

## THE LORF ORBIT EQUATION WITH FULL QUATERNIONS

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**Abstract:** The Local Orbital Reference Frame (LORF) is a Cartesian frame tangential to the trajectory of an orbiting satellite. The orbital motion of the satellite entrains the movement of the LORF around an Inertial Reference Frame (IRF). In this paper, the classical orbit equations (written in terms of inertial position and velocity vectors) are transformed into the LORF orbit equations, by means of full quaternions. Simple application examples are presented. *Copyright © 2005 IFAC*

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### 1. INTRODUCTION

Each scientific satellite mission adopts a so-called “mission reference frame”, that is typically a non-inertial local frame, to which the scientific measurements must be referred. One of the most known local reference frames employed in aerospace is the Local Orbital Reference Frame (LORF). This frame can also be classified as a tangential frame, since it is aligned with the inertial velocity of the orbiting satellite. As an example, the European satellite GOCE, that will fly at 250 Km height above the Earth, will be oriented along the LORF to measure gravity gradients and to provide global models of the Earth’s gravity field and of the geoid (see Canuto, *et al*, 2002a; Canuto, *et al*, 2002b).

The orientation of the LORF with respect to an Inertial Reference Frame (IRF) is univocally determined by the orbit position and velocity of the satellite. This means that the satellite trajectory can be described either in terms of inertial position and velocity (vectors), or in terms of the LORF attitude with respect to the IRF (quaternions). This is the basic principle of the presented work. By exploiting this approach, in the following it will be explained how the classical orbit state equation can be re-

written into an alternative quaternion form. Chapter 2 will be devoted to introduce a formal LORF definition and to describe how the orientation of the LORF with respect to the IRF can be represented either with a rotation matrix or by a unit quaternion. Next, in Chapter 3, the LORF kinematic equations will be presented both in the classical vector form and in the new quaternion form. Three simple examples will be provided in the Chapter 4.

The proposed method refers to a preceding work (Andreis and Canuto, 2004), concerning about unit and full quaternion notations as well as their elementary algebra (see also Chou, 1992; Wertz, 1978). Here will be recalled only the fundamental result, i.e.: the quaternion kinematic equation.

#### 1.1 The quaternion kinematic equation

A full quaternion can be represented as the product of a unit quaternion and a scalar (i.e. the norm).

$$\mathcal{Q} = |\mathcal{Q}| \underline{\mathcal{Q}}. \quad (1)$$

The kinematic equation of a full quaternion  $\mathcal{Q}$  is given both in quaternion and in matrix form by:

$$\begin{aligned}\dot{Q}(t) &= \mathcal{W}(t) \otimes Q(t), \quad Q(0) = Q_0 \\ \dot{Q}(t) &= [\mathcal{W}(t)^+] Q(t) = [Q(t)^-] \mathcal{W}(t),\end{aligned}\quad (2)$$

where  $\mathcal{W}$ , namely the generalized angular velocity of  $Q$ , is a full quaternion with components:

$$\begin{aligned}\mathcal{W} &= \begin{bmatrix} w_0 \\ 0 \end{bmatrix} + \underline{\mathcal{W}}, \quad w_0 = \frac{|\dot{Q}|}{|Q|}, \quad \underline{\mathcal{W}} = \begin{bmatrix} 0 \\ \mathbf{w} \end{bmatrix} = \underline{\dot{Q}} \otimes Q^* \\ \mathbf{w} &= [w_1 \quad w_2 \quad w_3]^T\end{aligned}\quad (3)$$

A significant consequence of (3) is that the kinematic equation of the unit quaternion is given by:

$$\begin{aligned}\dot{Q}(t) &= \underline{\mathcal{W}}(t) \otimes Q(t), \quad Q(0) = Q_0 \\ \dot{Q}(t) &= [\underline{\mathcal{W}}(t)^+] Q(t) = [Q(t)^-] \underline{\mathcal{W}}(t).\end{aligned}\quad (4)$$

## 2. THE LOCAL ORBITAL REFERENCE FRAME

In order to formalize the LORF definition, let consider an inertial reference frame (IRF)  $R = \{O, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ , with origin  $O$  placed in the Earth center of mass (COM) and unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  to form a right-handed Cartesian frame. The orbit trajectory of a point  $P$  having mass  $m$  and moving in the space subject to a force  $\mathbf{F}$ , can be represented, with respect to the IRF, by means of its inertial position vector  $\mathbf{r}$  and its inertial velocity vector  $\mathbf{v}$ . The point mass  $P$  can be thought as the COM of an Earth orbiting satellite.

