PARAMETRIZATION OF SUPPLY RATES FOR ESTABLISHING ISS AND PRESCRIBED DISSIPATIVE PROPERTIES OF NONLINEAR INTERCONNECTED SYSTEMS

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Abstract: In this paper, a novel technique of parametrization of supply rates is developed for establishing Input-to-State Stability(ISS) and dissipative properties of nonlinear interconnected systems. In the application of the ISS small-gain theorem, selecting supply rates to establish stability properties is not a straightforward task. This paper proposes useful tools for selecting supply rates by allowing some new freedoms in supply rates. The idea of the parametrization provides us with a set of various supply rates with which a fixed ISS small-gain condition can establish the ISS of the interconnected system. Dissipative properties of the interconnected system can be also prescribed in the parametrization.

Keywords: Nonlinear systems, Interconnected systems, Parametrization, Supply rates, Input-to-state stability, Small gain condition, Lyapunov function

1. INTRODUCTION

The task of establishing stability properties of interconnected systems is very important for a broad range of nonlinear systems control and design. There have been a lot of efforts put into the development of useful conditions which serve to establish stability properties of various classes of nonlinear systems effectively. The dissipative paradigm(Willems, 1972; Hill and Moylan, 1977) and the ISS small-gain theorem(Jiang *et al.*, 1994; Teel, 1996) are some of key contributions in this respect. Usually, the ISS small-gain theorem is explained in terms of trajectories, i.e., solutions, of systems. Since the input-to-state stability(ISS) is a special type of dissipation(Sontag and Wang, 1995), the ISS small-gain theorem can be also described without using solutions of the systems(Jiang *et al.*, 1996). The Lyapunov formulation of the ISS small-gain theorem establishes global stability properties of interconnected systems on the basis of nonlinear gain functions derived from supply rates which determine dissipation rates of individual subsystems. For nonlinear systems, it is reasonable that we do not assume knowledge of explicit solutions, and the idea dates back to the works of Lyapunov. The supply rates play a central role in getting rid of solutions computation in the application of the ISS small-gain theorem.

Roughly speaking, the ISS small-gain theorem applies to the interconnected system of the form

$$\Sigma_1 : \ \dot{x}_1 = f_1(x_1, u_1, r_1), \quad u_1 = x_2 \qquad (1)$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(x_2, u_2, r_2), \quad u_2 = x_1 \qquad (2)$$

It assumes the existence of continuous functions V_i, $\underline{\alpha}_i$, $\bar{\alpha}_i$, α_i and σ_i such that

$$\underline{\alpha}_i(|x_i|) \le V_i(x_i) \le \bar{\alpha}_i(|x_i|) \qquad (3)$$

$$\frac{\partial V_i}{\partial x_i} f_i(x_i,u_i,r_i) \le -\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|) \qquad (4)$$

$$\underline{\alpha}_i, \bar{\alpha}_i, \alpha_i \in \mathcal{K}_\infty, \quad \sigma_i, \sigma_{ri} \in \mathcal{K} \qquad (5)$$

hold. The ISS small-gain theorem is written as

$$\Gamma_2 \circ \Gamma_1(s) < 1, \quad \forall s \in (0, \infty) \qquad (6)$$

$$\Downarrow$$

$$\text{ISS with respect to } \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

where Γ_i's are nonlinear gain given by

$$\Gamma_i(s) = \underline{\alpha}_i^{-1} \circ \bar{\alpha}_i \circ \alpha_i^{-1} \circ c_i\sigma_i(s) \tag{7}$$

for $c_i > 1$(Sontag and Wang, 1995; Isidori, 1999). In order to invoke the ISS small-gain theorem, we need to pick the supply rates $-\alpha_i(|x_i|)+\sigma_i(|u_i|)+\sigma_{ri}(|r_i|)$ beforehand. We should be aware that the ISS small-gain *condition* (6) never leads us to the stability if one fails to find supply rates such that

- the supply rates are accepted by Σ_i, $i=1,2$;
- the supply rates satisfy the ISS small-gain condition

at the same time. Selecting supply rates fulfilling these two requirements simultaneously is not an easy task. The purpose of this paper is to develop useful tools for accomplishing this task.

This paper employs the unique idea of *parametrization of supply rates* which originates from Ito (2003), and proposes non-trivial extensions. The function in the form of (4) may not be the only supply rate that establishes the stability under the ISS small-gain *condition* (6). If we could obtain many candidates for a supply rate from a fixed small-gain condition, there would be more chances to come at a supply rate that fit the system. This idea leads to the following problem.

