A SYSTEM DECOMPOSITION FOR SENSOR LOCATION IN FAULT DETECTION AND ISOLATION

Christian Commault * **Jean-Michel Dion** *
Sameh Yacoub Agha *

* *Laboratoire d'Automatique de Grenoble. LAG-CNRS, ENSIEG-INPG BP 46
38402 Saint Martin d'Hères,
France.
Email: {Christian.Commault,Jean-Michel.Dion,Sameh.Yacoub-Agha}@inpg.fr*

Abstract: In this paper we consider linear systems with faults. We present a new system decomposition well suited for sensor location in the Fault Detection and Isolation problem. We deal with this problem when the system under consideration is structured, that is, the entries of the system matrices are either fixed zeros or free parameters. To such structured systems one can associate a graph. We present a structural decomposition of this graph which extends previous results. This decomposition is based on the analysis of particular separators. This finer structural decomposition allows to characterize all the solutions in terms of location of possible additional sensors.

Keywords: Linear systems, Structured systems, Fault detection, Sensor location.

1. INTRODUCTION

In this paper we consider linear systems with faults and we present a new system decomposition based on structural analysis which is well suited for sensor location in the Fault Detection and Isolation problem (FDI).

The FDI problem has received considerable attention in the past ten years (Chen and Patton, 1999; Frank, 1996). We are interested in building some auxiliary signals called residuals and obtaining a transfer from faults to residuals with a diagonal structure. When this FDI problem is not solvable with the existing sensors we look for a solution using a minimal number of additional sensors

We consider for this sensor location problem intrinsic solvability conditions depending on the internal structure of the system and not on the specific values of the parameters. An interesting tool for this purpose is the notion of structured system (Lin, 1974; Dion *et al.*, 2003). Solvability conditions were given recently in terms of the graph that can be associated in a natural way to a structured system (Commault *et al.*, 2002; Commault and Dion, 2003).

Using iteratively the Ford and Fulkerson algorithm in a specific way on the system's graph we get a sequence of separators of increasing size which lead to a new graph-based system decomposition which is the main contribution of this paper. This decomposition is an extension of previous results (Murota, 1987; van der Woude, 2000; Commault and Dion, 2003) which allows to characterize the solutions for the location of possible additional sensors.

The outline of this paper is as follows. The problem is formulated in section 2. The linear structured systems are presented in section 3. In section 4 we introduce the notion of separator and give the system decomposition defined in an iterative way. In section 5 we present the application of this decomposition to the sensor location in the FDI problem. In section 6 we present an illustrative example which emphasizes the interest of this decomposition for our sensor location problem. Some concluding remarks end the paper.

2. PROBLEM FORMULATION

2.1 Observer-based FDI problem

Let us consider the following linear time-invariant system :

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + Lf(t) \\ y(t) = Cx(t) + Mf(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f(t) \in \mathbb{R}^r$ the fault vector and $y(t) \in \mathbb{R}^p$ the measured output vector. A, C, L and M are matrices of appropriate dimensions.

Note that the control input effects are not considered here as, for any observer-based FDI problem, it is well known that these can be taken into account in the observer structure without loss of generality.

A dedicated residual set is designed using a bank of r observers for system (1), according to the dedicated observer scheme (Chen and Patton, 1999). Each residual will be designed to be sensitive to a single fault while remaining insensitive to the other faults.

The ith observer of this bank of r observers is designed for a system of type (1) as follows:

$$\dot{\hat{x}}^i(t) = A\hat{x}^i(t) + K^i(y(t) - C\hat{x}^i(t)) \tag{2}$$

where $\hat{x}^i(t) \in \mathbb{R}^n$ is the state of the ith observer, K^i is the observer gain to be designed such that $\hat{x}^i(t)$ asymptotically converges to $x(t)$, when no fault is considered.