### 2.1 The LORF definition

The LORF  $R_o = \{O_o, \mathbf{i}_o, \mathbf{j}_o, \mathbf{k}_o\}$  is a right-handed Cartesian frame defined as follows:

1. the origin  $O_o$  coincides with  $P$ ;
2. the unit vector  $\mathbf{i}_o$  lies along the velocity direction;
3. the unit vector  $\mathbf{j}_o$  is perpendicular to the instantaneous orbit plane (defined by the position and the velocity vectors);
4. the unit vector  $\mathbf{k}_o$  completes the frame:

$$\mathbf{i}_o \triangleq \frac{\mathbf{v}}{|\mathbf{v}|}, \quad \mathbf{j}_o \triangleq \frac{\mathbf{r} \times \mathbf{v}}{|\mathbf{r} \times \mathbf{v}|}, \quad \mathbf{k}_o = \mathbf{i}_o \times \mathbf{j}_o. \quad (5)$$

The symbol “ $\times$ ” in (5), stands for the cross product between three-dimensional vectors.

The LORF-to-IRF coordinate transformation matrix is defined as:

$${}_{IRF}R_{LORF} = [(\mathbf{i}_o)_{IRF} \quad (\mathbf{j}_o)_{IRF} \quad (\mathbf{k}_o)_{IRF}]. \quad (6)$$

In the following, it will be convenient to express the inertial position and velocity vectors both in LORF and IRF coordinates as follows:

$$\begin{aligned}(\mathbf{r})_{LORF} &= [{}_{IRF}R_{LORF}]^T (\mathbf{r})_{IRF} = [r_x \quad 0 \quad r_z]^T \\ (\mathbf{v})_{LORF} &= [{}_{IRF}R_{LORF}]^T (\mathbf{v})_{IRF} = [v \quad 0 \quad 0]^T.\end{aligned}\quad (7)$$

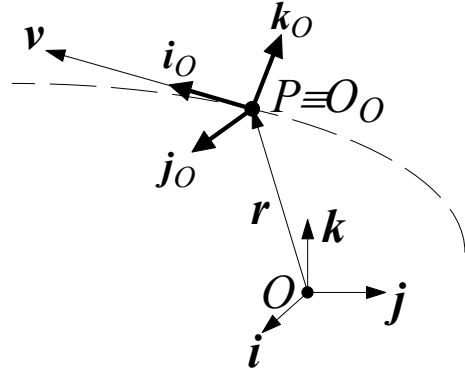


Fig. I. The LORF and the IRF frames.

The LORF and its orientation with respect to the IRF are depicted in Fig. I.

### 2.2 The LORF quaternion

The LORF orientation with respect to the IRF can be represented either by means of the rotation matrix  ${}_{LORF}R_{IRF} = ({}_{IRF}R_{LORF})^T$ , or equivalently with a unit quaternion, namely  $\underline{\mathcal{R}}_o$ . Two different conventions can be alternatively considered to define  $\underline{\mathcal{R}}_o$ :

#### 1. Convention A

Quaternion form:

$$(\mathbf{i}_o)_{IRF} = \underline{\mathcal{R}}_o \otimes (\mathbf{i}_o)_{LORF} \otimes \underline{\mathcal{R}}_o^*. \quad (8)$$

Matrix form:

$$\begin{aligned}\begin{bmatrix} 0 \\ (\mathbf{i}_o)_{IRF} \end{bmatrix} &= \underline{\mathcal{R}}_o^+ [\underline{\mathcal{R}}_o^-]^T \begin{bmatrix} 0 \\ (\mathbf{i}_o)_{LORF} \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 \\ 0 & {}_{IRF}R_{LORF} \end{bmatrix} \begin{bmatrix} 0 \\ (\mathbf{i}_o)_{LORF} \end{bmatrix} = \begin{bmatrix} 0 \\ {}_{IRF}R_{LORF,1} \end{bmatrix},\end{aligned}\quad (9)$$

where  ${}_{IRF}R_{LORF,1}$  is the first column of the transformation matrix  ${}_{IRF}R_{LORF}$ .

*Remark.* In this case  $\underline{\mathcal{R}}_o$  represents the orientation of the LORF with respect to the IRF.

#### 2. Convention B

Quaternion form:

$$(\mathbf{i}_o)_{LORF} = \underline{\mathcal{R}}_o \otimes (\mathbf{i}_o)_{IRF} \otimes \underline{\mathcal{R}}_o^* \quad (10)$$

Matrix form:

$$\begin{aligned}\begin{bmatrix} 0 \\ (\mathbf{i}_o)_{LORF} \end{bmatrix} &= \underline{\mathcal{R}}_o^+ [\underline{\mathcal{R}}_o^-]^T \begin{bmatrix} 0 \\ (\mathbf{i}_o)_{IRF} \end{bmatrix} = \\ &= \begin{bmatrix} 1 & 0 \\ 0 & {}_{LORF}R_{IRF} \end{bmatrix} \begin{bmatrix} 0 \\ (\mathbf{i}_o)_{IRF} \end{bmatrix}.\end{aligned}\quad (11)$$

*Remark.* In this case  $\underline{\mathcal{R}}_o$  represents the orientation of the IRF with respect to the LORF (opposite to Convention A).