Parametrization of supply rates: Suppose that a supply rate of Σ_2 is fixed *a priori*. Find a set of *multiple* supply rates for Σ_1 with which a desired stability property of the interconnected system can be proved under the *single* ISS small-gain condition (6).

The previous study has addressed only the global asymptotic stability as a stability property of the interconnected system(Ito, 2003). It is not yet known whether the technique of *parametrization of supply rates* can yield the ISS which is stronger than the global asymptotic stability. This paper not only provides a positive answer to this question, but also broaden the set of parametrized supply rates. Furthermore, this paper develops a new tool for the parametrization with which we can prescribe dissipative properties we require for the interconnected system *a priori*.

2. A MOTIVATING EXAMPLE

This section is devoted to providing a simple example that illustrates the idea and usefulness of the problem addressed by this paper.

Consider an interconnected system defined by

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -10x_1^5 - x_1^3 + x_1^4 x_2 + x_1 r_1^2 \\ e = x_1^2 \end{cases} \tag{8}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(x_2, x_1) \tag{9}$$

where x_1, x_2 and r_1 are scalar. It is supposed that we do not have information about the nonlinear system Σ_2 except that there exists a continuously differentiable function $V_2(x_2)$ satisfying

$$\dot{V}_2(x_2) = dV_2/dt \le -x_2^2 + \gamma_2^2 x_1^2, \quad \gamma_2 = 8 \tag{10}$$

along the trajectories of Σ_2, namely, the $\mathcal{L}_2$-gain between x_1 and x_2 is less than or equal to γ_2 (van der Schaft, 1999; Isidori, 1999). In this section, regarding properties of the interconnected system (8)-(9), we address questions of whether $[x_1, x_2]^T = 0$ is globally asymptotically stable in the absence of the exogenous input r_1, and whether the $\mathcal{L}_2$-gain between the input r_1 and the output e_1 is less than or equal to one.

We start with a quadratic function $V_1(x_1) = x_1^2$ which plays the role of a storage function(Willems, 1972) and an ISS Lyapunov function(Sontag and Wang, 1995) for Σ_1. The time-derivative of $V_1(x_1)$ along the trajectories of (8) is obtained as

$$\dot{V}_1(x_1) = 2x_1(-10x_1^5 - x_1^3 + x_1^4 x_2 + x_1 r_1^2) \tag{11}$$

It is verified easily from (11)-(10) that for any constant $c > 0$, there does not exist a positive definite function $\rho_e(x_1, x_2)$ such that

$$\dot{V}_1(x_1) + c\dot{V}_2(x_2) \le -\rho_e(x_1,x_2) + c\{-e^2 + r_1^2\}, \quad \forall x_1, x_2, r_1 \in \mathbb{R} \tag{12}$$

is satisfied. Notice that if (12) was achieved, the function $V_{cl}(x_1,x_2) = V_1(x_1) + cV_2(x_2)$ could be a storage function establishing the global asymptotic stability and the unit $\mathcal{L}_2$-gain between r_1 and e_1. Thus, the selection $V_{cl}(x_1,x_2) = V_1(x_1) + cV_2(x_2)$ is not successful in answering our questions of the stability and the $\mathcal{L}_2$-gain.

The time-derivative of $V_1(x_1)$ along the trajectories of (8) can be rewritten from (11) as

$$\dot{V}_1(x_1) \le \begin{cases} 2(-10x_1^6 + a x_1^6) & \text{for } |x_2| \le a|x_1| \\ 2(-10x_1^6 + a^{-5}x_2^6) & \text{for } |x_2| \ge a|x_1| \end{cases} \\ -2x_1^4 + 2x_1^2 r_1^2, \qquad a>0 \\ \le \{-2(10-a)x_1^6 - 2x_1^4 + 2a^{-5}x_2^6\} + 2x_1^2 r_1^2 \tag{13}$$