The residuals are defined as :

$$r_i(t) = Q^i(y(t) - C\hat{x}^i(t)), \text{ for } i = 1, \ldots, r \tag{3}$$

where Q^i is a $1 \times p$ matrix.

Definition 1. The bank of observers-based FDI problem consists in finding, if possible, matrices K^i and Q^i, such that, for $i = 1, 2, \ldots, r$, $A - K^iC$ is stable, and the fault to residual transfer matrix is non zero, proper and diagonal, i.e. the transfer form the faults to the residuals has the form

$$r(s) = \begin{bmatrix} t_{11}(s) & 0 & \cdots & 0 \\ 0 & t_{22}(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{rr}(s) \end{bmatrix} f(s) \tag{4}$$

where $t_{ii}(s) \neq 0$ for $i = 1, 2, \ldots, r$.

The solvability conditions for this problem will be detailed further. These conditions express in particular that there must exist a sufficient number of measured outputs to be able to detect and isolate the faults.

2.2 Sensor location for FDI

Consider again the system (1). In general the above defined FDI problem has no solution using only the existing sensors on the system. In this case we consider new sensors which could be implemented on the system. We assume that these new sensors are fault free. Define the new output vector z which collects the new measurements:

$$z(t) = Hx(t) + Pf(t), \tag{5}$$

$z(t) \in \mathbb{R}^q$, where $z_i(t)$ is the measure obtained from the i-th additional sensor.

Define now the composite system denoted by Σ^c.

$$\Sigma^c \begin{cases} \dot{x}(t) = Ax(t) + Lf(t) \\ y(t) = Cx(t) + Mf(t) \\ z(t) = Hx(t) + Pf(t) \end{cases} \tag{6}$$

In the next sections we will consider the following sensor location problem for FDI: in which parts of the system should we implement additional sensors in such a way that the FDI problem is solvable on the composite system? If these additional sensors are necessary we look for an implementation which minimizes their number, and gives results concerning the additional sensors location.

Our study will be achieved in the framework of structured systems that we introduce now.

3. LINEAR STRUCTURED SYSTEMS

In this part we recall some definitions and results on linear structured systems. More details can be found in (Dion *et al.*, 2003).

We consider linear systems as described in (1), but with parameterized entries and denoted by Σ_Λ

$$\Sigma_\Lambda \begin{cases} \dot{x}(t) = Ax(t) + Lf(t) \\ y(t) = Cx(t) + Mf(t) \end{cases} \tag{7}$$

This system is called a linear structured systems if the entries of the composite matrix $J = \begin{bmatrix} A & L \\ C & M \end{bmatrix}$

are either fixed zeros or independent parameters (not related by algebraic equations). $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ denotes the set of independent parameters of the composite matrix J. For the sake of simplicity the dependence of the system matrices on Λ will not be made explicit in the notation. A structured system represents a large class of parameter dependent linear systems. The structure is given by the location of the fixed zero entries of J.

For such systems one can study generic properties i.e. properties which are true for almost all values of the parameters collected in Λ (Murota, 1987; Reinschke, 1988). More precisely a property is said to be generic (or structural) if it is true for all values of the parameters (i.e. any $\Lambda \in \mathbb{R}^k$) outside a proper algebraic variety of the parameter space.

A directed graph $G(\Sigma_\Lambda) = (Z, W)$ can be easily associated to the structured system Σ_Λ of type (7) where the matrix $\begin{bmatrix} A & L \\ C & M \end{bmatrix}$ is structured:

- the vertex set is $Z = F \cup X \cup Y$ where F, X and Y are the fault, state and output sets given by $\{f_1, f_2, \ldots, f_r\}$, $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_p\}$ respectively,
- the arc set is $W = \{(f_i, x_j) | L_{ji} \neq 0\} \cup \{(x_i, x_j) | A_{ji} \neq 0\} \cup \{(x_i, y_j) | C_{ji} \neq 0\} \cup \{(f_i, y_j) | M_{ji} \neq 0\}$, where A_{ji} (resp. C_{ji},L_{ji},M_{ji}) denotes the entry (j, i) of the matrix A (resp. C,L,M).