In principle, it is the same to select one convention rather than the other. However, in the following, it will be demonstrated that the LORF quaternion kinematic equation would be written in a simpler form by exploiting the Convention B.

As explained by Andreis and Canuto (2004), it is possible to add a fourth degree of freedom (DOF) to the unit quaternion  $\underline{\mathcal{R}}_O$ , which is corresponding to let the LORF quaternion  $\mathcal{R}_O$  to have a non-unitary norm (full quaternion). If such norm is made equal to the square root of the inertial velocity vector norm:

$$\mathcal{R}_O = \sqrt{v} \underline{\mathcal{R}}_O, \quad v = |\mathbf{v}|, \quad (12)$$

then the *LORF quaternion definition* follows (for both the conventions A and B):

### 1. Convention A

$$\begin{aligned} (\mathbf{v})_{IRF} &= v(\mathbf{i}_O)_{IRF} = \mathcal{R}_O \otimes (\mathbf{i}_O)_{LORF} \otimes \mathcal{R}_O^* \\ (\mathbf{j}_O)_{IRF} &= \underline{\mathcal{R}}_O \otimes (\mathbf{j}_O)_{LORF} \otimes \underline{\mathcal{R}}_O^* \end{aligned} \quad (13)$$

### 2. Convention B

$$\begin{aligned} (\mathbf{v})_{LORF} &= v(\mathbf{i}_O)_{LORF} = \mathcal{R}_O \otimes (\mathbf{i}_O)_{IRF} \otimes \mathcal{R}_O^* \\ (\mathbf{j}_O)_{LORF} &= \underline{\mathcal{R}}_O \otimes (\mathbf{j}_O)_{IRF} \otimes \underline{\mathcal{R}}_O^* \end{aligned} \quad (14)$$

As already motivated in (Andreis and Canuto, 2004), there exist an infinite number of rotations satisfying the upper equations in (13) and (14). An additional constraint must be introduced for each of the two conventions in order to have a unique solution for  $\mathcal{R}_O$ . That constraint is given by the bottom equations in (13) and (14).

## 3. THE LORF KINEMATICS

The scope of this section is to rewrite the orbital equations for the point mass  $P$ , by exploiting the full quaternion notation defined previously. The proposed approach, if compared with the classical one, constitutes a direct way to express the movement of the LORF entrained by the point mass motion.

### 3.1 *The classical approach*

A common method to describe the motion of the point mass  $P$  subject to a force  $\mathbf{F}$ , is to assume the inertial position vector  $\mathbf{r}$  and the inertial velocity vector  $\mathbf{v}$  as state variables and to write the orbital state equations in IRF coordinates as follows:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} \dot{\mathbf{x}}_r \\ \dot{\mathbf{x}}_v \end{bmatrix}(t) = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_v \end{bmatrix}(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} \frac{(\mathbf{F})_{IRF}(t)}{m} \\ \mathbf{x}(0) &= \mathbf{x}_0 \\ \mathbf{x}_r(t) &= (\mathbf{r})_{IRF}(t), \quad \mathbf{x}_v(t) = (\mathbf{v})_{IRF}(t) \end{aligned} \quad (15)$$

The system has six DOF and the state equation in (15) is forced by a three-dimensional input vector, which is the force vector  $\mathbf{F}$  expressed in IRF coordinates.

At each time instant  $t$ , the orientation of the LORF with respect to the IRF can be obtained by construction of the LORF-to-IRF transformation matrix  ${}_{IRF}R_{LORF}$ , according to (6). For instance, such approach has been adopted to design a suitable reference trajectory generator to be embedded inside the digital control of the GOCE satellite (see Canuto, *et al.*, 2002b).

### 3.2 *The full quaternion approach*

First of all, let consider the LORF quaternion definition according with the convention A, as shown in equation (13). Taking the first derivative of both the upper and the bottom equations yields:

$$\begin{aligned} (\dot{\mathbf{v}})_{IRF} &= \frac{(\mathbf{F})_{IRF}}{m} = \dot{\mathcal{R}}_O \otimes (\mathbf{i}_O)_{LORF} \otimes \mathcal{R}_O^* + \\ &+ \mathcal{R}_O \otimes (\dot{\mathbf{i}}_O)_{LORF} \otimes \mathcal{R}_O^* + \mathcal{R}_O \otimes (\mathbf{i}_O)_{LORF} \otimes \dot{\mathcal{R}}_O^* \end{aligned} \quad (16)$$

$$\begin{aligned} (\dot{\mathbf{j}}_O)_{IRF} &= \dot{\underline{\mathcal{R}}}_O \otimes (\mathbf{j}_O)_{LORF} \otimes \underline{\mathcal{R}}_O^* + \\ &+ \underline{\mathcal{R}}_O \otimes (\dot{\mathbf{j}}_O)_{LORF} \otimes \underline{\mathcal{R}}_O^* + \underline{\mathcal{R}}_O \otimes (\mathbf{j}_O)_{LORF} \otimes \dot{\underline{\mathcal{R}}}_O^* \end{aligned} \quad (17)$$