The nonlinear gain function $\Gamma_1(s)$ of Σ_1 with respect to the input x_2 and the state x_1 can be calculated from (13) as the unique inverse map of

$$\Gamma_1^{-1}(s) = (1+\epsilon)^{-1}\left\{a^5(10-a)s^6 + a^5 s^4\right\}^{1/6}$$

for any $\epsilon > 0$ and $0 < a < 10$(Isidori, 1999; Sontag and Wang, 1995). Since the nonlinear gain of Σ_2 given in (10) is $\Gamma_2(s) = (1+\epsilon)8s$, we have

$$\frac{\Gamma_2(s)}{\Gamma_1^{-1}(s)} = (1+\epsilon)^2 \frac{8s}{\{a^5(10-a)s^6 + a^5 s^4\}^{1/6}}$$

$$\sup_{s\in\mathbb{R}_+} \frac{\Gamma_2(s)}{\Gamma_1^{-1}(s)} = \lim_{s\to\infty} \frac{\Gamma_2(s)}{\Gamma_1^{-1}(s)} = \frac{(1+\epsilon)^2 8}{[a^5(10-a)]^{1/6}}$$

The minimum is achieved with $a = 50/6$ as follows:

$$\min_{a>0}\left[\sup_{s\in\mathbb{R}_+} \frac{\Gamma_2(s)}{\Gamma_1^{-1}(s)}\right] \simeq (1+\epsilon)^2 \frac{8}{6.3727} \not\le 1$$

which implies that the inequality (6) does not hold for any $a > 0$. Thus, the ISS small-gain theorem (Jiang *et al.*, 1994) together with the ISS Lyapunov function $V_1(x_1) = x_1^2$ does not lead us to the global asymptotic stability.

Next, we calculate the time-derivative of $V_1(x_1)$ along the trajectories of (8) in the following way.

$$\dot{V}_1(x_1) \le 9x_1^4\left\{-x_1^2 - \frac{2}{11} + \gamma_1^2 x_2^2\right\} + 2x_1^2\left\{-x_1^4 - \frac{2}{11}x^2 + r_1^2\right\} \quad (14)$$

$$\le \frac{9x_1^2}{2}\left(2x_1^2 + \frac{4}{11}\right)\{-x_1^2 + \gamma_1^2 x_2^2\} + \left(2x_1^2 + \frac{4}{11}\right)\{-x_1^4 + r_1^2\} \quad (15)$$

The inequality (14) holds for $\gamma_1 > 1/9$. Motivated by $8 \cdot 1/9 < 1$, we define

$$\hat{V}_1(x_1) = \int_0^{V_1(x_1)} \left(\frac{9s}{2}\left(2s + \frac{4}{11}\right)\right)^{-1} ds \quad (16)$$

so that we have

$$\dot{\hat{V}}_1(x_1) = \left(\frac{9x_1^2}{2}\left(2x_1^2 + \frac{4}{11}\right)\right)^{-1} \dot{V}_1(x_1) \le \{-x_1^2 + \gamma_1^2 x_2^2\} + \frac{2}{9x_1^2}\{-e_1^2 + r_1^2\} \quad (17)$$

along the trajectories of (8). The function in (16) is, however, not qualified as a Lyapunov function. It is not integrable. It should be also stressed that a function in the form of

$$\hat{V}_1(x_1) = \int_0^{V_1(x_1)} \mu(s)ds$$

with a positive-valued function $\mu(s)$ decreasing faster than or as fast as $1/s^2$ toward ∞ is not radially unbounded, so that it cannot be used for proving global properties(Sontag and Teel, 1995). Thus, the $\mathcal{L}_2$ small-gain theorem together with (16) cannot lead us to the global asymptotic stability. The selection $V_{cl}(x_1, x_2) = \hat{V}_1(x_1) + cV_2(x_2)$ does not prove the desired $\mathcal{L}_2$-gain between r_1 and e_1 either due to the coefficient $2/9x_1^2$ in (17).

Finally, we look at the problem by constructing a Lyapunov function of the overall system. Assume that Σ_2 achieves (10) with $V_2(x_2) = x_2^2$. Define

$$V_{cl}(x_1, x_2) = \int_0^{V_1(x_1)} \frac{11}{22s+4} ds + \int_0^{V_2(x_2)} cs\, ds \quad (18)$$

for $c > 0$. Then, for $\gamma_1 > 1/9$, from (15) we obtain

$$\dot{V}_{cl}(x_1, x_2) \le \frac{9x_1^2}{2}\{-x_1^2 + \gamma_1^2 x_2^2\} + \{-x_1^4 + r_1^2\} + cx_2^2\{-x_2^2 + \gamma_2^2 x_1^2\} \le -\rho_e(x_1, x_2) + \{-e_1^2 + r_1^2\} \quad (19)$$

