression for the asymptotic variance (58) will be evaluated for a FIR-system. FIR is a special case of OE where $A(q) = 1$. The simulated system, with a zero at 1.3 is

$$y_t = (b_1 q^{-1} + \cdots + b_5 q^{-5}) u_t + e_t \tag{59}$$

$$u_t = \tfrac{f_0}{1 + g_1 q^{-1}} r_t - \tfrac{c_0 + c_1 q^{-1} + \cdots + c_3 q^{-3}}{1 + d_1 q^{-1} + \cdots + d_3 q^{-3}} y_t \tag{60}$$

where the parameters are given in Table 1. The system is simulated where r_t and e_t are zero mean white noises with variances 1 and 0.1. The system is identified in the model structure

$$\hat{y}_t = b_1 u_{t-1} \cdots + b_n u_{t-n}, \tag{61}$$

for different model orders. The variance of the estimated zero is evaluated by Monte Carlo simulations. The system is estimated 10,000 times with different noise realizations and each simulation has a duration of 10,000 time steps. The results are presented in Table 2. The last entry, denoted with ∞ shows the asymptotic variance given in (58). The simulations show good agreement with the asymptotic expression.

Table 2. Simulations of the FIR-system.

n	5	10	20	∞
$N \cdot \text{var}\ \hat{z}_k$	0.289	0.315	0.318	0.315

7. CONCLUSIONS

In this paper we have derived expressions for the asymptotic (in model order and data) variance of estimates of unstable poles and non-minimum phase zeros that are identified in closed loop ARX-systems. For high order ARX-models, direct and indirect identification turns out to be equally accurate. For some cases it is shown that closed loop and open loop experiments give the same variance when estimating poles and zeros. Output error- and FIR-systems are treated with a methodology that involves spectral factorization.

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APPENDIX A : SOME DEFINITIONS

The following notation is used in the text

$$\Pi_d = \begin{pmatrix} 1 & & 0 \\ d_1 & \ddots & \\ \vdots & \ddots & 1 \\ d_{n_d} & & d_1 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & d_{n_d} \\ 0 & & 0 \end{pmatrix}, \Pi_c = \begin{pmatrix} c_0 & & 0 \\ \vdots & \ddots & \\ c_{n_c} & & c_0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & c_{n_c} \\ 0 & & 0 \end{pmatrix}, \pi_d = \begin{pmatrix} d_1 \\ \vdots \\ d_{n_d} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\Pi_a = \left(\begin{smallmatrix} I & 0 \\ \Pi_d & \Pi_c \end{smallmatrix}\right), \Pi_b = \left(\begin{smallmatrix} \Pi_d & \Pi_c \\ 0 & I \end{smallmatrix}\right), \pi_a = \begin{pmatrix} 0 \\ \pi_d \end{pmatrix}, \pi_b = \begin{pmatrix} \pi_d \\ 0 \end{pmatrix}. \tag{62}$$