Then, by recalling the quaternion kinematics:

$$\begin{aligned} \dot{\mathcal{R}}_O &= \mathcal{W} \otimes \mathcal{R}_O \Leftrightarrow \dot{\mathcal{R}}_O^* = \mathcal{R}_O^* \otimes \mathcal{W}^* \\ \dot{\underline{\mathcal{R}}}_O &= \underline{\mathcal{W}} \otimes \underline{\mathcal{R}}_O \Leftrightarrow \dot{\underline{\mathcal{R}}}_O^* = \underline{\mathcal{R}}_O^* \otimes \underline{\mathcal{W}}^* \end{aligned} \quad (18)$$

equations (16) and (17) develop into:

$$\begin{aligned} (\dot{\mathbf{v}})_{IRF} &= \mathcal{W} \otimes (\mathbf{v})_{IRF} + (\mathbf{v})_{IRF} \otimes \mathcal{W}^* \\ (\dot{\mathbf{j}}_O)_{IRF} &= \underline{\mathcal{W}} \otimes (\mathbf{j}_O)_{IRF} + (\mathbf{j}_O)_{IRF} \otimes \underline{\mathcal{W}}^* = \\ &= 2\underline{\mathcal{W}} \otimes (\mathbf{j}_O)_{IRF} \end{aligned} \quad (19)$$

*Remark.* In (19), the generalized angular rate of the LORF quaternion must be expressed in IRF coordinates:

$$\mathcal{W} = (\mathcal{W})_{IRF}, \quad \underline{\mathcal{W}} = (\underline{\mathcal{W}})_{IRF} = \begin{bmatrix} 0 & (\mathbf{w})_{IRF}^T \end{bmatrix}^T. \quad (20)$$

By applying the matrix notation to (19), it is possible to find the relation between the angular rate  $(\mathcal{W})_{IRF}$  and the force vector  $(\mathbf{F})_{IRF}$ :

$$\begin{aligned} \frac{1}{m} \begin{bmatrix} 0 \\ (\mathbf{F})_{IRF} \end{bmatrix} &= \left\{ (\mathcal{W})_{IRF}^+ + [(\mathcal{W})_{IRF}^-]^T \right\} (\mathbf{v})_{IRF} \Rightarrow \\ \Rightarrow \frac{(\mathbf{F})_{IRF}}{m} &= 2\mathbf{w}_0 (\mathbf{v})_{IRF} + 2(\mathbf{w})_{IRF} \times (\mathbf{v})_{IRF} \\ (\dot{\mathbf{j}}_O)_{IRF} &= 2(\mathbf{w})_{IRF} \times (\mathbf{j}_O)_{IRF} \end{aligned} \quad (21)$$

The selection of the LORF quaternion definition according with the convention A suffers of at least two disadvantages:

1. it is not straightforward to obtain a simple analytical expression for  $(\dot{\mathbf{j}}_O)_{IRF}$  as a function of the force  $(\mathbf{F})_{IRF}$ ;
2. practical applications suggest to express the LORF angular rate in LORF coordinates rather than in IRF coordinates, since it could be used as

a reference trajectory for satellite attitude and angular rate control purposes.

The aforementioned drawbacks can be easily avoided by taking into account the LORF quaternion definition under the convention B. Then, let consider the definition in (14) and take the first derivative of both the upper and the bottom equations:

$$\begin{aligned} (\dot{\mathbf{v}})_{LORF} &= \dot{\mathcal{R}}_O \otimes (\mathbf{i}_O)_{IRF} \otimes \mathcal{R}_O^* + \\ &+ \mathcal{R}_O \otimes (\dot{\mathbf{i}}_O)_{IRF} \otimes \mathcal{R}_O^* + \mathcal{R}_O \otimes (\mathbf{i}_O)_{IRF} \otimes \dot{\mathcal{R}}_O^* \end{aligned} \quad (22)$$

$$\begin{aligned} (\dot{\mathbf{j}}_O)_{LORF} &= \dot{\mathcal{R}}_O \otimes (\mathbf{j}_O)_{IRF} \otimes \mathcal{R}_O^* + \\ &+ \mathcal{R}_O \otimes (\dot{\mathbf{j}}_O)_{IRF} \otimes \mathcal{R}_O^* + \mathcal{R}_O \otimes (\mathbf{j}_O)_{IRF} \otimes \dot{\mathcal{R}}_O^* \end{aligned} \quad (23)$$