along the trajectories of (8)-(9). It can be verified that there exists a constant $c > 0$ such that the inequality (19) holds with some positive definite $\rho_e(x_1, x_2)$ if and only if

$$\gamma_1 < 1/\gamma_2 \quad (20)$$

holds. Since $V_{cl}(x_1, x_2)$ defined by (18) is positive definite and radially unbounded, it proves the global asymptotic stability of $[x_1, x_2]^T = 0$. It also yields that the $\mathcal{L}_2$-gain between r_1 and e_1 is less than or equal to one since (19) implies

$$V_{cl}|_{t=T} - V_{cl}|_{t=t_0} \le \int_{t_0}^T -e_1^2 + r_1^2\, dt$$

Through the above discussions, the example has suggested the following two points.

A. the existence of a positive definite function $V_1(x_1)$ which is radially unbounded and satisfies

$$\frac{dV_1}{dt} \le \frac{9x_1^2}{2}\left(2x_1^2 + \frac{4}{11}\right)\{-x_1^2 + \gamma_1^2 x_2^2\} + \left(2x_1^2 + \frac{4}{11}\right)\{-x_1^4 + r_1^2\} \quad (21)$$

$$\gamma_2\gamma_1 < 1 \quad (22)$$

implies global asymptotic stability and $\mathcal{L}_2$-gain≤ 1 of the interconnected system (8)-(9);

B. In the case of $V_2(x_2) = x_2^2$, the function

$$V_{cl}(x_1, x_2) = \int_0^{V_1(x_1)} \frac{11}{22s+4} ds + \int_0^{V_2(x_2)} cs\, ds \quad (23)$$

establishes the the global asymptotic stability and the $\mathcal{L}_2$-gain≤ 1 simultaneously.

The claim **A** is trivial if (21) is replaced by

$$\frac{dV_1}{dt} \le \{-x_1^2 + \gamma_1^2 x_2^2\} + \{-x_1^4 + r_1^2\} \quad (24)$$

Indeed, it becomes the standard $\mathcal{L}_2$ small-gain theorem. The claim **A** suggests that the properties of global asymptotic stability and prescribed $\mathcal{L}_2$-gain may be often established even if the property (24) commonly used with the small-gain theorem is relaxed in the form of

$$\frac{dV_1}{dt} \le \hat{\lambda}(x_1)\{-x_1^2 + \gamma_1^2 x_2^2\} + \hat{\lambda}_P(x_1)\{-x_1^4 + r_1^2\} \quad (25)$$

It is desirable if we can predict when and what kind of functions $\hat{\lambda}(x_1)$ and $\hat{\lambda}_P(x_1)$ are allowed to be used before constructing a Lyapunov function $V_{cl}(x_1, x_2)$ for the entire interconnected system. In the above example, the usage of particular $\hat{\lambda}(x_1)$ and $\hat{\lambda}_P(x_1)$ is justified in a heuristic manner only when $V_2(x_2) = x_2^2$. It is greatly useful if the appropriateness of incorporating $\hat{\lambda}(x_1)$ and $\hat{\lambda}_P(x_1)$ is answered systematically without knowing V_2 and without invoking the construction of a Lyapunov function V_{cl} for the closed-loop system. The claim

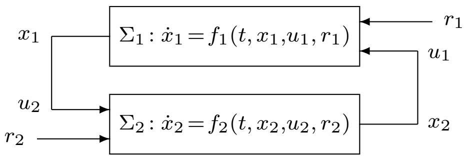

Fig. 1. Feedback interconnected system Σ

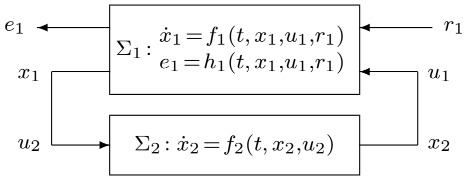

Fig. 2. Feedback interconnected system Σ_P

B presents a curious Lyapunov function of the closed-loop system when V_2 is given explicitly. In the example, the Lyapunov function is discovered in a heuristic way. In the rest of this paper,

- the claim **A** is justified independently of the choice of V_2;
- the idea of (25) is formulated precisely so that the problem of stability can be answered without constructing a Lyapunov function V_{cl} of the interconnected system;
- the idea of (25) is made applicable to a general class of supply rates and dissipative properties;
- a formula for constructing a Lyapunov function V_{cl} of the interconnected system is shown.