Moreover, recall that a directed path in $G(\Sigma_\Lambda)$ from a vertex $i_{\mu 0}$ to a vertex $i_{\mu q}$ is a sequence of arcs $(i_{\mu 0}, i_{\mu 1}), (i_{\mu 1}, i_{\mu 2}), \ldots, (i_{\mu q-1}, i_{\mu q})$ such that $i_{\mu t} \in Z$ for $t = 0, 1, \ldots, q$ and $(i_{\mu t-1}, i_{\mu t}) \in W$ for $t = 1, 2, \ldots, q$. Moreover, if $i_{\mu 0} \in F$ and, $i_{\mu q} \in Y$, P is called a fault-output path. A path which is such that $i_{\mu 0} = i_{\mu q}$ is called a circuit. If $i_0 \in V_1$ and, $i_l \in V_2$, where V_1 and V_2 are two subsets of Z, P is called a V_1-V_2 path. Moreover if the only vertices of P which belong to $V_1 \cup V_2$ are i_0 and i_l, P is called a *direct* V_1-V_2 path.

A set of paths with no common vertex is said to be vertex disjoint. A V_1-V_2 linking of size k is a set of k vertex disjoint V_1-V_2 paths. A linking is maximal when k is maximal.

All the previous definitions can be extended to a composite structured system Σ_Λ^c with associated graph $G(\Sigma_\Lambda^c)$ where Σ_Λ^c is defined as

$$\Sigma_\Lambda^c \begin{cases} \dot{x}(t) = Ax(t) + Lf(t) \\ y(t) = Cx(t) + Mf(t) \\ z(t) = Hx(t) + Pf(t) \end{cases} \tag{8}$$

As a first example of these results, recall the graph characterization of the structural observability, which will be useful later (Lin, 1974; Murota, 1987).

Proposition 2. Let Σ_Λ be the linear structured system defined by (7) with its associated graph $G(\Sigma_\Lambda)$. The system (in fact the pair (C, A)) is structurally observable if and only if:

- there exists a state-output path starting from any state vertex in X,
- there exists a set of vertex disjoint circuits and state-output paths which cover all state vertices.

Consider now the system Σ_Λ defined in (7) whose transfer matrix is $T_\Lambda(s) = C(sI - A)^{-1}L + M$. We can calculate the generic rank of $T_\Lambda(s)$ by using the following result (van der Woude, 1991).

Theorem 3. Let Σ_Λ be the linear structured system defined by (7) with its associated graph $G(\Sigma_\Lambda)$. The generic rank of $T_\Lambda(s)$ is equal to the size of a maximal fault-output linking in $G(\Sigma_\Lambda)$.

Give now the result concerning the diagonal FDI problem by using a bank of observers which was stated first in (Commault *et al.*, 2002).

Theorem 4. Consider the structurally observable system with r faults Σ_Λ as defined in (7) and the associated graph $G(\Sigma_\Lambda)$. The bank of observers-based diagonal FDI problem of Definition 1, is generically solvable if and only if:

$$k = r \tag{9}$$

where k is the size of a maximal linking in $G(\Sigma_\Lambda)$

Example 1 : Consider the following structured system Σ_Λ which is of type (7) with two faults and one output:

$$A = \begin{bmatrix} 0 & 0 \\ \lambda_1 & 0 \end{bmatrix}, L = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & \lambda_4 \end{bmatrix}, M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The non zero entries of these matrices are the free parameters $\Lambda = (\lambda_1, \ldots, \lambda_4)$. The associated graph $G(\Sigma_\Lambda)$ is given in Figure **??** It is clear that

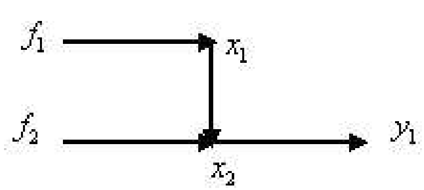

Fig. 1. Graph $G(\Sigma_\Lambda)$ of example 1

no solution for the *FDI* problem exists in our

example because the condition (9) is not satisfied, $k = 1 < r = 2$.
By measuring the state vertex x_1 for example with a new additional sensor, we will have $k = 2 = r$ and the FDI problem will become solvable.