Recalling again the quaternion kinematics in (18), the expressions in (22) and (23) become:

$$\begin{aligned} (\dot{\mathbf{v}})_{LORF} &= \mathcal{W} \otimes (\mathbf{v})_{LORF} + (\mathbf{v})_{LORF} \otimes \mathcal{W}^* + \\ &+ v [\mathcal{R}_O \otimes (\dot{\mathbf{i}}_O)_{IRF} \otimes \mathcal{R}_O^*] \end{aligned} \quad (24)$$

$$(\dot{\mathbf{j}}_O)_{LORF} = 2\mathcal{W} \otimes (\mathbf{j}_O)_{LORF} + \mathcal{R}_O \otimes (\dot{\mathbf{j}}_O)_{IRF} \otimes \mathcal{R}_O^*$$

*Remark.* In (24), the angular rate of the LORF quaternion must be expressed in LORF coordinates, according with the notation below:

$$\begin{aligned} \mathcal{W} &= (\mathcal{W})_{LORF} = \begin{bmatrix} w_0 \\ 0 \end{bmatrix} + (\underline{\mathcal{W}})_{LORF}, \\ w_0 &= |\dot{\mathcal{R}}_O| / |\mathcal{R}_O| = \dot{v} / 2v, \\ (\underline{\mathcal{W}})_{LORF} &= \begin{bmatrix} 0 \\ (\mathbf{w})_{LORF} \end{bmatrix} = \dot{\mathcal{R}}_O \otimes \mathcal{R}_O^*, \\ (\mathbf{w})_{LORF} &= [w_x \ w_y \ w_z]^T \end{aligned} \quad (25)$$

The value of the first derivatives of  $(\mathbf{i}_O)_{IRF}$  and  $(\mathbf{j}_O)_{IRF}$  have been reported in the equation below and demonstrated in the Appendix.

$$\begin{aligned} (\dot{\mathbf{i}}_O)_{IRF} &= \frac{(\mathbf{F})_{IRF}}{mv} - \frac{\dot{v}}{v} (\mathbf{i}_O)_{IRF} \\ (\dot{\mathbf{j}}_O)_{IRF} &= \frac{(\mathbf{r})_{IRF} \times (\mathbf{F})_{IRF}}{\lambda m} - \frac{\dot{\lambda}}{\lambda} (\mathbf{j}_O)_{IRF} \\ \lambda &= |(\mathbf{r})_{IRF} \times (\mathbf{v})_{IRF}| \end{aligned} \quad (26)$$

By substitution of (25) and (26) into (22) and (23), it is possible to write:

$$\begin{aligned} (\dot{\mathbf{v}})_{LORF} &= \frac{(\mathbf{F})_{LORF}}{m} - \dot{v} (\mathbf{i}_O)_{LORF} + \\ &+ (\mathcal{W})_{LORF} \otimes (\mathbf{v})_{LORF} + (\mathbf{v})_{LORF} \otimes (\mathcal{W})_{LORF}^* \end{aligned} \quad (27)$$

$$\begin{aligned} (\dot{\mathbf{j}}_O)_{LORF} &= 2(\underline{\mathcal{W}})_{LORF} \otimes (\mathbf{j}_O)_{LORF} + \\ &+ \frac{(\mathbf{r})_{LORF} \times (\mathbf{F})_{LORF}}{\lambda m} - \frac{\dot{\lambda}}{\lambda} (\mathbf{j}_O)_{LORF} \end{aligned}, \quad (28)$$

where the LORF coordinates of the force vector have been defined as  $(\mathbf{F})_{LORF} = [F_x \ F_y \ F_z]^T$ . At this point, it is convenient to use the matrix form of (27) and (28):

$$\begin{aligned} \begin{bmatrix} 0 \\ (\dot{\mathbf{v}})_{LORF} \end{bmatrix} &= \frac{1}{m} \begin{bmatrix} 0 \\ (\mathbf{F})_{LORF} \end{bmatrix} - \dot{v} \begin{bmatrix} 0 \\ (\mathbf{i}_O)_{LORF} \end{bmatrix} + \\ &+ \left\{ [(\mathcal{W})_{LORF}^+] + [(\mathcal{W})_{LORF}^-]^T \right\} \begin{bmatrix} 0 \\ (\mathbf{v})_{LORF} \end{bmatrix} \end{aligned} \quad (29)$$

$$\begin{aligned} \begin{bmatrix} 0 \\ (\dot{\mathbf{j}}_O)_{LORF} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2(\underline{\mathcal{W}})_{LORF}^+ \begin{bmatrix} 0 \\ (\mathbf{j}_O)_{LORF} \end{bmatrix} + \\ &+ \frac{1}{\lambda m} \begin{bmatrix} 0 \\ (\mathbf{r})_{LORF} \times (\mathbf{F})_{LORF} \end{bmatrix} - \frac{\dot{\lambda}}{\lambda} \begin{bmatrix} 0 \\ (\mathbf{j}_O)_{LORF} \end{bmatrix} \end{aligned} \quad (30)$$