3. PARAMETRIZATION OF SUPPLY RATES

This section presents main results. Consider the nonlinear interconnected system Σ shown in Fig.1. Suppose that Σ_1 and Σ_2 are described by

$$\Sigma_1 : \ \dot{x}_1 = f_1(t, x_1, u_1, r_1) \qquad (26)$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(t, x_2, u_2, r_2) \qquad (27)$$

These two systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. Assume that $f_1(t, 0, 0, 0) = 0$ and $f_2(t, 0, 0, 0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \geq 0$. The functions f_1 and f_2 are supposed to be piecewise continuous in t, and locally Lipschitz in the other arguments. The exogenous inputs $r_1 \in \mathbb{R}^{m_1}$ and $r_2 \in \mathbb{R}^{m_2}$ are packed into a single vector $r = [r_1^T, r_2^T]^T \in \mathbb{R}^m$. The state vector of the interconnected system Σ is $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ where $x_i \in \mathbb{R}^{n_i}$ is the state of Σ_i. We make the following assumption.

Assumption 1. There exist a $\mathbf{C}^1$ function $V_2 : \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$, continuous functions $\alpha_2 \in \mathcal{K}_\infty$ and $\sigma_2, \sigma_{r2} \in \mathcal{K}$ such that

$$\underline{\alpha}_2(|x_2|) \leq V_2(t, x_2) \leq \bar{\alpha}_2(|x_2|) \qquad (28)$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2, u_2, r_2) \leq -\alpha_2(|x_2|) + \sigma_2(|u_2|) + \sigma_{r2}(|r_2|) \qquad (29)$$

hold for all $x_2 \in \mathbb{R}^{n_2}$, $u_2 \in \mathbb{R}^{n_1}$, $r_2 \in \mathbb{R}^{m_2}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$.

The following is the first main result.

Theorem 2. Suppose that the system Σ_2 satisfies Assumption 1. Suppose that real numbers $c_1 > 1$, $c_2 > 1$ and class $\mathcal{K}_\infty$ functions α_1 and σ_1 satisfy

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \qquad (30)$$

If there exist a continuous function $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}$ and a $\mathbf{C}^1$ functions $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}$ such that

$$\hat{\lambda}(s) > 0, \ \forall s \in (0, \infty) \qquad (31)$$

$$\underline{\alpha}_1(|x_1|) \leq V_1(t, x_1) \leq \bar{\alpha}_1(|x_1|) \qquad (32)$$

$$\frac{\partial V_1(t, x_1)}{\partial t} + \frac{\partial V_1(t, x_1)}{\partial x_1} f_1(t, x_1, u_1, r_1) \leq \hat{\lambda}(V_1(t, x_1)) \left[-\alpha_1(|x_1|) + \sigma_1(|u_1|) + \sigma_{r1}(|r_1|)\right] \qquad (33)$$

hold for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_1$, $\bar{\alpha}_1 \in \mathcal{K}_\infty$ and some $\sigma_{r1} \in \mathcal{K}$, then the interconnected system Σ is ISS with respect to input r and state x.

Theorem 2 includes the standard version of the ISS small-gain theorem as a special case. In fact, when we pick $\hat{\lambda}(s) = 1$, the theorem reduced to the ISS small-gain theorem. The inequality (30) is often referred to as the ISS small-gain condition. Theorem 2 incorporates a free function $\hat{\lambda}$ into the stability test based on the ISS small-gain *condition*. The function $\hat{\lambda}$ in (33) provides flexibility in selecting a supply rate of Σ_1 to establish the ISS property of the interconnected system. The free function $\hat{\lambda}$ allows us to *scale* an initial supply rate $-\alpha_1(|x_1|) + \sigma_1(|u_1|) + \sigma_{r1}(|r_1|)$ which is chosen such that the ISS small-gain condition (30) is fulfilled. The function $\hat{\lambda}$ only needs to be continuous and positive in the sense of (31). Theorem 2, thereby, offers the technique of *parametrization of supply rates* for the ISS small-gain condition. Its usefulness is that the freedom of $\hat{\lambda}$ can be utilized to have (33) fulfilled by a given system Σ_1. The flexibility of $\hat{\lambda}$ in the supply rate offers more chances to come at a supply rate that fit the system Σ_1.