4. SYSTEM DECOMPOSITION

In this section we present a system decomposition based on an ordered sequence of separators. This is a new decomposition for the inconsistent parts in the M-decomposition of a system graph different from the one given in (Murota, 1987). It refines other decompositions presented in (van der Woude, 2000; Commault and Dion, 2003).

4.1 Basic notions

Most of the basic material of this subsection is based on (van der Woude, 2000). First consider again the graph $G(\Sigma_\Lambda) = (Z, W)$ of a structured system of type (7) with vertex set Z and edge set W. A separator S is a set of vertices such that any fault-output path has at least one vertex in S. Separators with a minimal number of vertices are called minimal. A classical result is that the minimal size of a separator is the maximal size of a fault-output linking. The set of essential vertices Z_{ess} is the set of vertices which belong to any maximal size linking. Construct now the set of vertices which contains for any fault-output path the first vertex which is in Z_{ess}, call this set S^*. It can be shown that S^* is a minimal separator. S^* is called the minimal input separator. S^* is indeed the first bottleneck between faults and outputs. S^* may contain fault, state and output vertices.
Using S^* we can reformulate Theorem 4 as follows.

Proposition 5. Consider the structurally observable system with r faults Σ_Λ as defined in (7) and the associated graph $G(\Sigma_\Lambda)$. The bank of observers-based diagonal FDI problem of Definition 1, is generically solvable if and only if: $S^* = F$.

4.2 Minimal input separator and the maximal flow problem

It is well known that the maximal size of a fault-output linking is closely related to the max flow-min cut problem in an auxiliary graph. The auxiliary graph of $G(\Sigma_\Lambda)$ is denoted $G_a(\Sigma_\Lambda)$ and defined as follows. We split each fault, state and output vertex v of $G(\Sigma_\Lambda)$ in two vertices v' and v'' of $G_a(\Sigma_\Lambda)$ with an edge (v', v'') connecting them of capacity one. Each edge of the form (v, w) in $G(\Sigma_\Lambda)$ is transformed in an edge (v'', w') in $G_a(\Sigma_\Lambda)$. We add then two vertices, a source s^+ with an edge from this source to all the fault vertices f_i', and a sink s^- with an edge from the output vertices y_j'' to this sink. The edges whose capacities have not been defined have an infinite capacity. Recall that a flow is a real number $f(e)$ associated with each edge e such that in each vertex (except s^+ and s^-) the first Kirchoff's law is satisfied i.e. the sum of the flows on the incoming edges is equal to the sum of the flows on the outcoming edges. Moreover the flows must satisfy the capacity constraints i.e. on each edge $0 \le f(e) \le c(e)$ where $f(e)$ is the flow on the edge e and $c(e)$ is the capacity of the edge. The flow is maximal when the total outcoming flow from s^+ (which is equal to the total incoming flow in s^-) is maximal. A cut is defined by a partition of the vertex set in two sets $Z = Z^+ \cup Z^-$ where $s^+ \in Z^+$ and $s^- \in Z^-$, the capacity of a cut is the sum of the capacities of edges with begin vertex in Z^+ and end vertex in Z^-.
We have the following result, see (Ford and Fulkerson, 1962; Yamada, 1988; Murota, 1987; Hovelaque *et al.*, 1996), which provides us with an efficient way to get the minimal input separator.