By making the computations, the equations (29) and (30) establish the static relation between the LORF coordinates of the generalized angular velocity  $\mathcal{W}$  and the force vector  $\mathbf{F}$ :

$$\begin{aligned} (\mathcal{W})_{LORF} &= \Gamma((\mathbf{r})_{LORF}, (\mathbf{v})_{LORF}, m) (\mathbf{F})_{LORF} \\ \Gamma((\mathbf{r})_{LORF}, (\mathbf{v})_{LORF}, m) &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\varepsilon \alpha & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{bmatrix} \end{aligned} \quad (31)$$

$$\alpha = 1/2mv, \quad \varepsilon = r_x/r_z$$

The previous result puts in evidence that the x-component of the angular velocity of the LORF quaternion depends on the parameter  $\varepsilon$ , that is an index of the eccentricity of the orbit. In fact, it is possible to observe that for a circular orbit, the position vector is always perpendicular to the velocity vector, and, since in that case the  $r_x$  position coordinate is null, it follows that  $\varepsilon = 0$  and finally that  $w_x = 0$ .

Once obtained the relation between the force and the generalized angular rate of the LORF quaternion, the kinematic state equation of  $\mathcal{R}_O$  is complete and is proposed in the matrix form by the following equation:

$$\dot{\mathcal{R}}_O = [\mathcal{W}_{LORF}^+] \mathcal{R}_O, \quad \mathcal{W}_{LORF} = \Gamma(\mathbf{F})_{LORF} \quad (32)$$

The state equation in (32) has four DOF and then it is not sufficient to completely describe the orbital motion of the point mass  $P$  in the three-dimensional space. The two lacking DOF originate from the point mass kinematic equation written in LORF coordinates:

$$\begin{aligned} (\dot{\mathbf{r}})_{LORF} &= \frac{d}{dt} \{ \mathcal{R}_O \otimes (\mathbf{r})_{IRF} \otimes \mathcal{R}_O^* \} = (\mathbf{v})_{LORF} + \\ &+ (\underline{\mathcal{W}})_{LORF} \otimes (\mathbf{r})_{LORF} + (\mathbf{r})_{LORF} \otimes (\underline{\mathcal{W}})_{LORF}^* \end{aligned} \quad (33)$$

The matrix form of (33) reads as:

$$\begin{bmatrix} 0 \\ \dot{r}_x \\ 0 \\ \dot{r}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2C(\mathbf{w}) \end{bmatrix} \begin{bmatrix} 0 \\ r_x \\ 0 \\ r_z \end{bmatrix} + \begin{bmatrix} 0 \\ v \\ 0 \\ 0 \end{bmatrix}, \quad (34)$$

and finally the point mass kinematic state equation is:

$$\begin{bmatrix} \dot{r}_x \\ \dot{r}_z \end{bmatrix} = \begin{bmatrix} 0 & 2w_y \\ -2w_y & 0 \end{bmatrix} \begin{bmatrix} r_x \\ r_z \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \quad (35)$$

#### 4. APPLICATIONS

*Remark.* In the case of uniform circular motion,  $r_x$  is null by definition and  $r_z = r$  is constant (the circle radius). Under these conditions, equation (35) turns into:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 & 2w_y \\ -2w_y & 0 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} 2w_y r + v \\ 0 \end{bmatrix} \Rightarrow \\ \Rightarrow v &= -2w_y r \Rightarrow w_y = -\frac{1}{2} \frac{v}{r} = -\frac{1}{2} \omega_o \end{aligned} \quad (36)$$

where  $\omega_o = 1/T_o$  is the orbit angular rate, i.e.: the inverse of the orbit period  $T_o$ .

The 6-DOF orbit state equation can be formalized in an alternative approach with respect to the classical one (15) by assuming the state vector to be defined as:

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_r \\ \mathbf{z}_q \end{bmatrix} = \begin{bmatrix} r_x \\ r_z \\ \mathcal{R}_o \end{bmatrix}, \quad \mathbf{z}_q = \mathcal{R}_o \in \mathbb{R}^4, \quad \mathbf{z}_r = \begin{bmatrix} r_x \\ r_z \end{bmatrix} \in \mathbb{R}^2 \quad (37)$$

The new formula of the orbit equation, written in terms of the LORF quaternion reads as:

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_q \end{bmatrix} = \begin{bmatrix} 0 & 2w_y & \mathbf{z}_q^T \\ -2w_y & 0 & 0 \\ 0 & 0 & (\mathcal{W})_{LORF}^+ \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \mathbf{z}_q \end{bmatrix} \quad (38) \\ \mathbf{z}(0) &= \mathbf{z}_0, \quad \mathbf{z}_q^T \mathbf{z}_q = v, \quad (\mathcal{W})_{LORF} = \Gamma(\mathbf{z}, m)(\mathbf{F})_{LORF} \end{aligned}$$

where the time dependence has been omitted for sake of simplicity.