Theorem 2 enables us to check the ISS property without constructing a Lyapunov function of the closed-loop system. A formula for the Lyapunov function proving the ISS property of the interconnected system is presented in the next section.

Remark 3. In the case of $\sigma_1 \in \mathcal{K} \backslash \mathcal{K}_\infty$, there exists $\hat{\sigma}_1 \in \mathcal{K}_\infty$ such that

$$\sigma_1(s) \leq \hat{\sigma}_1(s),$$

$$c_1\hat{\sigma}_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s)$$

are satisfied for all $s \in \mathbb{R}_+$. Therefore, Theorem 2 is applicable by replacing σ_1 with $\hat{\sigma}_1$.

Remark 4. For the set of functions $\hat{\lambda}$ allowed by Theorem 2, the choice $\int_0^{V_1} 1/\hat{\lambda}(s)ds$ is not guaranteed to be radially unbounded, so that it is *not* qualified to be a Lyapunov function proving the *global* properties such as the ISS. Indeed, Theorem 2 is developed by using a Lyapunov function different from $\int_0^{V_1} 1/\hat{\lambda}(s)ds$. Therefore, Theorem 2 is fundamentally beyond the technique in Sontag and Teel (1995) and Isidori (1999).

Next, consider the nonlinear interconnected system Σ_P shown in Fig.2 consisting of

$$\Sigma_1 : \begin{cases} \dot{x}_1 = f_1(t, x_1, u_1, r_1) \\ e_1 = h_1(t, x_1, u_1, r_1) \end{cases} \tag{34}$$

$$\Sigma_2 : \quad \dot{x}_2 = f_2(t, x_2, u_2) \tag{35}$$

These two systems are connected each other through $u_1 = x_2$ and $u_2 = x_1$. Assume that $f_1(t, 0, 0, 0) = 0$ and $f_2(t, 0, 0) = 0$ hold for all $t \in [t_0, \infty)$, $t_0 \geq 0$. The functions f_1 and f_2 are supposed to be piecewise continuous in t, and locally Lipschitz in the other arguments.

Assumption 5. There exist a $\mathbf{C}^1$ function $V_2 : \mathbb{R}_+ \times \mathbb{R}^{n_2} \to \mathbb{R}_+$, continuous functions $\alpha_2 \in \mathcal{K}_\infty$ and $\sigma_2 \in \mathcal{K}$ such that

$$\underline{\alpha}_2(|x_2|) \leq V_2(t, x_2) \leq \bar{\alpha}_2(|x_2|) \tag{36}$$

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x_2} f_2(t, x_2, u_2) \leq -\alpha_2(|x_2|) + \sigma_2(|u_2|) \tag{37}$$

hold for all $x_2 \in \mathbb{R}^{n_2}$, $u_2 \in \mathbb{R}^{n_1}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_2$, $\bar{\alpha}_2 \in \mathcal{K}_\infty$.

The following is the second main result.

Theorem 6. Suppose that the system Σ_2 satisfies Assumption 5. Suppose that real numbers $c_1 > 1$, $c_2 > 1$ and class $\mathcal{K}_\infty$ functions α_1 and σ_1 satisfy

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \bar{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1 \circ \bar{\alpha}_1^{-1} \circ \underline{\alpha}_1(s), \ \forall s \in \mathbb{R}_+ \tag{38}$$

and a continuous function $\rho_{cl}(e_1, r_1) : \mathbb{R}^{l_1} \times \mathbb{R}^{m_1} \to \mathbb{R}$ satisfies

$$\rho_{cl}(e_1, 0) \leq 0, \quad \forall e_1 \in \mathbb{R}^{l_1} \tag{39}$$

and there exist continuous functions $\hat{\lambda} : \mathbb{R}_+ \to \mathbb{R}$, $\hat{\lambda}_P : \mathbb{R}_+ \to \mathbb{R}$ and a $\mathbf{C}^1$ functions $V_1 : \mathbb{R}_+ \times \mathbb{R}^{n_1} \to \mathbb{R}$ such that