Theorem 6. The size of a maximal fault-output linking in $G(\Sigma_\Lambda)$ is the value of the maximal flow in $G_a(\Sigma_\Lambda)$. The minimal input separator S^* of $G(\Sigma_\Lambda)$ is given by the minimal cut obtained by the Ford and Fulkerson algorithm in $G_a(\Sigma_\Lambda)$.

4.3 Reduced system

The reduced system $\Sigma_{R\Lambda}$ is a structured system defined by its graph $G(\Sigma_{R\Lambda})$ with input set F_R, output set Y_R, state set X_R where $F_R = F/(F \bigcap S^*)$, $Y_R = S^*/(F \bigcap S^*)$ and X_R is the set of state vertices in any direct fault-output path from F_R to Y_R. The set of edges corresponds to the edges in any direct path from F_R to Y_R. Notice that Y_R is not in general a subset of Y.

Consider now that we can add a new sensor which provides us with a new measure $z_j(t) = H_j x(t) + P_j f(t)$. We say that a state or input variable is measured by this sensor if the corresponding entry in H_j or P_j is a non zero parameter. Denote by k the size of a maximal fault-output linking in the graph $G(\Sigma_\Lambda)$. We get the following result.

Theorem 7. Consider the linear structurally observable system Σ_Λ defined by (7) with its associated graph $G(\Sigma_\Lambda)$. Consider the structured system $\Sigma_\Lambda^{(j)}$ which is obtained from Σ_Λ by adding the additional sensor z_j, and its associated graph $G(\Sigma_\Lambda^{(j)})$ obtained from $G(\Sigma_\Lambda)$ by adding the output vertex z_j and incident edges. A maximal fault-output linking in $G(\Sigma_\Lambda^{(j)})$ has size $k+1$ if and only if the new sensor measures variables in $F_R \cup X_R$.

Measuring variables "after S^*" is therefore useless for FDI purpose. It is proved in (Commault and Dion, 2004) that efficient sensors should measure variables in the reduced system $\Sigma_{R\Lambda}$. From Theorem 7 any additional measure on this reduced system will increase the size of a maximal fault-output linking in the graph by one unit. In the next subsection we will decompose the system and the reduced system in order to determine the potential size increase of such maximal linkings when adding several additional measures.

4.4 System decomposition

In this subsection we will use iteratively Theorem 6 to get a set of input separators which will induce a decomposition of the system well suited for the sensor location FDI problem. We propose the following iterative procedure:
Initialisation: $i = 1$; G_a=$G_a(\Sigma_\Lambda)$
DO

- Apply the Ford and Fulkerson algorithm on G_a and get by Theorem 6 the separator S_i^*, denote k_i its cardinality.
- Set to infinity the capacities of the edges of G_a corresponding to $(S_i^*/S_i^* \cap F)$ as well as all the downstream edges (edges on a direct path from S_i^* to Y).
- i=i+1.

UNTIL $S_i^* = F$
This algorithm gives us a sequence of ν separators S_i^* for $i = 1, \ldots, \nu$ of respective cardinality k_i. Notice that at the first step $S_1^* = S^*$ which allows to construct the reduced system $\Sigma_{R\Lambda}$. At the last step $S_\nu^* = F$. These separators induce a natural decomposition of the system Σ_Λ. It is clear from the construction that $k_i < k_{i+1}$ for $i = 1, \ldots, \nu$. Moreover S_{i+1}^* is "closer to inputs" than S_i^*, which means that on any path from F to S_i^* there is a vertex in S_{i+1}^*. Considering the separator S_i^* we can define the sets T_i^+ for $i = 1, \ldots, \nu$.

Remark 8. The separators S_i^* are sets of k_i vertices of $G(\Sigma_\Lambda)$ such that there exists a F-S_i^* linking of size k_i in $G(\Sigma_\Lambda)$.