*Remark.* For the perfect circular orbit case, the dynamic matrix of the state equation (38) develops into a block matrix representing an harmonic oscillator at  $\omega_o = -2w_y$ :

$$\begin{aligned} (\mathbf{F})_{LORF} &= \begin{bmatrix} 0 \\ 0 \\ -m \frac{\mu_{\oplus}}{r^2} \end{bmatrix} \Rightarrow (\mathcal{W})_{LORF} = \begin{bmatrix} 0 \\ 0 \\ -\frac{\mu_{\oplus}}{2vr^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w_y \\ 0 \end{bmatrix} \\ \begin{bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_q \end{bmatrix} &= \begin{bmatrix} 0 & 2w_y & 0 & 0 & 0 & 0 \\ -2w_y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2w_y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2w_y \\ 0 & 0 & 2w_y & 0 & 0 & 0 \\ 0 & 0 & 0 & -2w_y & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \mathbf{z}_q \end{bmatrix} + \begin{bmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (39) \end{aligned}$$

where  $\mu_{\oplus} = 3.99 \cdot 10^{14} \text{ m}^3/\text{s}^2$  is the Earth gravitational constant.

It is possible also to develop an alternative form of (38), where the force input  $(\mathbf{F})_{LORF}$  is made explicit. The latter reads as:

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_q \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \mathbf{z}_q^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_r \\ \mathbf{z}_q \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2r_z & 0 \\ 0 & 0 & -2r_x & 0 \\ \mathbf{z}_q^T & & & \end{bmatrix} \Gamma(\mathbf{z}, m)(\mathbf{F})_{LORF} \quad (40) \\ \mathbf{z}(0) &= \mathbf{z}_0, \quad \mathbf{z}_q^T \mathbf{z}_q = v \end{aligned}$$

In order to provide better understanding on the results presented in the previous sections, some simple examples will be proposed. They both concern the orbit trajectory around the Earth of a point mass  $P$  (that can be thought as the COM of a Low Earth Orbiter satellite like GOCE) having a mass of about  $m = 1000 \text{ Kg}$ . The solution of the orbit state equation (38) or (40) has been numerically determined by means of a Matlab™ simulator in three different conditions, as summarized in Table 1.

The first orbit is a perfectly circular one, with condition equilibrium at 250Km of mean height from the Earth. The satellite, as well as in the other cases, is subject to the spherical term of the Earth's gravity field. The second example has been introduced to show the effects of an orbit decay induced by a force of 10mN counteracting the satellite motion (as it would have to be the atmospheric drag at such altitudes). The latter example consists on a highly eccentric orbit, subject only to the spherical term of the Earth's gravity field.

Table 1 Orbit conditions

Example N.	Orbit type	External force $(\mathbf{F})_{LORF}$
1	Circular	$(\mathbf{F})_{LORF} = -\frac{\mu_{\oplus} m}{r^3}(\mathbf{r})_{LORF}$
2	Circular	$(\mathbf{F})_{LORF} = -\frac{\mu_{\oplus} m}{r^3}(\mathbf{r})_{LORF} - \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \text{ mN}$
3	High eccentricity	$(\mathbf{F})_{LORF} = -\frac{\mu_{\oplus} m}{r^3}(\mathbf{r})_{LORF}$

The time series of the LORF quaternion in the case of a uniform circular orbit at a 250km height are depicted in Fig. II. As can be deduced from Fig. III, the angular velocity  $w_y$  of the LORF quaternion is about  $-5.85 \cdot 10^{-4} \text{ rad/s}$ . This verifies the equation (36), since the orbital angular rate is worth  $\omega_o = \sqrt{\mu_{\oplus}/r^3} = 1.17 \cdot 10^{-3} \text{ rad/s}$ . The effect of the orbit decay on the LORF quaternion components is clearly visible from Fig. IV, where  $\mathcal{R}_o$  shows an increasing trend, both in amplitude and in frequency. This means correctly that the inertial velocity and the orbit angular rate are increasing. In conclusion, the effect of the orbit eccentricity on the y-LORF coordinate of the angular velocity  $w_y$  is enlightened in Fig. V.

#### 5. CONCLUSIONS

The orbit equations describing the motion of a point mass in the three dimensional space have been formulated in terms of the LORF quaternion. LORF motion has been observed by means of a Matlab™ implementation of the LORF orbit equations. Among future developments, the design of a suitable LORF observer will cover an important role.