$$\hat{\lambda}(s) > 0, \ \forall s \in (0, \infty) \tag{40}$$

$$\hat{\lambda}_P(s) > 0, \ \forall s \in [0, \infty) \tag{41}$$

$$\int_1^\infty \frac{1}{\hat{\lambda}_P(s)} ds = \infty \tag{42}$$

$$\frac{\hat{\lambda}(s)}{\hat{\lambda}_P(s)} : \text{non-decreasing} \tag{43}$$

$$\frac{\hat{\lambda}(s)\left[\frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right]}{\hat{\lambda}_P(s)\left[\alpha_2 \circ \sigma_1^{-1} \circ \frac{1}{\tau_1}\alpha_1 \circ \bar{\alpha}_1^{-1}(s)\right]} : \text{non-decreasing} \tag{44}$$

$$\underline{\alpha}_1(|x_1|) \leq V_1(t, x_1) \leq \bar{\alpha}_1(|x_1|) \tag{45}$$

$$\frac{\partial V_1(t, x_1)}{\partial t} + \frac{\partial V_1(t, x_1)}{\partial x_1} f_1(t, x_1, u_1, r_1) \leq \hat{\lambda}(V_1(t, x_1))\left[-\alpha_1(|x_1|) + \sigma_1(|u_1|)\right] + \hat{\lambda}_P(V_1(t, x_1))\rho_{cl}(e_1, r_1) \tag{46}$$

hold for all $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{n_2}$, $r_1 \in \mathbb{R}^{m_1}$ and $t \in \mathbb{R}_+$ with some $\underline{\alpha}_1$, $\bar{\alpha}_1 \in \mathcal{K}_\infty$, where τ_1 is any real numbers satisfying

$$0 \leq m, \ 1 < \tau_1 < c_1, \ (\tau_1/c_1)^m \leq (\tau_1 - 1)(c_2 - 1) \tag{47}$$

Then, the equilibrium $x = 0$ of the interconnected system Σ_P is globally uniformly asymptotically stable for $r_1 \equiv 0$. Furthermore, there exist a $\mathbf{C}^1$ function $V_{cl} : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_{cl}$, $\bar{\alpha}_{cl}$ such that

$$\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t, x) \leq \bar{\alpha}_{cl}(|x|), \ \forall x \in \mathbb{R}^n, t \in \mathbb{R}_+ \tag{48}$$

is satisfied and the dissipation inequality

$$\dot{V}_{cl} \leq \rho_{cl}(e_1, r_1), \quad \forall x \in \mathbb{R}^n, r_1 \in \mathbb{R}^{m_1}, t \in \mathbb{R}_+ \tag{49}$$

holds along the trajectories of Σ_P.

Theorem 2 guarantees that the interconnected system is ISS. We are, however, not able to impose a particular strength of ISS *a priori*. For example, the nonlinear gain with respect to input r and state x may be very large. In contrast, Theorem 6 allows us to prescribe a dissipative property determined by the supply rate $\rho_{cl}(e_1, r_1)$. Theorem 6 shows that an additional parameter $\hat{\lambda}_P(s)$ can be used for obtaining the dissipative property.

Remark 7. An easy way to pick an initial supply rate $-\alpha_1(|x_1|) + \sigma_1(|u_1|)$ fulfilling the ISS small-gain condition(38) is to take copies of functions in the supply rates of Σ_2. If we choose σ_1 such that $\sigma_1(s) = k\alpha_2(s)$ holds for $k > 0$, the two constraints (43) and (44) become identical. Then, it is not necessary to calculate (47). The simple choice $\hat{\lambda}_P(s) = \hat{\lambda}(s)$ is one pair of functions fulfilling (43) in such a case.

Remark 8. There always exist τ_1 and m such that (47) holds. It is also possible to remove the intermediate variable τ_1 appearing in Theorem 6. Indeed, the constraint

$$\frac{s}{\alpha_2 \circ \sigma_1^{-1}(s)} : \text{non-decreasing} \tag{50}$$

which does not involve neither τ_1 nor m is sufficient for ensuring (44). Thus, the subordinate conditions (47) are removed completely.

Remark 9. Theorem 2 and Theorem 6 do not require explicit information of the ISS Lyapunov function V_2. They use only $\underline{\alpha}_2^{-1} \circ \bar{\alpha}_2(s)$ which is a composite map describing the gap between lower and upper bounding estimates for V_2. For instance, if V_2 is limited to quadratic functions, the function $\underline{\alpha}_2^{-1} \circ \bar{\alpha}_2(s)$ can be easily computed as a linear function ks with a real number $k > 0$.