Definition 9. Consider the structured system Σ_Λ with its associated graph $G(\Sigma_\Lambda)$. Consider the separators S_i^* for $i = 1, \ldots, \nu$ obtained in a constructive way as in the previous algorithm.
Define T_i^+ as the set of all vertices in any direct path from F to S_i^* in $G(\Sigma_\Lambda)$ except for the vertices of S_i^*.
The vertices sets T_i^+ are ordered for inclusion. $T_\nu^+ \subset \ldots T_i^+ \subset \ldots T_1^+$, and define a decomposition of $G(\Sigma_\Lambda)$. It turns out that the vertices of the reduced system graph $G(\Sigma_{R\Lambda})$ are $T_1^+ \cup S^*$.

5. APPLICATION TO THE SENSOR LOCATION FDI PROBLEM

We will now use the above system decomposition to tackle the sensor location problem. Using the algorithm of the subsection 4.4 we obtain our main Theorem.

Theorem 10. Consider the linear structurally observable system Σ_Λ defined by (7) with its associated graph $G(\Sigma_\Lambda)$. Consider the separators S_i^* and the sets T_i^+ defined previously. In any solution of the FDI problem and for $i = 1, \ldots, \nu$ there are at least $r - k_i$ additional sensors which measure vertices of T_i^+.

Remark 11. The proof can be done using (8) and computing a maximal size fault-output linking in the graph $G(\Sigma_\Lambda^c)$ of the composite system Σ_Λ^c defined in (8). Notice that for the case $i = 1$ we recover the result of (Commault and Dion, 2003).

6. ILLUSTRATIVE EXAMPLE

Example 2: Consider the structured system with 9 faults and 3 outputs whose graph is depicted in Figure 2. The same graph was studied in another context in (Murota, 1987). The corresponding auxiliary graph is given in Figure 3. Notice that for simplicity the splitting of vertices has not been done but instead we indicate for each vertex the corresponding capacity. For simplicity also the source and sink with adjacent edges have been omitted.
The infinite capacities are omitted.
The first step of our algorithm gives $S_1^* = S^* = \{y_1, y_2, y_3\}$ which shows that the system is in reduced form. In the same way in a second step, after the relaxation of capacities we get the separator $S_2^* = \{f_1, f_2, x_3, x_5\}$. The complete decomposition with the separators $S_1^*, S_2^*, S_3^*, S_4^*$ is given in Figure 4 with $S_3^* = \{f_1, f_2, f_3, f_4, f_5, f_6, x_7\}$ and $S_4^* = F$. The size of these separators is $k_1 = 3, k_2 = 4, k_3 = 7, k_4 = 9$.
The application of Theorem 10 to our example gives a number of interesting informations on the usefulness of additional sensors for FDI, for example:

- Since $k_1 = 3$, $r = 9$, in any solution of the FDI problem there are at least 6 additional sensors which measure vertices of $T_1^+ = F \cup X$.

- For $i = 3$, since $k_3 = 7$, in any solution of the FDI problem there are at least 2 additional sensors which measure vertices of $T_3^+ = \{f_7, f_8, f_9\}$.