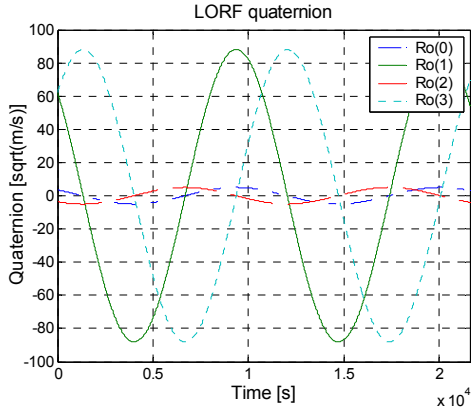


Fig. II. LORF quaternion – Example 1

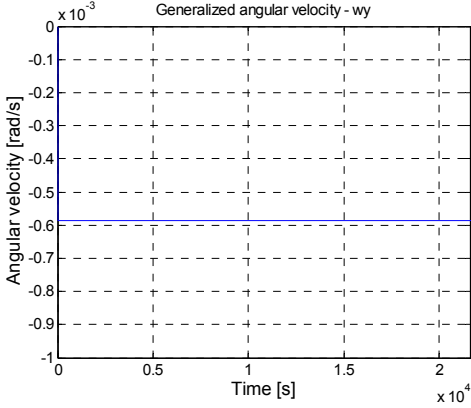


Fig. III. Angular velocity  $w_y$  - Example 1

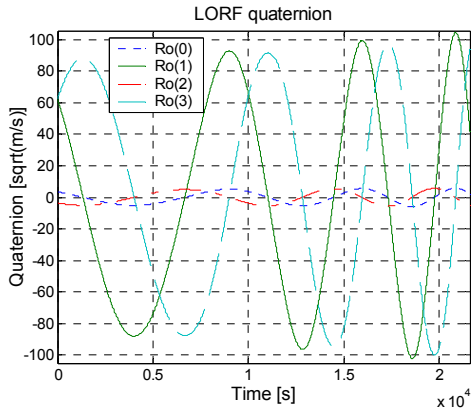


Fig. IV. LORF quaternion – Example 2

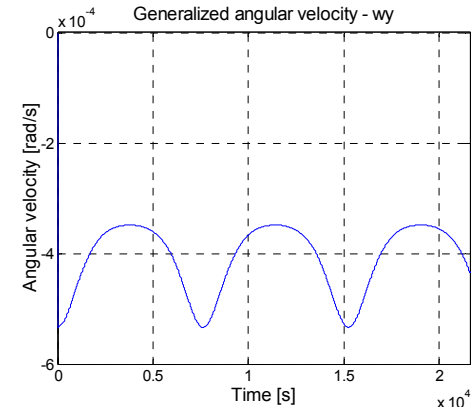


Fig. V. Angular rate  $w_y$  - Example 3

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## APPENDIX

*Proof of the equation (26).*

Remembering the definition of the first unit vector of the LORF in (5), it follows that:

$$\begin{aligned} (\mathbf{i}_O)_{IRF} &= \frac{d}{dt} \left( \frac{(\mathbf{v})_{IRF}}{v} \right), \quad v = |(\mathbf{v})_{IRF}| \\ (\mathbf{i}_O)_{IRF} &= \frac{(\dot{\mathbf{v}})_{IRF}}{v} - \frac{\dot{v}}{v} (\mathbf{i}_O)_{IRF} = \frac{(\mathbf{F})_{IRF}}{mv} - \frac{\dot{v}}{v} (\mathbf{i}_O)_{IRF} \end{aligned} \quad (41)$$

Then, starting from the definition of  $(\mathbf{j}_O)_{IRF}$ :

$$(\mathbf{j}_O)_{IRF} = \frac{(\mathbf{r})_{IRF} \times (\mathbf{v})_{IRF}}{|(\mathbf{r})_{IRF} \times (\mathbf{v})_{IRF}|}, \quad (42)$$

it is possible to label the denominator with the generic variable  $\lambda = |(\mathbf{r})_{IRF} \times (\mathbf{v})_{IRF}|$ . By taking the first derivative, the following relation yields:

$$\begin{aligned} (\mathbf{j}_O)_{IRF} &= \frac{(\dot{\mathbf{r}})_{IRF} \times (\mathbf{v})_{IRF}}{\lambda} + \frac{(\mathbf{r})_{IRF} \times (\dot{\mathbf{v}})_{IRF}}{\lambda} - \\ &= \frac{\dot{\lambda} (\mathbf{r})_{IRF} \times (\mathbf{v})_{IRF}}{\lambda} + \frac{(\mathbf{r})_{IRF} \times (\mathbf{F})_{IRF}}{m\lambda} - \frac{\dot{\lambda}}{\lambda} (\mathbf{j}_O)_{IRF} \end{aligned} \quad (43)$$

Furthermore, note that since:

$$\lambda = |(\mathbf{r})_{IRF} \times (\mathbf{v})_{IRF}| = |(\mathbf{r})_{LORF} \times (\mathbf{v})_{LORF}|, \quad (44)$$

the norm of the cross product can be expressed easily in LORF coordinates as follows:

$$\lambda = |(\mathbf{r})_{LORF} \times (\mathbf{v})_{LORF}| = r_z v. \quad (45)$$