4. LYAPUNOV FUNCTIONS

Although detailed proofs are omitted due to the space limitation, this section shows Lyapunov functions establishing the main results.

Sketch of the proof of Theorem 2 : Let τ_1 and m be any real numbers satisfying

$$0\leq m, \quad 1<\tau_1<c_1, \quad (\tau_1/c_1)^m \leq (\tau_1 - 1)(c_2 - 1)$$

Select a function $f : s \in \mathbb{R}_+ \to \mathbb{R}_+$ so that it is continuous at all $s \in [0, \infty)$ and satisfies

$$f(s) > 0, \quad \forall s \in (0, \infty)$$
$$f(s)\hat{\lambda}(s), f(s)\kappa(s) : \text{non-decreasing on } \mathbb{R}_+$$

where κ is a class $\mathcal{K}$ function given by

$$\kappa(s)=\left[\alpha_2\circ\sigma_1^{-1}\circ\frac{1}{\tau_1}\alpha_1\circ\bar{\alpha}_1^{-1}(s)\right]\left[\frac{1}{\tau_1}\alpha_1\circ\bar{\alpha}_1^{-1}(s)\right]^m$$

Such a function $f(s)$ always exists. Define

$$\lambda_0(s) = f(s)\kappa(s) \tag{51}$$

$$\lambda_2(s) = \frac{c_2}{(c_2-1)}\sqrt{\frac{c_1}{\tau_1}}\left[\nu \circ \sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right] \times \left[\sigma_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{m+1} \tag{52}$$

$$\nu(s)=\left[f\circ\bar{\alpha}_1\circ\alpha_1^{-1}(\tau_1 s)\right]\left[\hat{\lambda}\circ\bar{\alpha}_1\circ\alpha_1^{-1}(\tau_1 s)\right] \tag{53}$$

The $\mathbf{C}^1$ function

$$V_{cl}(t,x)=\int_0^{V_1(t,x_1)}\lambda_0(s)ds + \int_0^{V_2(t,x_2)}\lambda_2(s)ds \tag{54}$$

satisfies $\underline{\alpha}_{cl}(|x|) \leq V_{cl}(t,x) \leq \bar{\alpha}_{cl}(|x|)$ with some $\underline{\alpha}_{cl}, \bar{\alpha}_{cl} \in \mathcal{K}_\infty$ for all $x\in\mathbb{R}^n$ and $t\in\mathbb{R}_+$, and there exist $\alpha_{cl} \in \mathcal{K}_\infty$ and $\sigma_{cl} \in \mathcal{K}$ such that

$$\dot{V}_{cl} \leq -\alpha_{cl}(|x|)+\sigma_{cl}(|r|), \ \forall x\in\mathbb{R}^n, r\in\mathbb{R}^m, t\in\mathbb{R}_+$$

holds along the trajectories of Σ.

Sketch of the proof of Theorem 6 : Define $\lambda_0(s)$ and $\lambda_2(s)$ as (52)-(53) and

$$\lambda_0(s) = \frac{1}{\hat{\lambda}_P(s)}, \quad f(s) = \frac{1}{\hat{\lambda}_P(s)\kappa(s)} \tag{55}$$

The function $V_{cl}(t,x)$ given by (54) satisfies (48) with some $\underline{\alpha}_{cl}, \bar{\alpha}_{cl} \in \mathcal{K}_\infty$, and achieves (49).

5. CONCLUDING REMARKS

This paper has developed new tools to establish stability properties of interconnected nonlinear systems. The *parametrization of supply rates* has provided us with a set of supply rates with which a single ISS small-gain condition can establish the desired ISS or dissipative property of the interconnected system. The results in this paper have the following features:

- The ISS guaranteed by Theorem 2 in this paper is stronger than the asymptotic stability addressed in Ito (2003).
- Theorem 6 enables us to establish a desired dissipative property of the interconnected system, which was impossible in Ito (2003).
- Theorem 2 and Theorem 6 allow more flexible supply rates than the previous result presented in Ito (2003). We do not have to take copies of functions in the supply rate of Σ_2 for selecting initial supply rate of Σ_1.
- This paper has removed an assumption $(c_1 - 1)(c_2 - 1) > 1$ made in Ito (2003) which is not required in the original ISS small-gain theorem(Jiang *et al.*, 1994; Teel, 1996). Due to the removal, *the parametrization of supply rates* precisely includes the ISS small-gain theorem as a special case.

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