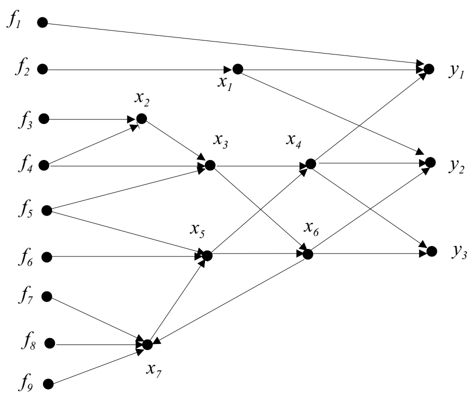

Fig. 2. Graph $G(\Sigma_\Lambda)$ of example 2

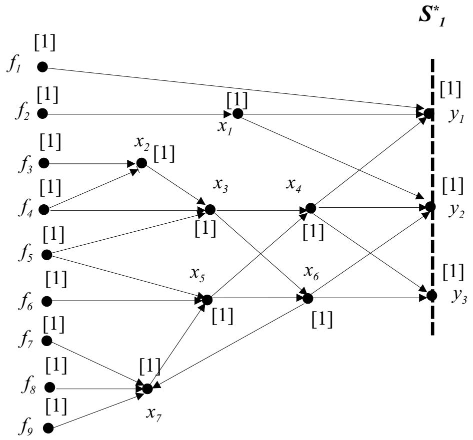

Fig. 3. The auxiliary graph and the separator S_1^*

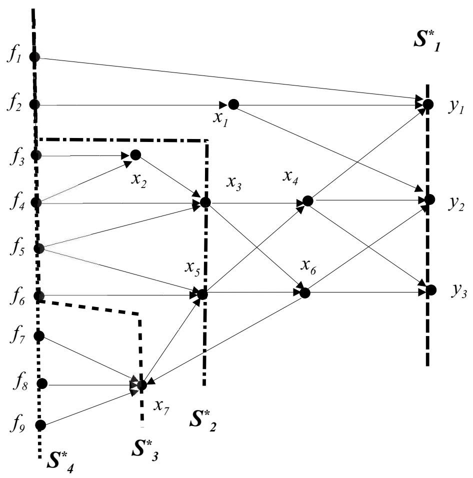

Fig. 4. System decomposition of example 2

7. CONCLUDING REMARKS

In this paper we have presented a new system decomposition which is well suited for the sensor location in the Fault Detection and Isolation problem (FDI).

We have dealt with this problem when the system under consideration is structured. The proposed decomposition is based on some new separators and extends and refines previous works. It can be obtained easily using standard and efficient algorithms of combinatorial optimization. It gives very useful informations on the possible location of additional sensors.

REFERENCES

Chen, J. and R.J. Patton (1999). *Robust model-based fault diagnosis for dynamic systems*. Kluwer academic publishers.

Commault, C. and J.M. Dion (2003). Sensor location for diagnosis in linear structured systems with disturbances. In: *IEEE CDC Conf.*. Maui, Hawaii.

Commault, C. and J.M. Dion (2004). Sensor location for diagnosis in linear systems: a structural analysis. Technical report. Laboratoire d'Automatique de Grenoble, France.

Commault, C., J.M. Dion, O. Sename and R. Motyeian (2002). Observer-based fault detection and isolation for structured systems. *IEEE Trans. Automat. Control* **47**(12), 2074–2079.

Dion, J.M., C. Commault and J. van der Woude (2003). Generic properties and control of linear structured systems: a survey. *Automatica* **39**(7), 1125–1144.

Ford, L.R. and D.R. Fulkerson (1962). *Flows in networks*. Princeton University Press.

Frank, P.M. (1996). Analytical and qualitative model-based fault diagnosis - a survey and some new results. *European Journal of Control* **2**, 6–28.

Hovelaque, V., C. Commault and J.M. Dion (1996). Analysis of linear systems using a primal-dual algorithm. *Systems and Control Letters* **27**, 73–85.

Lin, C.T. (1974). Structural controllability. *IEEE Trans. Automat. Control* **19**, 201–208.

Murota, K. (1987). *Systems Analysis by Graphs and Matroids*. Vol. 3 of *Algorithms and Combinatorics*. Springer-Verlag New-York, Inc.

Reinschke, K.J. (1988). *Multivariable control : A graph-theoretic approach*. Vol. 108 of *Lect. Notes in Control and Information Sciences*. Springer-Verlag.

van der Woude, J.W. (1991). A graph theoretic characterization for the rank of the transfer matrix of a structured system. *Math. Control Signals Systems* **4**, 33–40.

van der Woude, J.W. (2000). The generic number of invariant zeros of a structured linear system. *SIAM Journ. of Control* **38**, 1–21.

Yamada, T. (1988). A network flow algorithm to find an elementary i/o matching. *Networks* **18**, 105–109.